INSIGHT

Trial Exam Paper

2006

SPECIALIST MATHEMATICS

Written examination 2

Worked solutions

This book presents:

- worked solutions, giving you a series of points to show you how to work through the questions
- mark allocations.

SECTION 1

Question 1

If $2i$ is a solution of the equation $z^3 - 5z^2 + 4z - mi = 0$, then the value of m will be

A. $-2i$

B. **–20i**

C. -20

D. 20

E. $20i$

Answer is B

Worked solution

Let $z = 2i$

$(2i)^3 - 5(2i)^2 + 4(2i) - mi = 0$

$-8i + 20 + 8i - mi = 0$

$20 - mi = 0$

$m = \dfrac{20}{i}$

$m = \dfrac{20}{i} \times \dfrac{i}{i}$

$m = -20i$

Question 2

If $z = -1 + \sqrt{3}$, then Arg(z^2) equals

A. $-\dfrac{\mathbf{2\pi}}{\mathbf{3}}$

B. $-\dfrac{\pi}{3}$

C. $\dfrac{\pi}{3}$

D. $\dfrac{2\pi}{3}$

E. $\dfrac{4\pi}{3}$

Answer is A

Worked solution

$z = -1 + \sqrt{3} = r\text{cis}\theta$

$r = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$

$\theta = \tan^{-1}\left(\dfrac{\sqrt{3}}{-1}\right) = \pi - \dfrac{\pi}{3} = \dfrac{2\pi}{3}$

$z = 2\text{cis}\left(\dfrac{2\pi}{3}\right)$

$z^2 = 2^2\,\text{cis}\left(2 \times \dfrac{2\pi}{3}\right)$ by De Moivre's Theorem

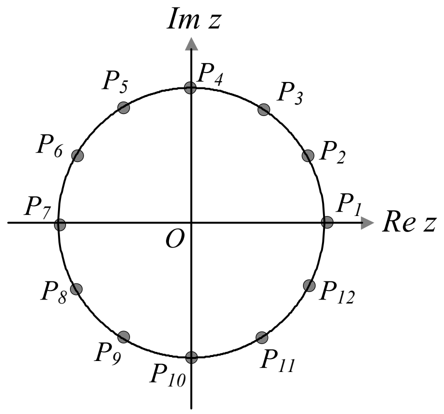
$z^2 = 4\text{cis}\left(\dfrac{4\pi}{3}\right)$

$z^2 = 4\text{cis}\left(-\dfrac{2\pi}{3}\right)$

$\therefore \text{Arg}(z^2) = -\dfrac{2\pi}{3}$

Question 3

Points P_1 to P_{12} are twelve equally spaced points around the circumference of a circle.



Point P_3 represents the complex number $z = a + ib$.

The complex number $i^{11}\bar{z}$ is represented by point

A. P_2

B. P_5

C. $\boldsymbol{P_8}$

D. P_9

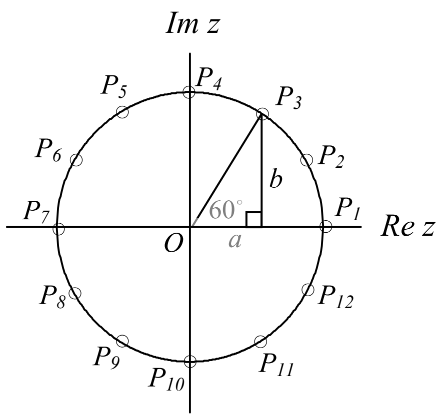
E. P_{11}

Answer is C

Worked solution

Find expressions for complex numbers P_2, P_5, P_8, P_9, P_{11} in terms of a and b

$\angle P_1OP_2 = \dfrac{360°}{12} = 30° \quad \Rightarrow \quad \angle P_1OP_3 = 60°$



$z = a + bi$

$\tan 60° = \dfrac{b}{a}$

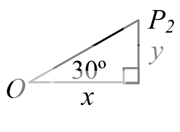
$\Rightarrow \sqrt{3} = \dfrac{b}{a}$

Taking reciprocals: $\dfrac{1}{\sqrt{3}} = \dfrac{a}{b}$

$\tan 30° = \dfrac{a}{b}$ K (1)

Finding P_2

Let $P_2 = x + yi$



$\tan 30° = \dfrac{y}{x}$

$\dfrac{a}{b} = \dfrac{y}{x}$ from (1)

$\therefore \quad a = y, \quad b = x$

Therefore $P_2 = b + ai$

By symmetry :
$P_5 = -a + bi, \quad P_6 = -b + ai$
$P_8 = -b - ai, \quad P_9 = -a - bi$
$P_{11} = a - bi, \quad P_{12} = b - ai$

Simplify $i^{11}\bar{z}$:
$i^{11}\bar{z} = i^{11}(a - bi)$
$= -i(a - bi)$
$= -ai + bi^2$
$= -ai - b$
$= -b - ai$
Point P_8

Question 4

The range of the function $f(x) = \cos^{-1}(x - \pi) - 1$ is

A. $[\pi - 1, \ \pi + 1]$

B. $\boldsymbol{[-1, \ \pi - 1]}$

C. $[0, \ \pi]$

D. $[-2, \ 0]$

E. $[-1, \ 1]$

Answer is B

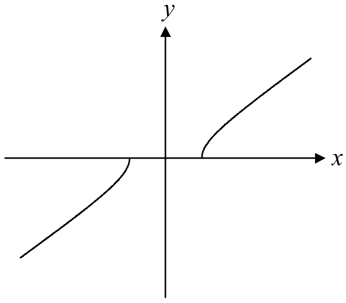
Worked solution

$g(x) = \cos^{-1}(x)$ has range $[0, \ \pi]$. This graph is translated horizontally by π units and vertically by -1 unit to give $f(x) = \cos^{-1}(x - \pi) - 1$
The range of this function is $[-1, \ \pi - 1]$

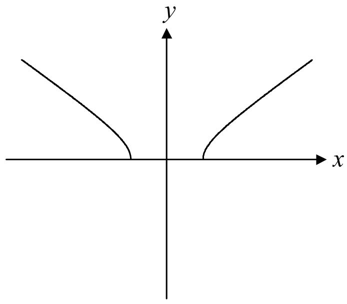
Question 5

A graph of the curve specified by the parametric equations $x = \sec(t)$, $y = \tan(t)$ where $t \in [0, \pi]$ could be

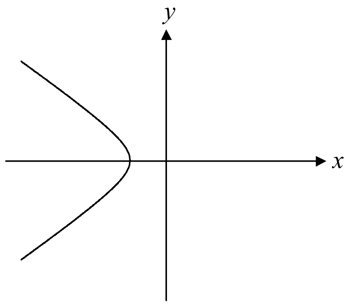
A.



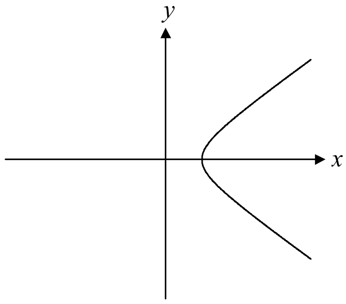
B.



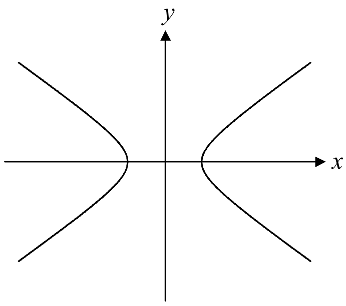
C.



D.



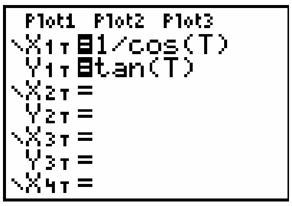
E.

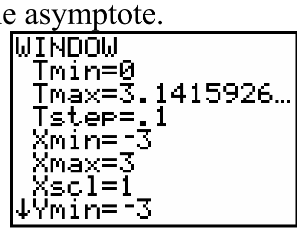


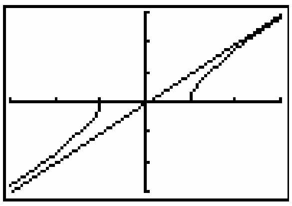
Answer is A

Worked solution

When $t \in \left[0, \frac{\pi}{2}\right)$, $x = \sec(t) = \frac{1}{\cos(t)}$ is positive and $y = \tan(t)$ is positive.
Since both x and y are positive, a branch of the graph will be in the first quadrant.
When $t \in \left(\frac{\pi}{2}, \pi\right]$, $x = \sec(t)$ is negative and $y = \tan(t)$ is negative.
Since both x and y are negative, a branch of the graph will be in the third quadrant.
A and E satisfy these conditions, but E shows four quadrants, i.e. $t \in [0, 2\pi]$
The solution can found by sketching the graph on a calculator in parametric mode.
The calculator draws the asymptote.







