
Question 1

a. $v = \sin(3t) - \frac{t}{2}$
 $a = 3 \cos(3t) - \frac{1}{2}$

(1 mark)

- b. $R = ma$ where R is the magnitude of the resultant force.

$$R = 2 \left(3 \cos(3t) - \frac{1}{2} \right)$$
$$= 6 \cos(3t) - 1$$

(1 mark)

R is a maximum when $\cos(3t) = 1$; that is, when $\cos(3t)$ equals its maximum value.

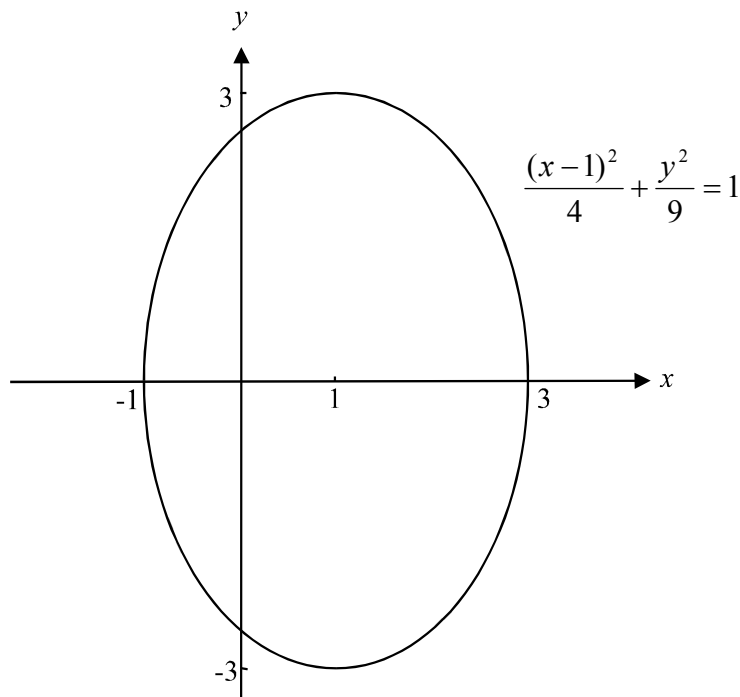
So, the maximum value of R is given by

$$R = 6 \times 1 - 1$$
$$= 5 \text{ Newtons}$$

(1 mark)

Question 2

a.



(1 mark) correct shape with points $(-1,0)$, $(3,0)$, $(1,-3)$, $(1,3)$ shown

(1 mark) correct centre

b. $\frac{(x-1)^2}{4} + \frac{y^2}{9} = 1$

$$9(x-1)^2 + 4y^2 = 36$$

$$18(x-1) + 8y \frac{dy}{dx} = 0$$

(implicit differentiation)

(1 mark) for differentiating x term and constant term

$$8y \frac{dy}{dx} = -18(x-1)$$

(1 mark) for differentiating y term

$$\frac{dy}{dx} = \frac{-18(x-1)}{8y}$$

$$= \frac{-9(x-1)}{4y}$$

(1 mark)

c. If $y > 0$ then $4y > 0$ _____ (A)

For $x \in (-1, 1)$, $x-1 < 0$

so $-9(x-1) > 0$ _____ (B)

(1 mark)

Using (A) and (B), we have

$$\frac{-9(x-1)}{4y} > 0$$

Therefore $\frac{dy}{dx} > 0$ since $\frac{dy}{dx} = \frac{-9(x-1)}{4y}$

(1 mark)

So for $y > 0$, $\frac{dy}{dx} > 0$ for $x \in (-1, 1)$.

