

INSIGHT

Trial Exam Paper

2006

SPECIALIST

MATHEMATICS

Written examination 1

Worked solutions

This book presents:

- worked solutions, giving you a series of points to show you how to work through the questions
- mark allocation details.

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Question 1

Consider the function $f(x) = \frac{1}{2x^2 - x - 3}$

1a. Determine the equations of the asymptotes of f .

Worked solution

$$\begin{aligned}\text{Vertical asymptotes: } 2x^2 - x - 3 &= 0 \\ (2x - 3)(x + 1) &= 0 \\ x &= \frac{3}{2}, \text{ and } x = -1\end{aligned}$$

1A

$$\text{Horizontal asymptote: } y = 0$$

1A

2 marks

1b. Find the coordinates of any intercepts and stationary points of f .

Worked solution

$$\text{y-intercept: } x = 0, y = -\frac{1}{3}$$

1A

$$\text{Stationary points: } f'(x) = -1(2x^2 - x - 3)^{-2}(4x - 1) = 0$$

1M

$$-\frac{4x - 1}{(2x^2 - x - 3)^2} = 0$$

$$4x - 1 = 0$$

$$x = \frac{1}{4}, \quad y = \frac{1}{2\left(\frac{1}{4}\right)^2 - \frac{1}{4} - 3} = -\frac{8}{25}$$

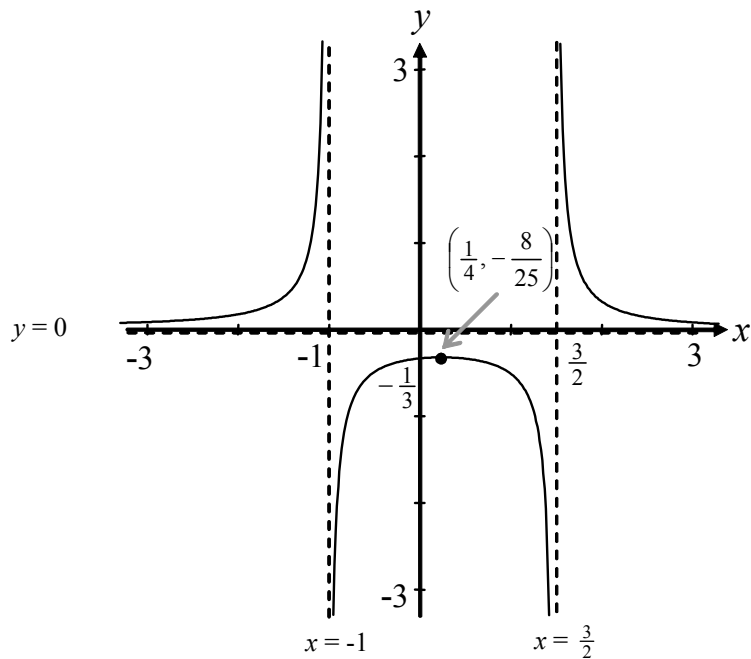
$$\text{There is a maximum turning point at } \left(\frac{1}{4}, -\frac{8}{25}\right)$$

1A

3 marks

1c. Sketch f on the axes below labeling all key features.

Answer



Shape and features 1A

1 mark

Question 2

2a. Show that $\frac{1}{\cos^4 x - \sin^4 x} = \sec(2x)$

Worked solution

$$\begin{aligned}
 & \frac{1}{\cos^4 x - \sin^4 x} \\
 &= \frac{1}{(\cos^2 x + \sin^2 x)(\cos^2 x - \sin^2 x)} && 1M \\
 &= \frac{1}{1 \times (\cos^2 x - \sin^2 x)} \\
 &= \frac{1}{\cos(2x)} && 1A \\
 &= \sec(2x)
 \end{aligned}$$

2 marks

2b. Hence find the exact values of x for which $\frac{1}{\cos^4 x - \sin^4 x} = 2, \quad x \in [0, 2\pi]$

Worked solution

$$\frac{1}{\cos^4 x - \sin^4 x} = 2$$

$$\sec(2x) = 2 \quad (\text{from a.})$$

$$\cos(2x) = \frac{1}{2} \quad 1A$$

$$2x = \frac{\pi}{3}, \quad 2\pi - \frac{\pi}{3}, \quad 2\pi + \frac{\pi}{3}, \quad 4\pi - \frac{\pi}{3}$$

$$2x = \frac{\pi}{3}, \quad \frac{5\pi}{3}, \quad \frac{7\pi}{3}, \quad \frac{11\pi}{3}$$

$$x = \frac{\pi}{6}, \quad \frac{5\pi}{6}, \quad \frac{7\pi}{6}, \quad \frac{11\pi}{6} \quad 1A$$

2 marks

Question 3

Find the fourth roots of $16i$ in exact polar form.