Question 6

Consider the function $f : R \to R$ where $f(x) = 4x^3 - 3x^4$

Which one of the following statements is not true?

A. f has two stationary points

B. f has two points of inflexion

C. $\boldsymbol{f'}$ **is maximum when** $\boldsymbol{x = \frac{2}{3}}$

D. $\frac{1}{f}$ has three asymptotes

E. $f = \frac{1}{f}$ has three solutions

Answer is C

Worked solution

Graphing $f(x) = 4x^3 - 3x^4$ shows that stationary points occur at $x = 0$ and $x = 1$.
Therefore A is true.

$f'(x) = 12x^2 - 12x^3$

$f''(x) = 24x - 36x^2$

Points of inflexion occur where $f''(x) = 0$.

$24x - 36x^2 = 0$

$12x(2 - 3x) = 0$

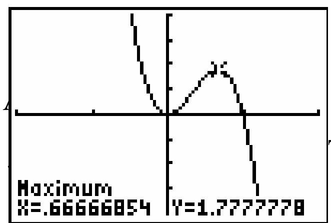
$x = 0, \quad x = \frac{2}{3}$

$f(x)$ has two points of inflexion. Therefore B is true.

$x = 0$ is a stationary point of inflexion.

$x = \frac{2}{3}$ is the point of maximum gradient over the interval $\left(-\frac{1}{3}, \quad \infty\right)$.

Sketching a graph of $f'(x)$ shows a local maximum at $x = \frac{2}{3}$; however this is not the maximum value of $f'(x)$ over $x \in R$.

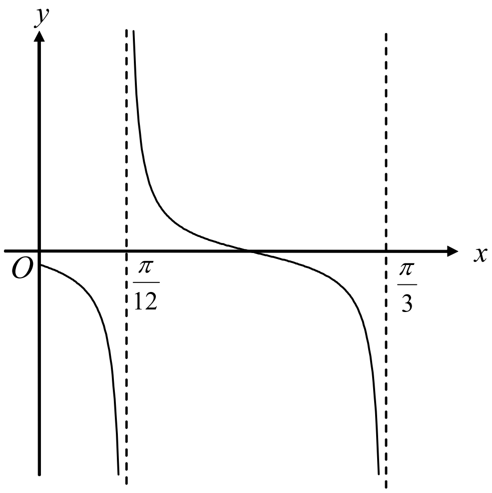


Therefore C is not true.

Graphing $f(x)$ and $\frac{1}{f(x)}$ will show that both D and E are true.

Question 7

A graph of $f:\left[0, \ \frac{\pi}{3}\right) \text{ where } f(x)=\cot\left(nx-\frac{\pi}{3}\right)$ is sketched below.



The value of n could be

A. $\frac{1}{4}$

B. $\frac{1}{3}$

C. 3

D. **4**

E. 8

Answer is D

Worked solution

Period of $y=\cot(nx-\frac{\pi}{3})$ is $\frac{\pi}{n}$

Period of this graph is $\frac{\pi}{3}-\frac{\pi}{12}=\frac{\pi}{4}$

$\Rightarrow \frac{\pi}{n}=\frac{\pi}{4}$

$\therefore \quad n=4$

Question 8

The gradient of the curve $y^2 = 4x + 6y - 5$ is $-\frac{2}{3}$ at the point where y equals

A. **0**

B. 0.15

C. 1.25

D. 5

E. 6

Answer is A

Worked solution

$y^2 = 4x + 6y - 5$

Using implicit differentiation:

$2y\frac{dy}{dx} = 4 + 6\frac{dy}{dx} + 0$

$(2y - 6)\frac{dy}{dx} = 4$

$\frac{dy}{dx} = \frac{4}{2y - 6}$

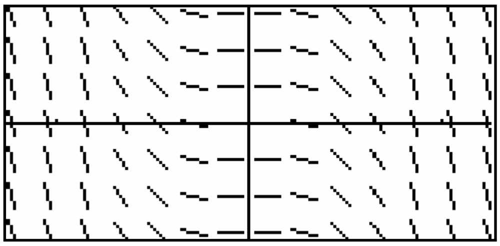
$-\frac{2}{3} = \frac{4}{2y - 6}$

$-4y + 12 = 12$

$y = 0$

Question 9

The slope field from a first order differential equation is shown below.



If $a \in R$, a solution of this differential equation could be

A. $y = a\log_e(x)$

B. $y = a\cos(x)$

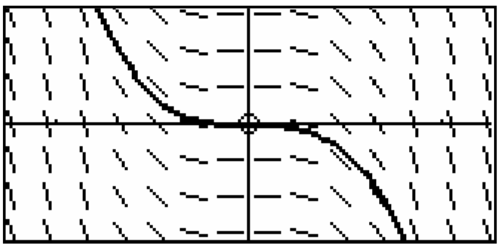
C. $y = a\tan^{-1}(x)$

D. $y = \frac{a}{x^2}$

E. $\boldsymbol{y = ax^3}$

Answer is E

Worked solution



$y = ax^3$ where $a \in (-\infty, 0)$

Question 10

Given $\dfrac{dy}{dx} = \sqrt{\sin(2x)}$ and $y = \sqrt{2}$ when $x = \dfrac{\pi}{12}$.

The value of y when $x = \dfrac{\pi}{3}$ is

A. 0.2500

B. 0.7298

C. 0.9306

D. **1.4369**

E. 2.1440

Answer is D

Worked solution

$$y = \int \sqrt{\sin(2x)}\, dx$$
$$y = f(x) + c$$

When $\quad x = \dfrac{\pi}{12}, \quad y = \dfrac{1}{\sqrt{2}}$

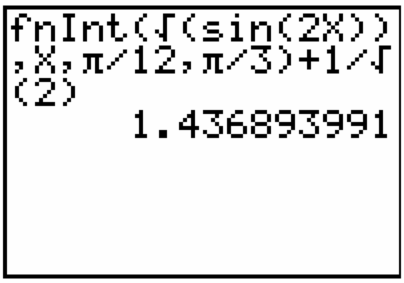
$$\frac{1}{\sqrt{2}} = f\left(\frac{\pi}{12}\right) + c$$
$$c = -f\left(\frac{\pi}{12}\right) + \frac{1}{\sqrt{2}} \qquad \ldots(1)$$

When $\quad x = \dfrac{\pi}{3}, \quad y = f\left(\dfrac{\pi}{3}\right) + c \qquad \ldots(2)$

Substitute (1) into (2):

$$y = f\left(\frac{\pi}{3}\right) - f\left(\frac{\pi}{12}\right) + \frac{1}{\sqrt{2}}$$
$$y = \int_{\frac{\pi}{12}}^{\frac{\pi}{3}} \sqrt{\sin(2x)}\, dx + \frac{1}{\sqrt{2}}$$

Using fnInt on calculator:



Question 11

Using a suitable substitution, $\int_5^{10} \frac{1}{x^2} e^{\frac{10}{x}}\, dx$ can be expressed as

A. $\int_1^2 \frac{100}{u^2} e^u\, du$

B. $100 \int_1^2 u^2 e^u\, du$

C. $-10 \int_2^1 e^u\, du$

D. $\boldsymbol{\frac{1}{10} \int_1^2 e^u\, du}$

E. $-\frac{1}{10} \int_5^{10} e^u\, du$

Answer is D

Worked solution

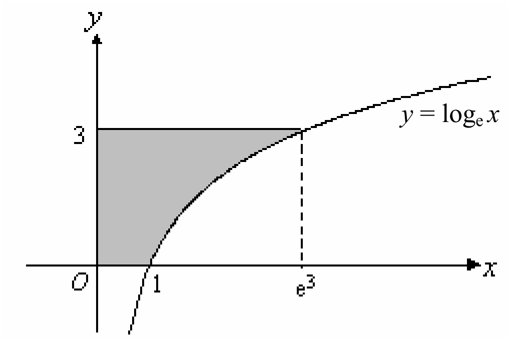
Let $u = \frac{10}{x}$, $\frac{du}{dx} = -\frac{10}{x^2}$ $\Rightarrow du = -\frac{10}{x^2} dx$

