Specialist Mathematics Trial Exam 1 Solutions 2006
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Q1. $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$

Since $\overline{OB}$ bisects $\overline{AC}$, $\therefore \overrightarrow{OB} = \frac{1}{2}\left(\overrightarrow{OC} + \overrightarrow{OA}\right)$.

Since $\overline{OB}$ is perpendicular to $\overline{AC}$, $\therefore \overrightarrow{OB} \bullet \overrightarrow{AC} = 0$,

$\therefore \frac{1}{2}\left(\overrightarrow{OC} + \overrightarrow{OA}\right) \bullet \left(\overrightarrow{OC} - \overrightarrow{OA}\right) = 0$, $\therefore \overrightarrow{OC} \bullet \overrightarrow{OC} - \overrightarrow{OA} \bullet \overrightarrow{OA} = 0$,

$\therefore \overline{OC}^2 - \overline{OA}^2 = 0$ and $\therefore \overline{OC} = \overline{OA}$. Hence ΔOAC is isosceles.

Q2. $z^3 - 3iz^2 + 3z + 9i^3 = 0$, $z^3 - 3iz^2 + 3z - 9i = 0$,

$\left(z^3 - 3iz^2\right) + \left(3z - 9i\right) = 0$, $z^2(z - 3i) + 3(z - 3i) = 0$,

$(z - 3i)\left(z^2 + 3\right) = 0$, $(z - 3i)\left(z - i\sqrt{3}\right)\left(z + i\sqrt{3}\right) = 0$,

$\therefore z = 3i$, $i\sqrt{3}$ or $-i\sqrt{3}$.

Q3a. $xy - y^2 = 2$.

By implicit differentiation: $\frac{d}{dx}\left(xy - y^2\right) = 0$,

$\frac{d}{dx}(xy) - \frac{d}{dx}\left(y^2\right) = 0$, $y + x\frac{dy}{dx} - 2y\frac{dy}{dx} = 0$,

$y = (2y - x)\frac{dy}{dx}$, $\therefore \frac{dy}{dx} = \frac{y}{2y - x}$.

Q3b. When $x = 3$, $3y - y^2 = 2$, $y^2 - 3y + 2 = 0$,

$(y-1)(y-2) = 0$, $\therefore y = 1$ or 2. $\therefore$ the relation contains two points with $x = 3$. They are $(3,1)$ and $(3,2)$.

At $(3,1)$, $\frac{dy}{dx} = \frac{1}{2(1) - 3} = -1$; at $(3,2)$, $\frac{dy}{dx} = \frac{2}{2(2) - 3} = 2$.

Q4. Let $u = \log_e\left(x^2 + 1\right)$, $\frac{du}{dx} = \frac{1}{x^2 + 1} \times 2x = \frac{2x}{x^2 + 1}$.

When $x = 0$, $u = 0$; when $x = \sqrt{e - 1}$, $u = 1$.

$\therefore \int_0^{\sqrt{e-1}} \left(\frac{2x \log_e\left(x^2 + 1\right)}{x^2 + 1}\right) dx = \int_0^{\sqrt{e-1}} u\frac{du}{dx}\,dx = \int_0^1 u\,du = \left[\frac{u^2}{2}\right]_0^1 = \frac{1}{2}$.

Q5a. $y = 1 + \left(x^2 - 2x + 2\right)\tan^{-1}(x - 1)$,

$\frac{dy}{dx} = 0 + \left(x^2 - 2x + 2\right)\frac{d}{dx}\left(\tan^{-1}(x - 1)\right) + \tan^{-1}(x - 1)\frac{d}{dx}\left(x^2 - 2x + 2\right)$

$= \left(x^2 - 2x + 2\right)\frac{1}{1 + (x-1)^2} + (2x - 2)\tan^{-1}(x - 1)$

$= \left(x^2 - 2x + 2\right)\frac{1}{x^2 - 2x + 2} + 2(x - 1)\tan^{-1}(x - 1)$

$= 1 + 2(x - 1)\tan^{-1}(x - 1)$.

Q5b. $\frac{dy}{dx} = 1 + 2(x - 1)\tan^{-1}(x - 1)$,

$\therefore \frac{d^2y}{dx^2} = 2(x - 1)\frac{1}{1 + (x - 1)^2} + 2\tan^{-1}(x - 1)$.

At $(1,1)$, $x = 1$, $\frac{d^2y}{dx^2} = 0 + 2\tan^{-1}(0) = 0$, $\therefore (1,1)$ is a point of inflection.

Q6a. $\boldsymbol{r} = \sin(2t)\,\boldsymbol{i} + \cos(2t)\,\boldsymbol{j} + \left(10 - t^2\right)\boldsymbol{k}$,

$\boldsymbol{v}(t) = \frac{d}{dt}\boldsymbol{r} = 2\cos(2t)\,\boldsymbol{i} - 2\sin(2t)\,\boldsymbol{j} - 2t\,\boldsymbol{k}$.

At $t = 0$, $\boldsymbol{v} = 2\cos(0)\,\boldsymbol{i} - 2\sin(0)\,\boldsymbol{j} = 2\,\boldsymbol{i}$.