Question 3

a. Let $y = \arctan(e^{2x})$
 $= \arctan(u)$ where $u = e^{2x}$

$$\frac{dy}{du} = \frac{1}{1+u^2} \quad \frac{du}{dx} = 2e^{2x}$$

$$= \frac{1}{1+e^{4x}}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad (\text{chain rule})$$

$$= \frac{1}{1+e^{4x}} \times 2e^{2x}$$

$$= \frac{2e^{2x}}{1+e^{4x}}$$

as required. **(1 mark)**

b. From **a.** $\frac{d}{dx}(\arctan(e^{2x})) = \frac{2e^{2x}}{1+e^{4x}}$

so, $\int \frac{d}{dx}(\arctan(e^{2x})) dx = \int \frac{2e^{2x}}{1+e^{4x}} dx$

$$\arctan(e^{2x}) + c = 2 \int \frac{e^{2x}}{1+e^{4x}} dx \quad c \text{ is a constant}$$

Now $\int_0^{\log_e(5)} \frac{e^{2x}}{1+e^{4x}} dx = \frac{1}{2} [\arctan(e^{2x})]_0^{\log_e 5}$ **(1 mark)**

$$= \frac{1}{2} \{ \arctan(e^{2 \log_e(5)}) - \arctan(e^0) \}$$

$$= \frac{1}{2} \{ \arctan(e^{\log_e(5^2)}) - \arctan(1) \}$$
 (1 mark)

$$= \frac{1}{2} \left(\arctan(25) - \frac{\pi}{4} \right)$$

(1 mark)

Question 4

- a. If $z = \sqrt{3}i$ is a solution to the equation $z^4 - 2z^3 + 5z^2 - 6z + a = 0$ then

$$\begin{aligned} (\sqrt{3}i)^4 - 2(\sqrt{3}i)^3 + 5(\sqrt{3}i)^2 - 6(\sqrt{3}i) + a &= 0 \\ 9 + 6\sqrt{3}i - 15 - 6\sqrt{3}i + a &= 0 \\ -6 + a &= 0 \\ a &= 6 \end{aligned}$$

(1 mark)

- b. Since all the coefficients of the equation are real, one other solution is $z = -\sqrt{3}i$ since the solutions occur in conjugate pairs (conjugate root theorem).

Now $(z - \sqrt{3}i)(z + \sqrt{3}i) = z^2 + 3$ is a quadratic factor.

(1 mark)Method 1

$$\begin{aligned} \text{Let } p(z) &= z^4 - 2z^3 + 5z^2 - 6z + 6 \\ &= (z^2 + 3)z^2 + (z^2 + 3)(-2z) + (z^2 + 3)(2) \\ &= (z^2 + 3)(z^2 - 2z + 2) \end{aligned}$$

(1 mark)Method 2

$$\begin{array}{r} \overline{z^2 - 2z + 2} \\ z^2 + 3 \overline{z^4 - 2z^3 + 5z^2 - 6z + 6} \\ \underline{z^4 + 3z^2} \\ - 2z^3 + 2z^2 - 6z \\ \underline{- 2z^3 - 6z} \\ 2z^2 + 6 \\ \underline{2z^2 + 6} \end{array}$$

$$\begin{aligned} z^4 - 2z^3 + 5z^2 - 6z + 6 \\ = (z^2 + 3)(z^2 - 2z + 2) \end{aligned}$$

(1 mark)

$$\begin{aligned} \text{Now } z^2 - 2z + 2 \\ = ((z^2 - 2z + 1) - 1 + 2) \\ = (z - 1)^2 + 1 \\ = (z - 1)^2 - i^2 \\ = (z - 1 - i)(z - 1 + i) \end{aligned}$$

All the solutions to $p(z) = 0$ are therefore $z = \pm\sqrt{3}i$ and $z = 1 \pm i$.

(1 mark)

Question 5

a.
$$\int \left(\frac{\sec(2x)}{\tan(2x)} \right)^2 dx = \int \frac{\sec^2(2x)}{\tan^2(2x)} dx$$

$$= \frac{1}{2} \int \frac{du}{dx} u^{-2} dx \quad u = \tan(2x)$$

$$= \frac{1}{2} \int u^{-2} du \quad \frac{du}{dx} = 2 \sec^2(2x)$$

$$= -\frac{1}{2} u^{-1} + c$$

$$= \frac{-1}{2 \tan(2x)} + c$$

(1 mark)