Finding the terminals of integration: When $x = 10$, $u = \frac{10}{10} = 1$

When $x = 5$, $u = \frac{10}{5} = 2$

$$\begin{aligned}
\int_5^{10} \frac{1}{x^2} e^{\frac{10}{x}}\, dx &= -\frac{1}{10} \int_5^{10} -\frac{10}{x^2} e^{\frac{10}{x}}\, dx \\
&= -\frac{1}{10} \int_5^{10} e^{\frac{10}{x}} \left(-\frac{10}{x^2}\, dx \right) \\
&= -\frac{1}{10} \int_2^1 e^u\, du \\
&= \frac{1}{10} \int_1^2 e^u\, du
\end{aligned}$$

Question 12



The graph of $y = \log_e x$ is shown above. The volume of the solid of revolution formed when the shaded region is rotated around the y-axis is given by

A. $\pi\int_0^3 (3-\log_e x)^2\,dx$

B. $\pi\int_1^{e^3} (\log_e x)^2\,dx$

C. $\pi\int_1^{e^3} (3-e^y)^2\,dy$

D. $\pi\int_0^3 e^y\,dy$

E. $\boldsymbol{\pi\int_0^3 e^{2y}\,dy}$

Answer is E

Worked solution

Rotating around the y-axis: Volume $= \pi\int_0^3 x^2\,dy$

Finding expression for x^2: $y = \log_e x$

$x = e^y$

$x^2 = (e^y)^2$

$V = \pi\int_0^3 (e^y)^2\,dy$

$V = \pi\int_0^3 e^{2y}\,dy$

Question 13

A spherical ice ball initially has radius 0.9 cm. It is placed in a drink and melts at a constant rate of 1.5 cm^3/minute. When the radius is 0.6 cm, the rate, in cm/minute, at which the radius is decreasing is

A. $\frac{5}{24\pi}$

B. $\frac{25}{72\pi}$

C. $\boldsymbol{\frac{25}{24\pi}}$

D. $\frac{54\pi}{25}$

E. $\frac{36\pi}{25}$

Answer is C

Worked solution

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

$1.5 = 4\pi r^2 \times \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{1.5}{4\pi r^2}$

When $r = 0.6$ cm

$\frac{dr}{dt} = \frac{1.5}{4\pi \times 0.6^2} = \frac{25}{24\pi}$

Question 14

A tank initially contains 200 litres of pure water. A salt solution with a concentration of 0.2 kg/litre is poured into the tank at a rate of 5 litres/minute. The mixture is kept uniform by stirring and flows out of the tank at a rate of 3 litres/minute.

Let Q be the amount of salt in the tank after t minutes.

$\frac{dQ}{dt}$ is equal to

A. $5-\frac{3Q}{200+2t}$

B. $5-\frac{3Q}{200}$

C. $(5-3t)\frac{Q}{200}$

D. $1-\frac{3Q}{200-2t}$

E. $\mathbf{1-\frac{3Q}{200+2t}}$

Answer is E

Worked solution

The volume of mixture in the tank after t minutes is $200+2t$ litres

The concentration of salt in the tank after t minutes is $\frac{Q}{200+2t}$ kg/litre

Rate of inflow of salt is $5\times 0.2=1$ kg/minute

Rate of outflow of salt is $3\times\frac{Q}{200+2t}$ kg/minute

$\frac{dQ}{dt}=$ rate of inflow – rate of outflow

$\frac{dQ}{dt}=1-\frac{3Q}{200+2t}$ kg/minute

Question 15

Let $\underset{\sim}{u} = 6\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}$ and $\underset{\sim}{v} = 2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$.

The vector resolute of $\underset{\sim}{u}$ in the direction of $\underset{\sim}{v}$ is

A. $\frac{1}{49}(2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k})$

B. $\frac{1}{7}(2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k})$

C. $\mathbf{\frac{1}{14}(2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k})}$

D. $\frac{1}{\sqrt{14}}(2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k})$

E. $\frac{1}{7\sqrt{14}}(2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k})$

Answer is C

Worked solution

Vector resolute of $\underset{\sim}{u}$ in the direction of $\underset{\sim}{v}$ is

$$(\underset{\sim}{u}.\hat{\underset{\sim}{v}})\hat{\underset{\sim}{v}} = \left((6\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}).\frac{(2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k})}{\sqrt{4+1+9}}\right).\frac{(2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k})}{\sqrt{4+1+9}}$$

$$= \left(\frac{12-2-9}{\sqrt{14}}\right)\frac{(2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k})}{\sqrt{14}}$$

$$= \frac{1}{14}(2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k})$$

Question 16

Points A, B and C are collinear such that $AB : BC = 1 : 3$

If $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OC} = \underset{\sim}{c}$ then $\overrightarrow{OB}$ equals

A. $\mathbf{\frac{1}{4}(3\underset{\sim}{a} + \underset{\sim}{c})}$

B. $\frac{1}{4}(\underset{\sim}{a} + 3\underset{\sim}{c})$

C. $\frac{1}{4}(5\underset{\sim}{a} - \underset{\sim}{c})$

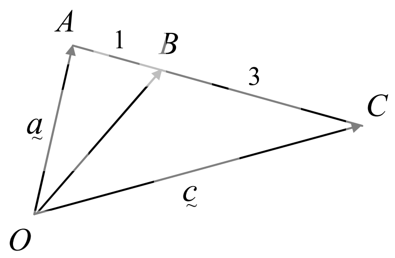
D. $\frac{1}{3}(2\underset{\sim}{a} + \underset{\sim}{c})$

E. $\frac{1}{3}(\underset{\sim}{a} - 3\underset{\sim}{c})$

Answer is A

Worked Solution

Draw a vector diagram



$\overrightarrow{AC} = -\underset{\sim}{a} + \underset{\sim}{c}$

$\overrightarrow{AB} = \frac{1}{4}\overrightarrow{AC} = \frac{1}{4}(-\underset{\sim}{a} + \underset{\sim}{c})$

$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$

$\overrightarrow{OB} = \underset{\sim}{a} + \frac{1}{4}(-\underset{\sim}{a} + \underset{\sim}{c})$

$\overrightarrow{OB} = \frac{3}{4}\underset{\sim}{a} + \frac{1}{4}\underset{\sim}{c}$

$\overrightarrow{OB} = \frac{1}{4}(3\underset{\sim}{a} + \underset{\sim}{c})$

Question 17

The position of a particle at time t is given by $\underset{\sim}{r}(t) = (t^3 + 2t)\underset{\sim}{i} + 5t\underset{\sim}{j} - t^2\underset{\sim}{k}$.

The magnitude of its acceleration when $t = 1$ is

A. $3\underset{\sim}{i} + 5\underset{\sim}{j} - \underset{\sim}{k}$

B. $6\underset{\sim}{i} - 2\underset{\sim}{k}$

C. $\mathbf{2\sqrt{10}}$

D. $3\sqrt{6}$

E. $\sqrt{35}$

Answer is C

Worked solution

$\underset{\sim}{r}(t) = (t^3 + 2t)\underset{\sim}{i} + 5t\underset{\sim}{j} - t^2\underset{\sim}{k}$

$\dot{\underset{\sim}{r}}(t) = (3t^2 + 2)\underset{\sim}{i} + 5\underset{\sim}{j} - 2t\underset{\sim}{k}$

$\ddot{\underset{\sim}{r}}(t) = 6t\underset{\sim}{i} - 2\underset{\sim}{k}$

$\ddot{\underset{\sim}{r}}(1) = 6\underset{\sim}{i} - 2\underset{\sim}{k}$

$\left|\ddot{\underset{\sim}{r}}(1)\right| = \sqrt{6^2 + (-2)^2} = \sqrt{40} = 2\sqrt{10}$

Question 18

A particle is moving in a straight line with an acceleration of $-20x+20$ m/s^2, where x is its displacement, in metres, from a fixed point O. If the particle is travelling with a velocity of 6 m/s when it is 3 metres to the right of O, its maximum speed, in m/s, is

