

Q1a 20 litres in and 10 litres out per minute. The volume increases by 10 litres per minute. At time t minutes, the volume of solution in the tank = $10 + 10t$ litres.

$$\therefore \text{concentration} = \frac{x}{10 + 10t} = \frac{x}{10(1+t)} \text{ grams per litre.}$$

Q1b Rate of inflow of chemical = $\frac{2}{1+t^2} \times 20 = \frac{40}{1+t^2}$ grams

per minute. Rate of outflow of chemical = $\frac{x}{10(1+t)} \times 10 = \frac{x}{1+t}$ grams per minute.

Rate of change of chemicals = rate of inflow – rate of outflow

i.e. $\frac{dx}{dt} = \frac{40}{1+t^2} - \frac{x}{1+t}$. Hence $\frac{dx}{dt} + \frac{x}{1+t} = \frac{40}{1+t^2}$.

Q1ci $x = \frac{40}{1+t} \tan^{-1}(t) + \frac{20}{1+t} \log_e(1+t^2)$,
 $\frac{dx}{dt} = -\frac{40}{(1+t)^2} \tan^{-1}(t) + \frac{40}{1+t} \times \frac{1}{1+t^2} - \frac{20}{(1+t)^2} \log_e(1+t^2) + \frac{20}{1+t} \times \frac{2t}{1+t^2}$,
 $= \frac{40(1+t)}{(1+t)(1+t^2)} - \frac{40}{(1+t)^2} \tan^{-1}(t) - \frac{20}{(1+t)^2} \log_e(1+t^2)$
 $= \frac{40}{(1+t^2)} - \frac{40}{(1+t)^2} \tan^{-1}(t) - \frac{20}{(1+t)^2} \log_e(1+t^2).$

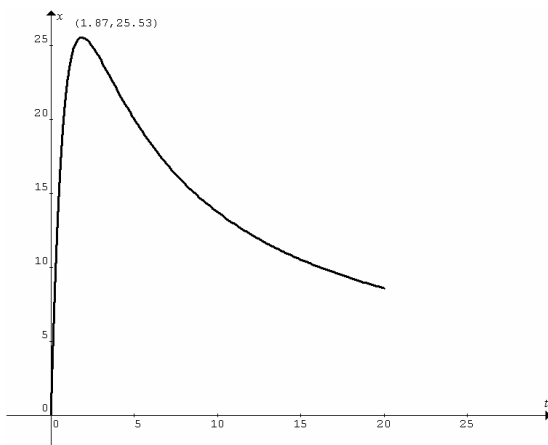
Q1cii $\frac{dx}{dt} = \frac{40}{1+t^2} - \frac{x}{1+t}$.

$$RHS = \frac{40}{1+t^2} - \frac{\frac{40}{1+t} \tan^{-1}(t) + \frac{20}{1+t} \log_e(1+t^2)}{1+t}$$

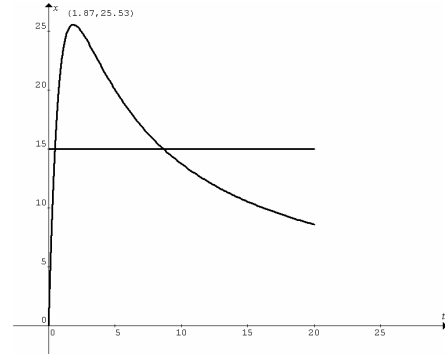
$$= \frac{40}{(1+t^2)} - \frac{40}{(1+t)^2} \tan^{-1}(t) - \frac{20}{(1+t)^2} \log_e(1+t^2) = LHS$$

$\therefore x$ satisfies the differential equation.

Q1d Sketch and find stationary points by graphics calculator.

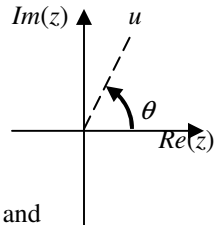


Q1ei Sketch $x = 15$ (horizontal line) on the last graph, find the first intersection $t = 0.485$ min.



Q1eii Find the second intersection $t = 8.655$ min.
Duration = $8.655 - 0.485 = 8.17$ minutes.