Worked solution

$$\text{Let } z^4 = 0 + 16i$$

$$z^4 = 16\text{cis}\left(\frac{\pi}{2}\right)$$

$$z = \left(16\text{cis}\left(\frac{\pi}{2} + 2k\pi\right)\right)^{\frac{1}{4}}, \quad k \in Z \quad 1A$$

$$z = 16^{\frac{1}{4}} \text{cis} \frac{1}{4} \left(\frac{\pi}{2} + 2k\pi\right) \quad \text{By De Moivre's Theorem}$$

$$z = 2\text{cis}\left(\frac{\pi}{8} + \frac{2k\pi}{4}\right) \quad 1M$$

$$k = 0, \quad z = 2\text{cis}\left(\frac{\pi}{8}\right), \quad k = 1, \quad z = 2\text{cis}\left(\frac{5\pi}{8}\right) \quad 1A$$

$$k = -1, \quad z = 2\text{cis}\left(-\frac{3\pi}{8}\right) \quad k = -2, \quad z = 2\text{cis}\left(-\frac{7\pi}{8}\right)$$

$$\text{The four roots are: } 2\text{cis}\left(-\frac{7\pi}{8}\right), \quad 2\text{cis}\left(-\frac{3\pi}{8}\right), \quad 2\text{cis}\left(\frac{\pi}{8}\right), \quad 2\text{cis}\left(\frac{5\pi}{8}\right) \quad 1A$$

4 marks

Question 4

Determine the rate of change of y with respect to x on the curve $y = x - 5xy^2$ at the point where $y = 1$.

Worked solution

Find x when $y = 1$:

$$y = x - 5xy^2$$

$$1 = x - 5x(1)^2$$

$$1 = -4x$$

$$x = -0.25$$

1A

Find the rate of change using implicit differentiation:

$$\frac{dy}{dx} = 1 - 5 \left(1 \times y^2 + x \times 2y \frac{dy}{dx} \right)$$

1M

$$\frac{dy}{dx} = 1 - 5y^2 - 10xy \frac{dy}{dx}$$

$$(1 + 10xy) \frac{dy}{dx} = 1 - 5y^2$$

$$\frac{dy}{dx} = \frac{1 - 5y^2}{1 + 10xy}$$

1A

When $y = 1$, $x = -0.25$:

$$\frac{dy}{dx} = \frac{1 - 5(1)^2}{1 + 10 \times -0.25 \times 1}$$

$$\frac{dy}{dx} = \frac{-4}{-1.5}$$

$$\frac{dy}{dx} = \frac{8}{3}$$

1A

4 marks

Question 5

5a. i. Give the domain over which $\frac{d}{dx} \left(\arcsin(2x) + 2x\sqrt{1-4x^2} \right)$ is defined.

Worked solution

Domain is $x \in \left(-\frac{1}{2}, \frac{1}{2} \right)$.

1A

The derivative is not defined at the end points of this range.

5a. ii. Show that $\frac{d}{dx}(\arcsin(2x) + 2x\sqrt{1-4x^2}) = 4\sqrt{1-4x^2}$

Worked solution

$$\begin{aligned}
 & \frac{d}{dx}(\arcsin(2x) + 2x\sqrt{1-4x^2}) \\
 &= \frac{d}{dx}(\arcsin(2x)) + \frac{d}{dx}(2x\sqrt{1-4x^2}), \quad x \in \left(-\frac{1}{2}, \frac{1}{2}\right) \\
 &= \frac{1}{\sqrt{1-(2x)^2}} \times 2 + 2\sqrt{1-4x^2} + 2x \left(\frac{1}{2}(1-4x^2)^{-\frac{1}{2}}\right)(-8x) \quad 1M \\
 &= \frac{2}{\sqrt{1-4x^2}} + 2\sqrt{1-4x^2} - \frac{8x^2}{\sqrt{1-4x^2}} \\
 &= \frac{2}{\sqrt{1-4x^2}} - \frac{8x^2}{\sqrt{1-4x^2}} + 2\sqrt{1-4x^2} \\
 &= \frac{2-8x^2}{\sqrt{1-4x^2}} + 2\sqrt{1-4x^2} \\
 &= \frac{2(1-4x^2)}{\sqrt{1-4x^2}} + 2\sqrt{1-4x^2} \\
 &= 2\sqrt{1-4x^2} + 2\sqrt{1-4x^2} \quad 1A
 \end{aligned}$$

1 + 2 = 3 marks

5b. Hence, find the exact area enclosed by the curve $4x^2 + y^2 = 1$.