A. 6.0

B. 9.8

C. 10.0

D. **10.8**

E. 12.0

Answer is D

Worked solution

$$a=-20x+20$$

$$\frac{d\left(\frac{1}{2}v^2\right)}{dx}=-20x+20$$

$$\frac{1}{2}v^2=\frac{-20}{2}x^2+20x+c$$

When $x=3,\quad v=6$:

$$\frac{1}{2}\times 6^2=-10\times 3^2+20\times 3+c$$

$$18=-90+60+c$$

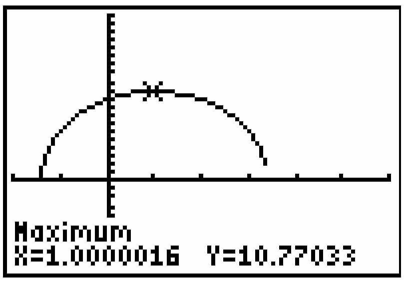
$$c=48$$

$$\Rightarrow\quad \frac{1}{2}v^2=-10x^2+20x+48$$

$$v^2=-20x^2+40x+96$$

$$v=\sqrt{-20x^2+40x+96}$$

Draw a graph of velocity on calculator:



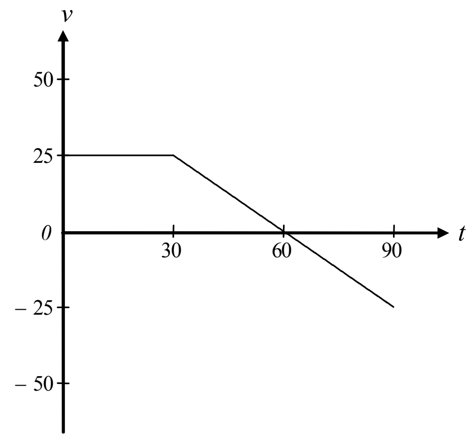
The maximum speed is 10.8 m/s. This occurs when the particle is 1 m from O.

Question 19

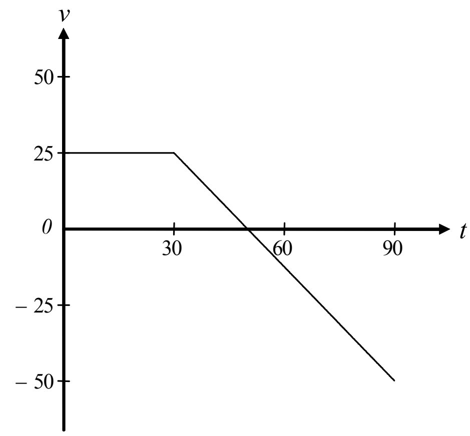
A particle travels in a straight line with a constant velocity of 25 m/s for 30 seconds. It then decelerates for 60 seconds and returns to its original position.

The velocity-time graph that best represents the motion of the particle is

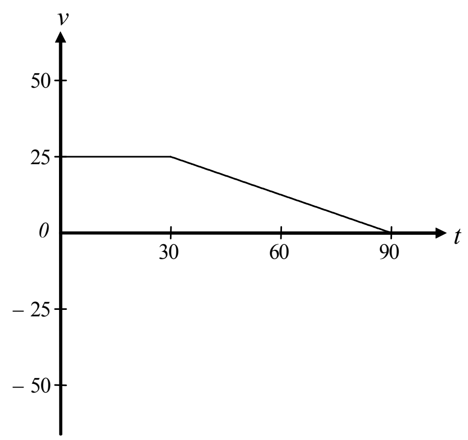
A.



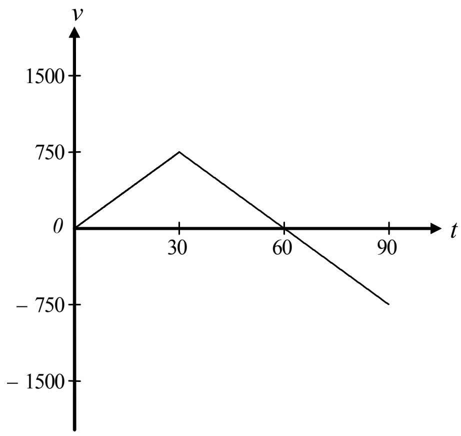
B.



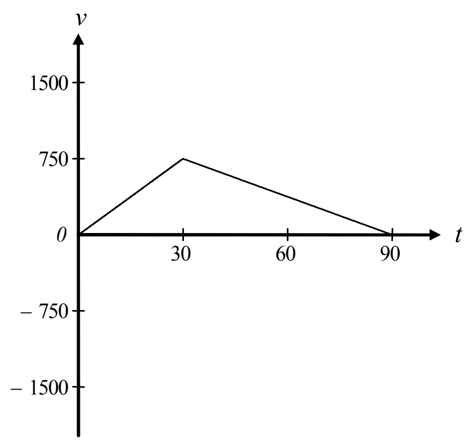
C.



D.



E.



Answer is B

Worked solution

Velocity is constant for the first 30 seconds, therefore discard D and E, as these graphs show constant acceleration.

The total signed area under velocity-time graph must be zero after 90 seconds since the particle returns to its original position.

Discard A and C since the total signed area is not zero.

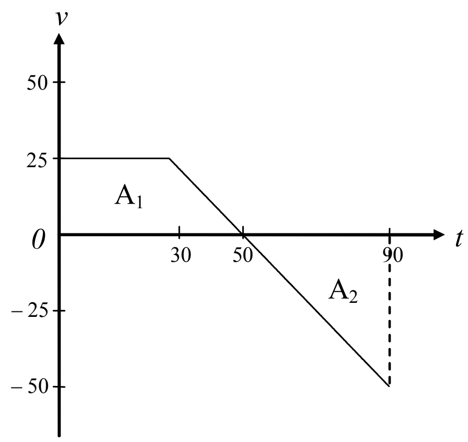
For B, calculate the t intercept of the line segment joining (30, 25) and (90, –50):

$$\frac{25-0}{30-t}=\frac{25+50}{30-90}$$

$$\frac{25}{30-t}=\frac{75}{-60}$$

$$-1500=2250-75t$$

$$t=50$$



Calculate the signed area of trapezium A_1.

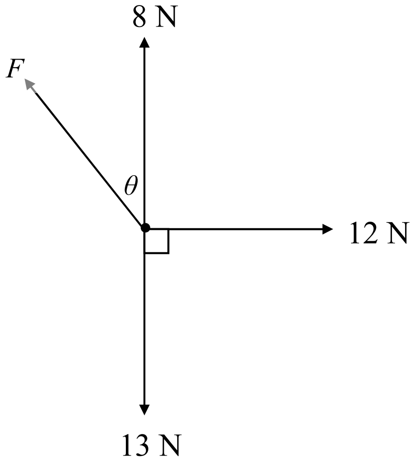
$$A_1=\frac{1}{2}(50+30)\times 25=1000$$

Calculate the signed area of triangle A_2

$$A_2=\frac{1}{2}\times 40\times -50=-1000$$

Total area $A_1+A_2=0$

Question 20



Four forces are acting on a particle as shown in the diagram above.

The particle will be in equilibrium when F, measured in newtons, is equal to

A. $5\cos\theta$

B. $12\sin\theta$

C. $\dfrac{\cos\theta}{12}$

D. 5

E. **13**

Answer is E

Worked solution

Resolving forces in a horizontal direction:

$$F\sin\theta = 12 \quad \text{K (1)}$$

Resolving forces in a vertical direction:

$$F\cos\theta + 8 = 13$$

$$F\cos\theta = 5 \quad \text{K (2)}$$

From (1) and (2):

$$(F\sin\theta)^2 + (F\cos\theta)^2 = 12^2 + 5^5$$

$$F^2(\sin^2\theta + \cos^2\theta) = 144 + 25$$

$$F^2 = 169$$

$$F = 13$$

Question 21

A motorbike is travelling at a speed of 60 km/hr on a straight road. A school zone is observed in the distance and over the next 10 seconds it reduces speed to 40 km/hr.

If the mass of the motorbike is 900 kg, the change in momentum, measured in kg m/s, in the direction of motion is

A. −6480

B. **−5000**

C. −1800

D. −500

E. −180

Answer is B

Worked solution

Speed must be converted to m/s.

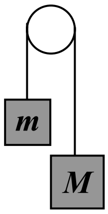
$$\Delta \underset{\sim}{p} = m\underset{\sim}{v}_2 - m\underset{\sim}{v}_1$$

$$\Delta \underset{\sim}{p} = 900 \times 40 \times \frac{1}{3.6} - 900 \times 60 \times \frac{1}{3.6}$$

$$\Delta \underset{\sim}{p} = -5000 \text{ kg m/s}$$

Question 22

A mass of m kg is attached to a second mass of M kg, $m < M$, by a light string passing over a smooth pulley as shown below. The tension in the string is T newtons.