Q2ai

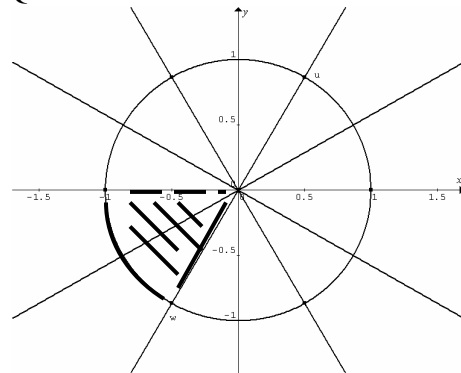


$u = r \text{cis} \theta$, where $r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1$ and

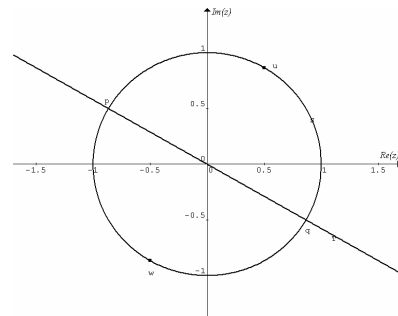
$\theta = \tan^{-1}\left(\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right) = \frac{\pi}{3}$. $\therefore u = \text{cis}\left(\frac{\pi}{3}\right)$.

Q2aii $u^6 = \text{cis}\left(6 \times \frac{\pi}{3}\right) = \text{cis}(2\pi) = 1$.

Q2aiii and Q2b

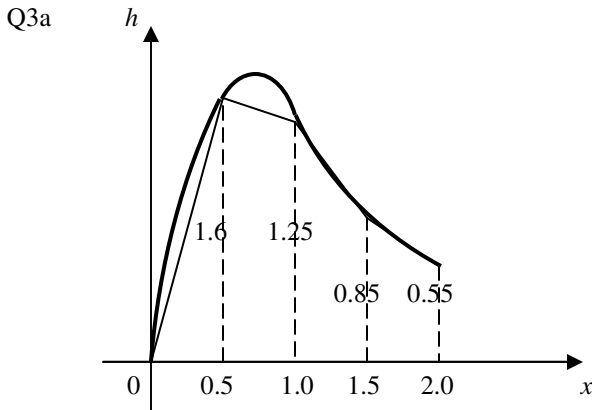


Q2ci and ii



Q2ciii Complex numbers $p = \text{cis}\left(\frac{5\pi}{6}\right)$ and $q = -p$. The

coordinates are $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ and $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ respectively.



$$\text{Area} \approx \frac{1}{2}(0+1.6)0.5 + \frac{1}{2}(1.6+1.25)0.5 + \frac{1}{2}(1.25+0.85)0.5 + \frac{1}{2}(0.85+0.55)0.5$$

$$= 1.9875 \approx 2 \text{ m}^2.$$

Q3b

$$\frac{10x}{(x^2+1)(3x+1)} = \frac{x+A}{x^2+1} + \frac{B}{3x+1} = \frac{(x+A)(3x+1)+B(x^2+1)}{(x^2+1)(3x+1)}.$$

Equate the numerators: $10x = (x+A)(3x+1) + B(x^2+1)$

Let $x = 0$, $0 = A + B \dots\dots\dots(1)$

Let $x = 1$, $10 = (1+A)4 + 2B$, $\therefore 3 = 2A + B \dots\dots\dots(2)$

$(2) - (1)$, $3 = A$, $\therefore B = -3$.

Q3c

$$\text{Area} = \int_0^2 \frac{x+3}{x^2+1} - \frac{3}{3x+1} dx$$

$$= \int_0^2 \frac{x}{x^2+1} + \frac{3}{x^2+1} - \frac{3}{3x+1} dx$$

$$= \left[\frac{1}{2} \log_e (x^2+1) + 3 \tan^{-1}(x) - \log_e (3x+1) \right]_0^2$$

$$= \frac{1}{2} \log_e 5 + 3 \tan^{-1}(2) - \log_e 7 = 2.18 \text{ m}^2.$$

Q3d

$$h(x) = \frac{10x}{(x^2+1)(3x+1)}. \text{ At } x = 2, h = 0.57143.$$

The other position where $h = 0.57143$ is $x = 0.06937$, found by

sketching $h(x) = \frac{10x}{(x^2+1)(3x+1)}$ and $h = 0.57143$, and finding

the first intersection. At the base of a panel the length that is overlapped by the next panel is 0.06937 . $\therefore$ distance between any two panels is $2.0 - 0.06937 = 1.93063$.