Worked solution

$4x^2 + y^2 = 1$ is an ellipse.

Express y in terms of x

$$y = \pm\sqrt{1-4x^2}$$

Sketch a graph.

Area of ellipse = 4 × shaded region

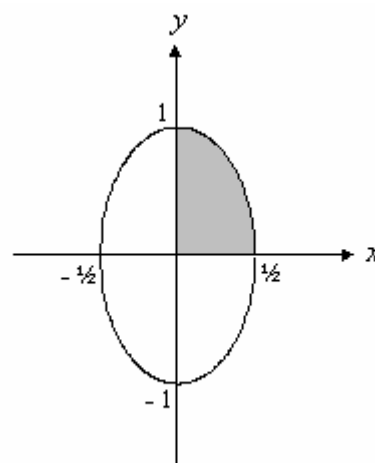
$$A = 4 \int_0^{\frac{1}{2}} \sqrt{1-4x^2} \, dx$$

$$A = \int_0^{\frac{1}{2}} 4\sqrt{1-4x^2} \, dx$$

$$A = \left[\arcsin(2x) + 2x\sqrt{1-4x^2} \right]_0^{\frac{1}{2}}$$

$$A = \arcsin(1)$$

$$A = \frac{\pi}{2} \text{ square units}$$



1A

1A

1A

3 marks

Question 6

Find constants m and n such that $y = \frac{\log_e |x|}{x}$, $x \neq 0$ is a solution of the differential equation

$$x^2 \frac{d^2 y}{dx^2} + mx \frac{dy}{dx} + ny = 0$$

Worked solution

$$y = \frac{\log_e |x|}{x}$$

Using the quotient rule

$$\frac{dy}{dx} = \frac{\frac{1}{x} \cdot x - 1 \cdot \log_e |x|}{x^2}$$

$$\frac{dy}{dx} = \frac{1 - \log_e |x|}{x^2}$$

1A

$$\frac{d^2 y}{dx^2} = \frac{-\frac{1}{x} \cdot x^2 - (1 - \log_e |x|) \cdot 2x}{x^4}$$

$$\frac{d^2 y}{dx^2} = \frac{-x - 2x + 2x \log_e |x|}{x^4}$$

$$\frac{d^2 y}{dx^2} = \frac{2 \log_e |x| - 3}{x^3}$$

1A

Substitute into the given differential equation

$$x^2 \left(\frac{2 \log_e |x| - 3}{x^3} \right) + mx \left(\frac{1 - \log_e |x|}{x^2} \right) + n \left(\frac{\log_e |x|}{x} \right) = 0 \quad x \neq 0$$

1M

$$\frac{2 \log_e |x| - 3}{x} + \frac{m(1 - \log_e |x|)}{x} + \frac{n \log_e |x|}{x} = 0$$

$$2 \log_e |x| - 3 + m - m \log_e |x| + n \log_e |x| = 0$$

$$(2 - m + n) \log_e |x| + m - 3 = 0$$

Therefore:

$$m - 3 = 0 \quad \text{and} \quad 2 - m + n = 0$$

$$\therefore m = 3 \quad 2 - 3 + n = 0$$

$$\therefore n = 1$$

1A

4 marks

Question 7

Given $\frac{dy}{dx} = x\sqrt{1+x^2}$ and $y = 1$ when $x = 0$.

Find the value of y when $x = \sqrt{3}$.

Worked solution

$$\frac{dy}{dx} = x\sqrt{1+x^2}$$

$$y = \int x\sqrt{1+x^2} dx$$

$$\text{Let } u = 1+x^2 \Rightarrow \frac{du}{dx} = 2x$$

$$y = \frac{1}{2} \int \sqrt{1+x^2} (2x dx)$$

$$du = 2x dx$$

1M

$$y = \frac{1}{2} \int u^{\frac{1}{2}} du$$

$$y = \frac{1}{2} \times \frac{2}{\frac{3}{2}} u^{\frac{3}{2}} + c$$

c is a constant

1A

$$y = \frac{1}{3} (1+x^2)^{\frac{3}{2}} + c$$

$$\text{Find } c: \text{ When } x = 0, y = 1 \Rightarrow 1 = \frac{1}{3} (1+(0)^2)^{\frac{3}{2}} + c$$