The acceleration, in m/s^2, of the M kg mass is

A. g

B. Mg

C. $\dfrac{Mg - T}{m}$

D. $\boldsymbol{\dfrac{g(M - m)}{(M + m)}}$

E. $\dfrac{g(M + m)}{(M - m)}$

Answer is D

Worked solution

Let a m/s^2 be the acceleration of the system.
Since $m < M$, mass M is accelerating downwards.

Resolving forces:

m kg mass: $T - mg = ma$ …..(1)

M kg mass: $Mg - T = Ma$ …..(2)

Solving (1) and (2) for a:

$$Mg - mg = Ma + ma$$

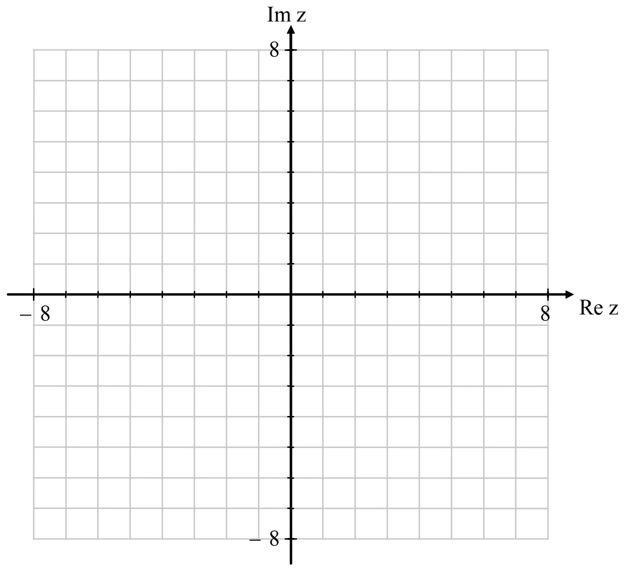
$$g(M - m) = a(M + m)$$

$$a = \frac{g(M - m)}{(M + m)}$$

END OF SECTION 1

SECTION 2

Question 1



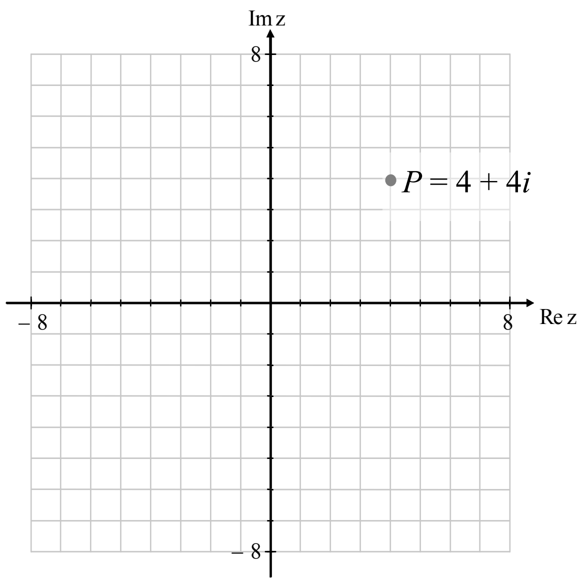
1a. Let $P = 4\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)$.

Express P in Cartesian form and plot and label this point in the Argand plane above.

Worked solution

$P = 4\sqrt{2}\cos\left(\frac{\pi}{4}\right) + 4\sqrt{2}\sin\left(\frac{\pi}{4}\right)i$

$P = 4 + 4i$ Point must be plotted correctly in Argand plane. 1A



1 mark

1b. **i.** Find an equivalent Cartesian equation for
$\{z : |z+2-4i| = |z-2|, z \in C\}$

Worked solution

$|z+2-4i| = |z-2|$

Let $z = x + yi$

$|x+yi+2-4i| = |x+yi-2|$

$\sqrt{(x+2)^2+(y-4)^2} = \sqrt{(x-2)^2+y^2}$ 1M

$x^2+4x+4+y^2-8y+16 = x^2-4x+4+y^2$

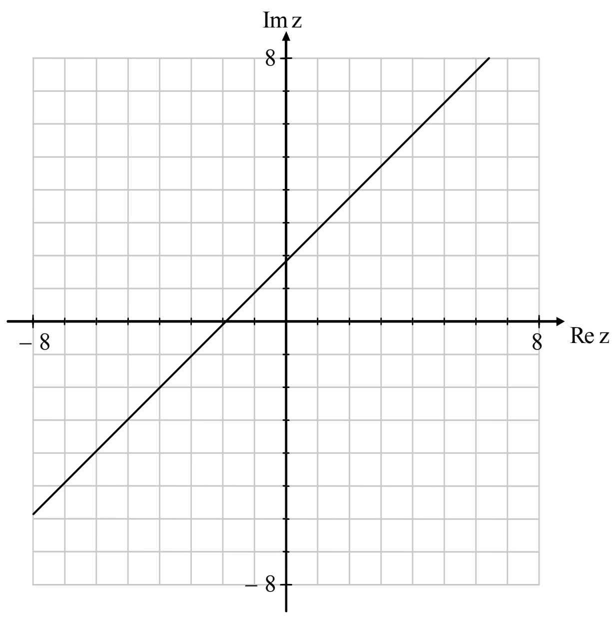
$8x-8y+16=0$

$y = x+2$ 1A

2 marks

1b. **ii.** Hence sketch $\{z : |z+2-4i| = |z-2|, z \in C\}$ on the Argand plane below.

Worked solution



Graph of $y = x+2$ sketched above. 1H

1 mark

1c. Describe the key features of the relation defined by $\{z : |z-i| = 5\}$

Worked solution

$|x+yi-i| = 5$

$|x+(y-1)i| = 5$

$\sqrt{x^2+(y-1)^2} = 5$

$x^2+(y-1)^2 = 25$

The relation represents a circle with centre $0+i$. 1A
The circle has radius 5 units. 1A

2 marks

1d. M and N are the points of intersection of the relations $\{z : |z - i| = 5\}$ and $\{z : |z + 2 - 4i| = |z - 2|\}$. Determine points M and N in Cartesian form using your graphics calculator.

Answer

The points are $M = -4 - 2i$, $N = 3 + 5i$ (or vice versa). 2A

2 marks

1e. Use vectors to prove that points M, N and P are the vertices of a right-angled triangle.

Worked solution

The vectors are:

$\vec{OP} = 4\underset{\sim}{i} + 4\underset{\sim}{j}$, $\vec{OM} = -4\underset{\sim}{i} - 2\underset{\sim}{j}$, $\vec{ON} = 3\underset{\sim}{i} + 5\underset{\sim}{j}$

Find vectors $\vec{MN}$ and $\vec{NP}$:

$\vec{MN} = \vec{MO} + \vec{ON}$ $\vec{NP} = \vec{NO} + \vec{OP}$

$\vec{MN} = -(-4\underset{\sim}{i} - 2\underset{\sim}{j}) + 3\underset{\sim}{i} + 5\underset{\sim}{j}$ $\vec{NP} = -(3\underset{\sim}{i} + 5\underset{\sim}{j}) + 4\underset{\sim}{i} + 4\underset{\sim}{j}$

$\vec{MN} = 7\underset{\sim}{i} + 7\underset{\sim}{j}$ $\vec{NP} = \underset{\sim}{i} - \underset{\sim}{j}$ 1A

Find the dot product of the vectors:

$\vec{MN} . \vec{NP} = (7\underset{\sim}{i} + 7\underset{\sim}{j}).(\underset{\sim}{i} - \underset{\sim}{j})$ 1M

$\vec{MN} . \vec{NP} = 7 - 7$

$\vec{MN} . \vec{NP} = 0$

Since the dot product is zero, the angle between $\vec{MN}$ and $\vec{NP}$ is 90°. 1A

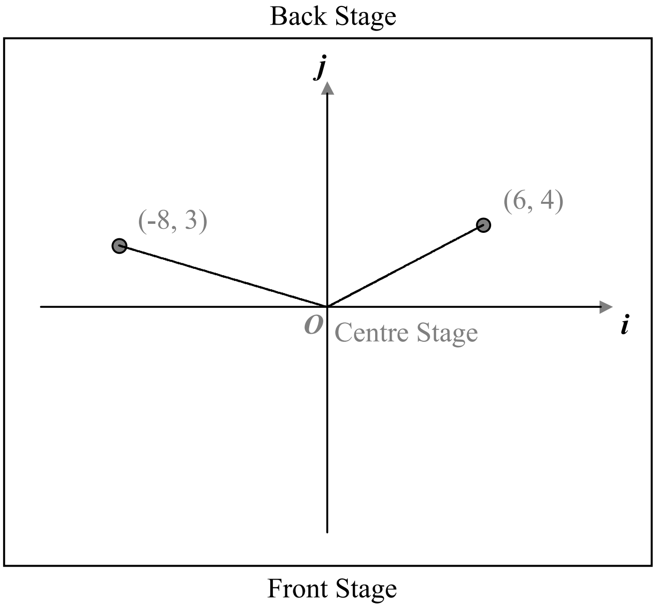
$\therefore$ Points M, N and P are the vertices of a right-angled triangle.