Q6b. $\boldsymbol{v}(t) = 2\cos(2t)\,\boldsymbol{i} - 2\sin(2t)\,\boldsymbol{j} - 2t\,\boldsymbol{k}$,

$\boldsymbol{a}(t) = \frac{d}{dt}\boldsymbol{v} = -4\sin(2t)\,\boldsymbol{i} - 4\cos(2t)\,\boldsymbol{j} - 2\,\boldsymbol{k}$,

$a = \sqrt{(-4\sin(2t))^2 + (-4\cos(2t))^2 + (-2)^2}$

$= \sqrt{16\sin^2(2t) + 16\cos^2(2t) + 4}$

$= \sqrt{16\left(\sin^2(2t) + \cos^2(2t)\right) + 4} = \sqrt{16 + 4} = \sqrt{20} = 2\sqrt{5}$.

Q7 $y = \frac{2}{\sqrt{2 - x^2}}$, $0 \le x \le 1$ is rotated about the x-axis.

Volume of the solid of revolution

$V = \int_0^1 \pi y^2\,dx = 4\pi\int_0^1 \frac{1}{2 - x^2}\,dx$.

Change $\frac{1}{2 - x^2}$ to partial fractions,

$\frac{1}{2 - x^2} = \frac{A}{\sqrt{2} - x} + \frac{B}{\sqrt{2} + x} = \frac{\sqrt{2}}{4}\left(\frac{1}{\sqrt{2} - x} + \frac{1}{\sqrt{2} + x}\right)$.

$\therefore V = \sqrt{2}\pi\int_0^1 \left(\frac{1}{\sqrt{2} - x} + \frac{1}{\sqrt{2} + x}\right)dx$

$= \sqrt{2}\pi\left[-\log_e\left(\sqrt{2} - x\right) + \log_e\left(\sqrt{2} + x\right)\right]_0^1$

$= \sqrt{2}\pi\left[\log_e\left(\frac{\sqrt{2} + x}{\sqrt{2} - x}\right)\right]_0^1 = \sqrt{2}\pi\log_e\left(\frac{\sqrt{2} + 1}{\sqrt{2} - 1}\right)$.

Q8. $\frac{dy}{dx} = \sin(2x)\sqrt{1+\sin(x)}$.

$y = \int \sin(2x)\sqrt{1+\sin(x)}dx = \int 2\sin(x)\cos(x)\sqrt{1+\sin(x)}dx$.

Let $u = 1+\sin(x)$, $\therefore \sin(x) = u-1$ and $\frac{du}{dx} = \cos(x)$.

$\therefore y = \int 2(u-1)\sqrt{u}\frac{du}{dx}dx = \int\left(2u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right)du$

$= \frac{4}{5}u^{\frac{5}{2}} - \frac{4}{3}u^{\frac{3}{2}} + C = \frac{4}{5}(1+\sin(x))^{\frac{5}{2}} - \frac{4}{3}(1+\sin(x))^{\frac{3}{2}} + C$.

$y = \frac{7}{15}$ when $x = 0$, $\therefore \frac{7}{15} = \frac{4}{5} - \frac{4}{3} + C$, $C = 1$,

$\therefore y = \frac{4}{5}(1+\sin(x))^{\frac{5}{2}} - \frac{4}{3}(1+\sin(x))^{\frac{3}{2}} + 1$.

Q9. $\{z : |z+i| + |z-i| = 4\}$. Let $z = x + yi$,

$|x+(y+1)i| = 4 - |x+(y-1)i|$,

$|x+(y+1)i|^2 = (4 - |x+(y-1)i|)^2$,

$|x+(y+1)i|^2 = 16 - 8|x+(y-1)i| + |x+(y-1)i|^2$,

$x^2 + (y+1)^2 = 16 - 8|x+(y-1)i| + x^2 + (y-1)^2$,

$8|x+(y-1)i| = 16 - 4y$, $\therefore 2|x+(y-1)i| = 4 - y$,

$(2|x+(y-1)i|)^2 = (4-y)^2$,

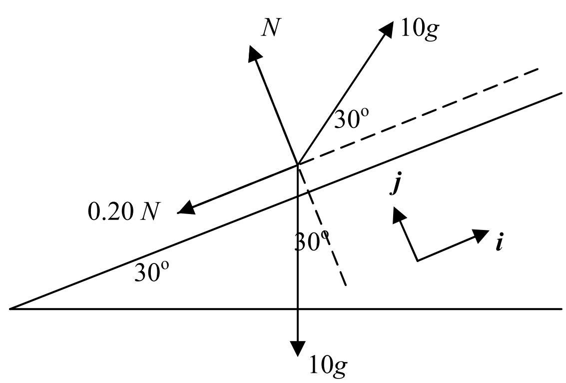
$4\left(x^2 + (y-1)^2\right) = 16 - 8y + y^2$,

$4x^2 + 4y^2 - 8y + 4 = 16 - 8y + y^2$,

$4x^2 + 3y^2 = 12$, $\therefore \frac{x^2}{3} + \frac{y^2}{4} = 1$, i.e. $\frac{(\operatorname{Re} z)^2}{3} + \frac{(\operatorname{Im} z)^2}{4} = 1$.

Hence $p = 3$ and $q = 4$.

Q10a.



j-component: Resultant force $= 0$,

$N + 10g\sin 30^o - 10g\cos 30^o = 0$,

$\therefore N + 5g - 5\sqrt{3}g = 0$,

$\therefore N = 5\left(\sqrt{3}-1\right)g$.

Q10b.

i-component: Resultant force $= ma$,

$10g\cos 30^o - 10g\sin 30^o - 0.20N = 10a$,

$5\sqrt{3}g - 5g - 0.20\left(5\left(\sqrt{3}-1\right)g\right) = 10a$,

$4\left(\sqrt{3}-1\right)g = 10a$,

$\therefore a = \frac{2g}{5}\left(\sqrt{3}-1\right)$.

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