b.
$$\int_0^1 \frac{x}{\sqrt{2-x}} dx = \int_2^1 u^{-\frac{1}{2}} \times -1 \frac{du}{dx} \times (2-u) dx$$

$$= -1 \int_2^1 \left(2u^{-\frac{1}{2}} - u^{\frac{1}{2}} \right) du \quad \text{(1 mark) for integrand}$$

$$= \left[4u^{\frac{1}{2}} - \frac{2}{3} u^{\frac{3}{2}} \right]_1^2 \quad \text{(1 mark) for terminals}$$

$$= \left\{ \left(4\sqrt{2} - \frac{2}{3} 2^{\frac{3}{2}} \right) - \left(4 - \frac{2}{3} \right) \right\}$$

$$= \sqrt{2} \left(4 - \frac{4}{3} \right) - \frac{10}{3}$$

$$= \frac{8\sqrt{2}}{3} - \frac{10}{3}$$

$$= \frac{2(4\sqrt{2} - 5)}{3}$$

(1 mark)

Question 6

$$\underline{u} = \underline{i} + \sqrt{2} \underline{j} + \underline{k}$$

$$\underline{v} = \underline{i} + a \underline{j} - \underline{k}$$

$$\underline{u} \cdot \underline{v} = |\underline{u}| |\underline{v}| \cos\left(\frac{\pi}{3}\right) \quad (1 \text{ mark})$$

$$1 + \sqrt{2}a - 1 = \sqrt{1+2+1} \sqrt{1+a^2+1} \times \frac{1}{2}$$

$$\sqrt{2}a = \sqrt{2+a^2} \quad - (*) \quad (1 \text{ mark})$$

$$2a^2 = a^2 + 2 \quad (\text{Square both sides})$$

$$a^2 = 2$$

$$a = \pm\sqrt{2}$$

$$\text{Check } a = \sqrt{2} \text{ in } - (*)$$

$$LS = \sqrt{2} \times \sqrt{2}$$

$$= 2$$

$$RS = \sqrt{2+2}$$

$$= 2$$

$$\text{Check } a = -\sqrt{2} \text{ in } - (*)$$

$$LS = \sqrt{2} \times -\sqrt{2}$$

$$= -2$$

$$RS = \sqrt{2+2}$$

$$= 2$$

$$LS \neq RS \text{ so reject } a = -\sqrt{2}$$

$$\text{So } a = \sqrt{2}$$

(1 mark)**(1 mark)** for rejecting $a = -\sqrt{2}$.

(Note – when you square both sides of an equation it is important that you verify any resulting solutions.)

Question 7

$$\frac{dy}{dx} = \frac{x^2 + 7}{x^2 + 4}$$

$$y = \int \frac{x^2 + 7}{x^2 + 4} dx$$

$$= \int \left(\frac{x^2 + 4}{x^2 + 4} + \frac{3}{x^2 + 4} \right) dx$$

$$= \int \left(1 + \frac{3}{2} \times \frac{2}{x^2 + 4} \right) dx \quad (1 \text{ mark})$$

$$y = x + \frac{3}{2} \arctan\left(\frac{x}{2}\right) + c \quad (1 \text{ mark})$$

Now $y(0) = 0$

so $c = 0$

$$y = x + \frac{3}{2} \arctan\left(\frac{x}{2}\right)$$

(1 mark)**Question 8**

$$\text{volume} = \pi \int_0^{\frac{\pi}{2}} y^2 dx$$

$$= \pi \int_0^{\frac{\pi}{2}} (2 - 2\sin(x))^2 dx \quad (1 \text{ mark})$$

$$= \pi \int_0^{\frac{\pi}{2}} (4 - 8\sin(x) + 4\sin^2(x)) dx$$

$$= 4\pi \int_0^{\frac{\pi}{2}} \left(1 - 2\sin(x) + \frac{1}{2}(1 - \cos 2x) \right) dx \quad (1 \text{ mark})$$

$$= 4\pi \left[\frac{3x}{2} + 2\cos(x) - \frac{1}{4}\sin(2x) \right]_0^{\frac{\pi}{2}} \quad (1 \text{ mark})$$

$$= 4\pi \left\{ \left(\frac{3\pi}{4} + 0 - 0 \right) - (0 + 2 - 0) \right\}$$

$$= 4\pi \left(\frac{3\pi}{4} - 2 \right)$$

$$= \pi(3\pi - 8) \text{ cubic units}$$

(1 mark)