3 marks

Total 11 marks

Question 2

Two dancers, Ari, A, and Ben, B, are standing on stage at the start of a performance. Their position coordinates, in metres, in relation to point O at the centre of the stage are shown in the diagram below.



2a. Write vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ in terms of $\underset{\sim}{i}$ and $\underset{\sim}{j}$ to describe the positions of Ari and Ben at the start of the performance.

Worked solution

$\overrightarrow{OA} = 6\underset{\sim}{i} + 4\underset{\sim}{j}$

$\overrightarrow{OB} = -8\underset{\sim}{i} + 3\underset{\sim}{j}$ 1A

1 mark

2b. Find the obtuse angle AOB in degrees correct to one decimal place.

Worked solution

$$\cos\theta = \frac{\overrightarrow{OA}.\overrightarrow{OB}}{|OA|\,|OB|}$$

$$\cos\theta = \frac{(6\underset{\sim}{i} + 4\underset{\sim}{j}).(-8\underset{\sim}{i} + 3\underset{\sim}{j})}{\sqrt{6^2 + 4^2}.\sqrt{(-8)^2 + 3^2}}$$ 1M

$$\cos\theta = \frac{-48 + 12}{\sqrt{52}.\sqrt{73}}$$

$$\theta = \cos^{-1}\left(\frac{-36}{\sqrt{52}.\sqrt{73}}\right)$$

$$\theta = 125.8°$$ 1A

2 marks

As the performance starts spotlight, r, is beamed onto the stage. The path the spotlight follows around the stage is given by the equation $\underset{\sim}{r} = 10\cos(t)\underset{\sim}{i} + 5\sin(t)\underset{\sim}{j}, \quad t \geq 0$.

2c. Write a vector that describes the position of spotlight r initially.

Worked solution

When $t = 0$:

$\underset{\sim}{r} = 10\cos(0)\underset{\sim}{i} + 5\sin(0)\underset{\sim}{j}$

$\underset{\sim}{r} = 10\underset{\sim}{i}$ 1A

1 mark

2d. Show that both Ari and Ben are standing in the path traced out by spotlight r.

Worked solution

Method 1

If Ari is standing in the path of the spotlight then $10\cos(t)\underset{\sim}{i} + 5\sin(t)\underset{\sim}{j} = 6\underset{\sim}{i} + 4\underset{\sim}{j}$.

Equate the $\underset{\sim}{i}$ and $\underset{\sim}{j}$ components and solve for t: 1M

$10\cos(t) = 6$ and $5\sin(t) = 4$

$\cos(t) = 0.6$ $\sin(t) = 0.8$

$t = 0.9273$ $t = 0.9273$ 1A

The value of t is the same for $\underset{\sim}{i}$ and $\underset{\sim}{j}$

$\therefore$ Ari is standing in the path of the spotlight.

If Ben is standing in the path of the spotlight then $10\cos(t)\underset{\sim}{i} + 5\sin(t)\underset{\sim}{j} = -8\underset{\sim}{i} + 3\underset{\sim}{j}$.

Equate the $\underset{\sim}{i}$ and $\underset{\sim}{j}$ components and solve for t:

$10\cos(t) = -8$ and $5\sin(t) = 4$

$\cos(t) = -0.8$ $\sin(t) = 0.6$

$\cos(t)$ is negative and $\sin(t)$ is positive, therefore t is in the second quadrant.

$t = 2.4981$ $t = \pi - 0.6435 = 2.4981$ 1A

The value of t is the same for $\underset{\sim}{i}$ and $\underset{\sim}{j}$.

$\therefore$ Ben is standing in the path of the spotlight.

Method 2 (alternative)

Change $\underset{\sim}{r} = 10\cos(t)\underset{\sim}{i} + 5\sin(t)\underset{\sim}{j}$ to Cartesian form:

$x = 10\cos(t)$ $\quad\quad$ $y = 5\sin(t)$

$\Rightarrow \cos(t) = \dfrac{x}{10}$ K (1) $\quad\quad$ $\Rightarrow \sin(t) = \dfrac{y}{5}$ K (2)

From (1) and (2):

$$\left(\frac{x}{10}\right)^2 + \left(\frac{y}{5}\right)^2 = \cos^2(t) + \sin^2(t)$$

$$\therefore \quad \frac{x^2}{100} + \frac{y^2}{25} = 1$$

Spotlight follows an elliptical path.

Ari's position coordinates are (6, 4). Substitute $x = 6$ and $y = 4$ into $\dfrac{x^2}{100} + \dfrac{y^2}{25}$:

$\dfrac{6^2}{100} + \dfrac{4^2}{25} = \dfrac{36}{100} + \dfrac{16}{25} = 1$ $\quad\quad$ $\Rightarrow$ Point (6, 4) lies on ellipse.

Ben's position coordinates are (–8, 3). Substitute these into the equation for the ellipse:

$$\frac{(-8)^2}{100} + \frac{3^2}{25} = \frac{64}{100} + \frac{9}{25} = 1$$

$\Rightarrow$ Point (–8, 3) lies on ellipse

3 marks

2e. How long after the spotlight passes Ari does it reach Ben? Write your answer in seconds correct to two decimal places.

Worked solution

From part d, method 1:

Spotlight passed Ari when $t = 0.9273$ and Ben when $t = 2.4981$ $\quad$ 1M

$\therefore$ Spotlight reaches Ben 1.57 seconds after it passed Ari. $\quad$ 1A

2 marks

A second spotlight, *s*, starts moving at the same time as spotlight *r*. It follows a path given by the equation $\underset{\sim}{s} = 5\sin(t)\underset{\sim}{i} + 10\cos(t)\underset{\sim}{j}, \quad t \geq 0$.

2f. Find the times and position coordinates of the points on stage where the spotlights meet. Write your answers correct to two decimal places.

Worked solution

The spotlights meet when $\underset{\sim}{r} = \underset{\sim}{s}$.

$10\cos(t)\underset{\sim}{i} + 5\sin(t)\underset{\sim}{j} = 5\sin(t)\underset{\sim}{i} + 10\cos(t)\underset{\sim}{j}$ $\quad$ 1M

Equating $\underset{\sim}{i}$ components:

$10\cos(t) = 5\sin(t)$

$\Rightarrow$

$\dfrac{\sin(t)}{\cos(t)} = \dfrac{10}{5}$

$\tan(t) = 2$

$t = 1.1071, \quad \pi + 1.1071$

t = 1.1071 and 4.2487 seconds 1A

Equating $\underset{\sim}{j}$ components:

$5\sin(t) = 10\cos(t)$ (same equation)

When $t = 1.1071$, $\underset{\sim}{r} = 5\sin(1.1071)\underset{\sim}{i} + 10\cos(1.1071)\underset{\sim}{j}$