$$1 = \frac{1}{3} + c$$

$$c = \frac{2}{3}$$

1A

$$\therefore y = \frac{1}{3} (1+x^2)^{\frac{3}{2}} + \frac{2}{3}$$

$$\text{When } x = \sqrt{3}, y = \frac{1}{3} \left(1 + (\sqrt{3})^2 \right)^{\frac{3}{2}} + \frac{2}{3}$$

$$y = \frac{1}{3} \times 4^{\frac{3}{2}} + \frac{2}{3}$$

$$y = \frac{1}{3} \times 8 + \frac{2}{3}$$

$$y = \frac{10}{3}$$

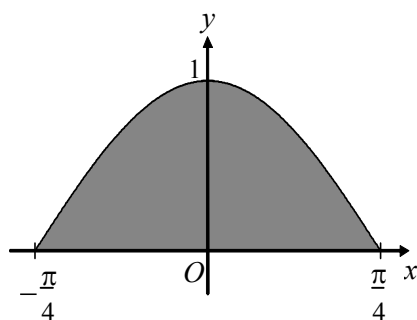
1H

$$\therefore y(\sqrt{3}) = \frac{10}{3}$$

4 marks

Question 8

The graph below shows the region bounded by the curve $y = \cos(2x)$ and the x -axis.



Find the exact value of the volume of the solid of revolution formed when this region is rotated around the x -axis.

Worked solution

$$\begin{aligned}
 V &= \pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} y^2 dx = 2\pi \int_0^{\frac{\pi}{4}} y^2 dx \\
 &= 2\pi \int_0^{\frac{\pi}{4}} \cos^2(2x) dx && 1A \\
 &= 2\pi \int_0^{\frac{\pi}{4}} \frac{1}{2} (\cos(4x) + 1) dx && 1A \\
 &= \pi \int_0^{\frac{\pi}{4}} (\cos(4x) + 1) dx \\
 &= \pi \left[\frac{1}{4} \sin(4x) + x \right]_0^{\frac{\pi}{4}} && 1M \\
 &= \pi \left[\frac{1}{4} \sin(\pi) + \frac{\pi}{4} \right] \\
 &= \pi \left[0 + \frac{\pi}{4} \right] \\
 &= \frac{\pi^2}{4} \text{ cubic units.} && 1A
 \end{aligned}$$

4 marks

Question 9

A particle moves in such a way that its position vector at time t seconds is given by

$$\underline{r} = 2t \underline{i} + \cos(2\pi t) \underline{j} + \sin(2\pi t) \underline{k}$$

9a. Find the constant speed at which the particle is moving.

Worked solution

$$\underline{r} = 2t \underline{i} + \cos(2\pi t) \underline{j} + \sin(2\pi t) \underline{k}$$

$$\dot{\underline{r}} = 2 \underline{i} - 2\pi \sin(2\pi t) \underline{j} + 2\pi \cos(2\pi t) \underline{k}$$

1A

$$\begin{aligned} \text{speed} &= \sqrt{2^2 + (-2\pi \sin(2\pi t))^2 + (2\pi \cos(2\pi t))^2} \\ &= \sqrt{4 + 4\pi^2 \sin^2(2\pi t) + 4\pi^2 \cos^2(2\pi t)} \\ &= \sqrt{4 + 4\pi^2 (\sin^2(2\pi t) + \cos^2(2\pi t))} \\ &= \sqrt{4 + 4\pi^2} \\ &= 2\sqrt{1 + \pi^2} \text{ units/sec} \end{aligned}$$

1A

2 marks

9b. Show that the velocity and acceleration are always perpendicular.

Worked solution

$$\ddot{\underline{r}}(t) = -4\pi^2 \cos(2\pi t) \underline{j} - 4\pi^2 \sin(2\pi t) \underline{k}$$

1A

$$\dot{\underline{r}} \cdot \ddot{\underline{r}} = \left(2 \underline{i} - 2\pi \sin(2\pi t) \underline{j} + 2\pi \cos(2\pi t) \underline{k} \right) \cdot \left(0 \underline{i} - 4\pi^2 \cos(2\pi t) \underline{j} - 4\pi^2 \sin(2\pi t) \underline{k} \right)$$

$$\dot{\underline{r}} \cdot \ddot{\underline{r}} = 0 + 8\pi^3 \sin(2\pi t) \cos(2\pi t) - 8\pi^3 \cos(2\pi t) \sin(2\pi t)$$

$$\dot{\underline{r}} \cdot \ddot{\underline{r}} = 0$$

1A

$\therefore$ The velocity and acceleration are always perpendicular.

2 marks

END OF WORKED SOLUTIONS