Question 9

a. i.

$$y = \frac{1}{x^2 - 2x - 3}$$

$$= (x^2 - 2x - 3)^{-1}$$

$$\frac{dy}{dx} = -1(x^2 - 2x - 3)^{-2} \times (2x - 2)$$

$$= \frac{2(1 - x)}{(x^2 - 2x - 3)^2}$$

(1 mark)

For a stationary point $\frac{dy}{dx} = 0$

$$2(1 - x) = 0$$

$$x = 1$$

When $x = 1$,

$$y = \frac{1}{(1)^2 - 2(1) - 3}$$

$$= \frac{1}{-4}$$

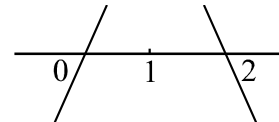
$\left(1, -\frac{1}{4}\right)$ is the stationary point.

(1 mark)

ii. For $x = 0$, $\frac{dy}{dx} = \frac{2}{9}$
 > 0

For $x = 2$, $\frac{dy}{dx} = \frac{-2}{9}$
 < 0

There is a maximum turning point at $\left(1, -\frac{1}{4}\right)$.

**(1 mark)**

b.
$$\frac{1}{x^2 - 2x - 3} = \frac{1}{(x-3)(x+1)}$$

Let
$$\frac{1}{(x-3)(x+1)} \equiv \frac{A}{(x-3)} + \frac{B}{(x+1)}$$

$$\equiv \frac{A(x+1) + B(x-3)}{(x-3)(x+1)}$$

True iff $1 \equiv A(x+1) + B(x-3)$ **(1 mark)**

Put $x = -1$, $1 = -4B$, $B = -\frac{1}{4}$

Put $x = 3$, $1 = 4A$, $A = \frac{1}{4}$

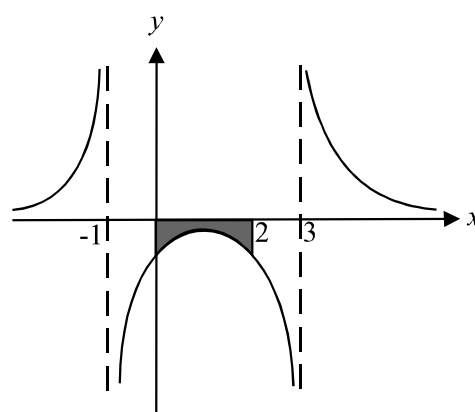
So,
$$\frac{1}{x^2 - 2x - 3} = \frac{1}{4(x-3)} - \frac{1}{4(x+1)}$$
 (1 mark)

(Check
$$\frac{1}{4(x-3)} - \frac{1}{4(x+1)} = \frac{x+1-(x-3)}{4(x-3)(x+1)}$$

$$= \frac{1}{(x-3)(x+1)}$$
)

- c.** Do a fast sketch.
 From **a.** we know that
 there is a max. tp. at $\left(1, -\frac{1}{4}\right)$
 and no other stationary points.
 There are asymptotes at $x = 3$ and $x = -1$.
 The area required is shaded in the diagram.

$$\begin{aligned} \text{area} &= -\int_0^2 \frac{1}{(x-3)(x+1)} dx \\ &= -\frac{1}{4} \int_0^2 \left(\frac{1}{x-3} - \frac{1}{x+1} \right) dx \text{ from part b.} \\ &= -\frac{1}{4} [\log_e |x-3| - \log_e |x+1|]_0^2 \\ &= -\frac{1}{4} \{(\log_e(1) - \log_e(3)) - (\log_e(3) - \log_e(1))\} \\ &= -\frac{1}{4} (-2 \log_e(3)) \\ &= \frac{1}{2} \log_e(3) \text{ square units} \end{aligned}$$



(1 mark)

(1 mark)