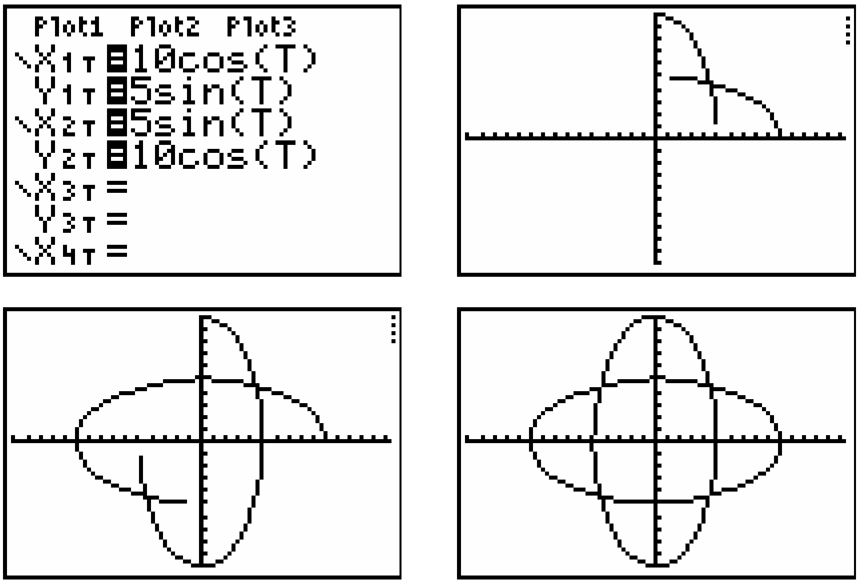
$$\underset{\sim}{r} = 4.4721\underset{\sim}{i} + 4.4721\underset{\sim}{j}$$

When $t = 4.2487$, $\underset{\sim}{r} = 5\sin(4.2487)\underset{\sim}{i} + 10\cos(4.2487)\underset{\sim}{j}$

$$\underset{\sim}{r} = -4.4721\underset{\sim}{i} - 4.4721\underset{\sim}{j}$$

Position coordinates are (4.47, 4.47) and (-4.47, -4.47) 1A

Note that the paths of the spotlights cross in four places, but the spotlights only *meet* (i.e., are in the same position at the same time) on two occasions. This can be seen by graphing the curves simultaneously on a calculator using parametric mode.



3 marks

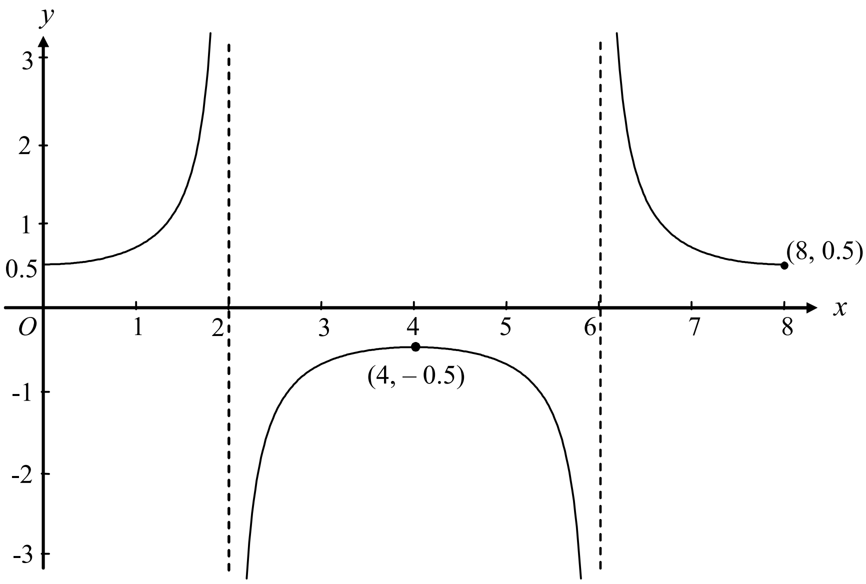
Total 12 marks

Question 3

Consider the function $f: D \to R$ where $f(x) = 0.5\text{cosec}\left(\dfrac{\pi}{4}(2-x)\right)$

3a. i. On the axes below, sketch a graph of f over the interval $[0, 8]$, labelling all features clearly.

Worked solution



Correct shape	1A
Local maximum of (4, – 0.5), Asymptotes and endpoints	1A

2 marks

3a. ii. Determine the domain and range of f over this interval.

Worked solution

Domain	$[0, 8]\backslash\{2,6\}$	or	$[0, 2)\cup(2, 6)\cup(6, 8]$	1A
Range	$R\backslash(-0.5, 0.5)$	or	$(-\infty, -0.5]\cup[0.5, \infty)$	1A

2 marks

3b. An equivalent rule for f is $f_1(x) = \dfrac{1}{a\cos(bx+c)}$ where $a, b, c \in R$

Give values for a, b, and c.

Worked solution

The simplest solution is

$a = 2$

$b = \dfrac{\pi}{4}$ — Two values correct 1A

$c = 0$ — All three values correct 1A

There are many other solutions – for example: $a = -2$, $b = \dfrac{\pi}{4}$, $c = -\pi$,

or $a = 2$, $b = \dfrac{\pi}{4}$, $c = 2\pi$.

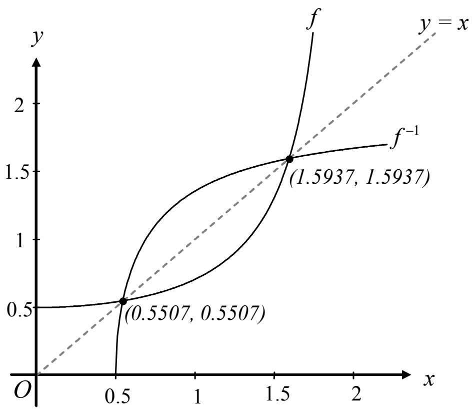
2 marks

3c. Let $D = [0, 2)$.

Sketch f and f^{-1} on the axes below, clearly showing the key features.

Worked solution

To graph f^{-1}, reflect graph of f in the line $y = x$.



Position and shape 1A

1 mark

3d. Write a definite integral that will give the area enclosed by f and f^{-1}. Using your graphics calculator, evaluate this integral correct to three decimal places.

Worked solution

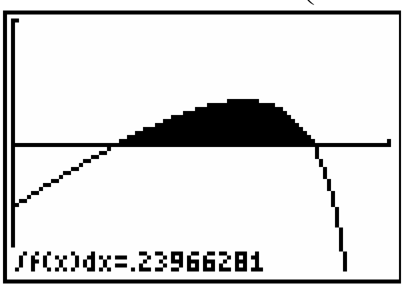
f and f^{-1} intersect at (0.5507, 0.5507) and (1.5937, 1.5937). 1H

These coordinates are found by graphing $y = x$ and $y = 0.5\text{cosec}\left(\frac{\pi}{4}(2-x)\right)$ on a calculator.

The area enclosed by f and f^{-1} is twice the area between $y = x$ and $y = 0.5\cos ec\left(\frac{\pi}{4}(2-x)\right)$.

Area $= 2 \times \int_{0.5507}^{1.5937} \left[x - 0.5\text{cosec}\left(\frac{\pi}{4}(2-x)\right)\right] dx$ 1A

Graph $y = x - 0.5\text{cosec}\left(\frac{\pi}{4}(2-x)\right)$ on calculator and find area above the x-axis.



Area enclosed by f and f^{-1} is $2 \times 0.2397 = 0.479$ square units. 1A

3 marks

Total 10 marks

SECTION 2 – continued

Question 4

A box of mass *m* kg is dropped from a hot air balloon. Its motion is retarded by a variable force of $\frac{mv}{5}$ newton, where *v* m/s is the velocity of the box *t* seconds after it is dropped.

4a. Taking vertically downwards as positive, show that the differential equation $\frac{dv}{dt} = \frac{5g - v}{5}$, where $g = 9.8$ m/sec^2 is the acceleration due to gravity, applies to this situation.

Worked solution

$$ma = mg - \frac{mv}{5} \qquad \text{1A}$$

$$a = g - \frac{v}{5}$$

$$\frac{dv}{dt} = \frac{5g - v}{5} \qquad \text{1M}$$

2 marks

4b. Hence, show that $t = 5\log_e\left(\frac{5g}{5g - v}\right)$

Worked solution

$$\frac{dv}{dt} = \frac{5g - v}{5}$$

$$\frac{dt}{dv} = \frac{5}{5g - v}$$

$$t = \int \frac{5}{5g - v}\,dv \qquad \text{1M}$$

$$t = -5\int \frac{-1}{5g - v}\,dv$$

$$t = -5\log_e(5g - v) + c$$

When $t = 0, \quad v = 0$:

$$c = 5\log_e(5g)$$

$$t = -5\log_e(5g - v) + 5\log_e(5g) \qquad \text{1M}$$

$$t = 5\log_e\left(\frac{5g}{5g - v}\right)$$

2 marks

4c. Show that at time t the velocity of the box is $5g\left(1-e^{-0.2t}\right)$ m/s.

Worked solution

Transpose $t=5\log_e\left(\dfrac{5g}{5g-v}\right)$ to make v the subject

$e^{\frac{t}{5}}=\dfrac{5g}{5g-v}$ 1M

$e^{0.2t}\left(5g-v\right)=5g$

$5g-v=5g\,e^{-0.2t}$

$v=5g-5g\,e^{-0.2t}$

$v=5g\left(1-e^{-0.2t}\right)$ 1M

2 marks

4d. Write an expression for the limiting velocity of the box. Show how you deduced your result.

Worked solution

At time t seconds, the velocity of the box is $v=5g\left(1-e^{-0.2t}\right)$ m/s

As $t\to\infty$, $e^{-0.2t}\to 0$, therefore $v\to 5g\left(1+0\right)=5g$ 1A

The limiting velocity is $5g$ m/s (49 m/s). 1A

2 marks

4e. Determine the time taken for the box to reach half its limiting velocity. Write your answer in seconds correct to two decimal places.

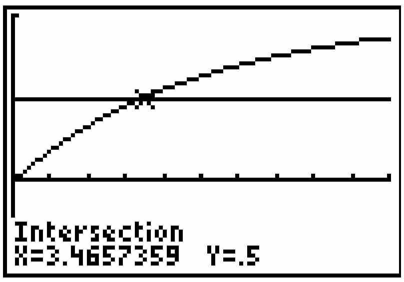
Worked solution

Finding t when $v=\dfrac{5g}{2}$

$\dfrac{5g}{2}=5g\left(1-e^{-0.2t}\right)$ 1M

$\dfrac{1}{2}=1-e^{-0.2t}$

Equation can be solved graphically on calculator



$e^{-0.2t}=\dfrac{1}{2}$

$t=-\dfrac{1}{0.2}\log_e\left(\dfrac{1}{2}\right)$

$t=5\log_e\left(2\right)$

$t=3.47$ seconds

It takes 3.47 seconds for the box to reach half its limiting velocity. 1H

2 marks

4f. Find the distance travelled by the box in the first 10 seconds of motion. Write your answer correct to the nearest metre.

Worked solution

$\frac{dx}{dt} = 5g\left(1 - e^{-0.2t}\right)$

$x = 5g\int\left(1 - e^{-0.2t}\right)dt$

$x = 5g\left(t + 5e^{-0.2t}\right) + c$ 1M

When $t = 0$, $x = 0$:

$0 = 5g\left(0 + 5e^{0}\right) + c$

$c = -25g$

$x = 5g\left(t + 5e^{-0.2t}\right) - 25g$ 1A

$\therefore \quad x = 5g\left(t + 5e^{-0.2t} - 5\right)$

When $t = 10$:

$x = 5 \times 9.8\left(10 + 5e^{-0.2\times10} - 5\right)$

$x = 278$ metres 1A

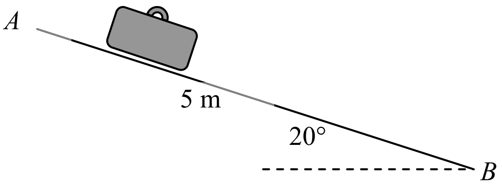
The box travels 278 metres in the first 10 seconds.

3 marks

Total 13 marks

Question 5

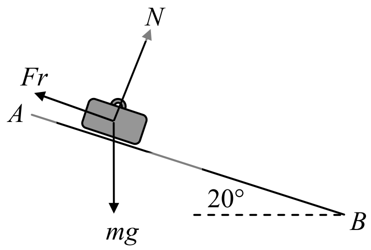
Baggage handlers use ramps to transport luggage. Ramp *AB* is 5 metres in length and inclined at an angle of 20° to the horizontal. A 20 kg suitcase, initially at rest at *A*, slides down ramp *AB* under the force of gravity. The coefficient of friction between the suitcase and the ramp is 0.2. Take $g = 9.8$ m/sec^2.



5a. On the diagram above, draw all forces acting on the suitcase as it slides down the ramp.

Worked solution

The forces are: normal reaction *N*, weight force *mg* (20*g*), friction *Fr*.

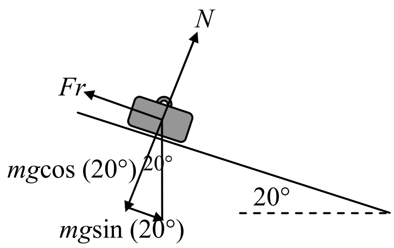


1A

1 mark

5b. Show that the suitcase slides down the ramp with an acceleration of 1.51 m/s^2.

Worked solution



Resolving forces acting perpendicular to the ramp:

$N = mg\cos(20°)$

Resolving forces acting parallel to the ramp:

$ma = mg\sin(20°) - Fr$

$ma = mg\sin(20°) - \mu N$ 1M

$ma = mg\sin(20°) - 0.2mg\cos(20°)$

$a = g\left(\sin(20°) - 0.2\cos(20°)\right)$ 1A

$a = 1.51$ m/s^2

2 marks

SECTION 2 – continued

5c. Find the time taken for the suitcase to reach point B. Write your answer in seconds correct to two decimal places.

Worked solution

The suitcase is moving under constant acceleration.

$u = 0 \quad a = 1.51 \quad s = 5$

$s = ut + \frac{1}{2}at^2$

$5 = 0 + \frac{1}{2} \times 1.51t^2$ 1M

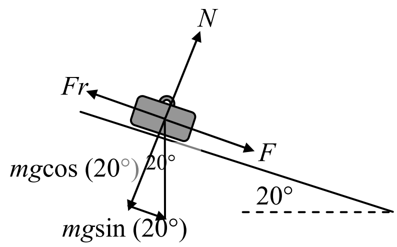
$t = \sqrt{\dfrac{5}{0.755}}$

$t = 2.57$ seconds 1A

2 marks

5d. Some time later, an identical 20 kg suitcase, initially at rest at A, is pushed down the ramp with a force of $100 - 200t$ newtons for the first 0.5 seconds of motion. Show that at time t, $0 < t < 0.5$, the acceleration of this suitcase is $6.51 - 10t$ m/s^2.

Worked solution



Let $F = 100 - 200t$

Resolving forces acting perpendicular to the ramp:

$N = mg\cos(20°)$

Resolving forces acting parallel to the ramp:

$ma = F + mg\sin(20°) - Fr$

$ma = 100 - 200t + mg\sin(20°) - \mu N$ 1M

$ma = 100 - 200t + mg\sin(20°) - 0.2mg\cos(20°)$

$20a = 100 - 200t + 20g\sin(20°) - 0.2 \times 20g\cos(20°)$

$20a = 130.2 - 200t$ 1A

$a = 6.51 - 10t$ m/s^2

2 marks

5e. Find the speed of the suitcase when $t = 0.5$. Write your answer in m/s, correct to two decimal places.

Worked solution

$v = \int a\ dt$

$v = \int (6.51 - 10t)dt$

$v = 6.51t - 5t^2 + c$

When $t = 0, v = 0 : c = 0$

$v = 6.51t - 5t^2$ 1A

When $t = 0.5,\ v = 2.01$.

After 0.5 seconds of motion the suitcase is moving at a speed of 2.01m/s. 1A

2 marks

5f. Determine the speed of this suitcase when it reaches point *B*. Write your answer in m/s, correct to two decimal places.

Worked solution

Find how far the suitcase travels whilst being pushed.

$x = \int v\ dt$

$x = \int (6.51t - 5t^2)dt$

$x = 3.255t^2 - 1.\dot{6}t^3 + c$ 1M

When $t = 0$, $x = 0$, so $c = 0$.

$x = 3.255t^2 - \frac{5}{3}t^3$

When $t = 0.5, x = 0.605$.

The suitcase travels 0.61 m while being pushed. 1A

For the remaining distance down the ramp, the suitcase travels with constant acceleration of 1.51 m/s^2.

$u = 2.01, \quad a = 1.51, \quad s = 5 - 0.605 = 4.395$

$v^2 = u^2 + 2as$

$v^2 = 2.01^2 + 2 \times 1.51 \times 4.395$

$v = \sqrt{17.31}$

$v = 4.16$ m/s 1A

The speed of this suitcase at point *B* is 4.16 m/s.

3 marks

12 marks

END OF SOLUTIONS BOOK