$$\text{Minimum number of panels required} = \frac{100}{1.93063} = 51.8, \text{ i.e. } 52.$$

Q4a

$$\underline{u} = 12 \cos 60^\circ \underline{i} + 12 \sin 60^\circ \underline{j} = 6\underline{i} + 6\sqrt{3}\underline{j}.$$

Q4b

$$\underline{v} = \int \underline{\ddot{r}} dt = -0.05t^2 \underline{i} - (gt - 0.05t^2) \underline{j} + \underline{c}.$$

At $t = 0$, $\underline{v} = 6\underline{i} + 6\sqrt{3}\underline{j}$, $\therefore \underline{c} = 6\underline{i} + 6\sqrt{3}\underline{j}$.

Hence $\underline{v} = -0.05t^2 \underline{i} - (gt - 0.05t^2) \underline{j} + 6\underline{i} + 6\sqrt{3}\underline{j}$

$$\therefore \underline{v} = (6 - 0.05t^2) \underline{i} + (6\sqrt{3} - gt + 0.05t^2) \underline{j}.$$

$$\underline{r} = \int \underline{\dot{r}} dt = \left(6t - \frac{0.05t^3}{3} \right) \underline{i} + \left(6\sqrt{3}t - \frac{gt^2}{2} + \frac{0.05t^3}{3} \right) \underline{j} + \underline{d}.$$

At $t = 0$, $\underline{r} = \underline{0}$, $\therefore \underline{d} = \underline{0}$.

$$\therefore \underline{r} = \left(6t - \frac{0.05t^3}{3} \right) \underline{i} + \left(6\sqrt{3}t - \frac{gt^2}{2} + \frac{0.05t^3}{3} \right) \underline{j} \text{ or}$$

$$\therefore \underline{r}(t) = \left(6t - \frac{t^3}{60} \right) \underline{i} + \left(6\sqrt{3}t - \frac{gt^2}{2} + \frac{t^3}{60} \right) \underline{j}, \text{ where } 0 \leq t \leq T.$$

Q4c At $t = T$, skier lands on down-slope and skier's position vector makes -45° with the horizontal (positive x-axis).

$$\therefore \underline{r}(T) = \left(6T - \frac{T^3}{60} \right) \underline{i} + \left(6\sqrt{3}T - \frac{gT^2}{2} + \frac{T^3}{60} \right) \underline{j},$$

$$\frac{6\sqrt{3}T - \frac{gT^2}{2} + \frac{T^3}{60}}{6T - \frac{T^3}{60}} = \tan(-45^\circ), \frac{6\sqrt{3}T - \frac{gT^2}{2} + \frac{T^3}{60}}{6T - \frac{T^3}{60}} = -1,$$

$$6\sqrt{3}T - \frac{gT^2}{2} + \frac{T^3}{60} = -6T + \frac{T^3}{60}, (6\sqrt{3} + 6)T - \frac{gT^2}{2} = 0,$$

$$T \left((6\sqrt{3} + 6) - \frac{gT}{2} \right) = 0. \text{ Since } T \neq 0, \therefore T = \frac{12}{g}(\sqrt{3} + 1).$$

Q4d

$$\underline{v}(t) = (6 - 0.05t^2) \underline{i} + (6\sqrt{3} - gt + 0.05t^2) \underline{j},$$

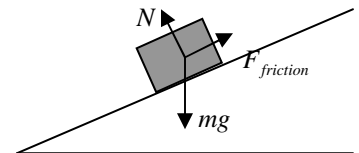
At $t = T = \frac{12}{g}(\sqrt{3} + 1) = 3.3454$, $\underline{v} = 5.44\underline{i} - 21.83\underline{j}$,

$$\text{Speed} = |\underline{v}| = \sqrt{5.44^2 + (-21.83)^2} = 22.5 \text{ ms}^{-1}.$$

Q5a $R = 0$,

$$F_{\text{friction}} - 5g \sin 30^\circ = 0,$$

$$F_{\text{friction}} = \frac{5g}{2} = 24.5 \text{ newtons}$$



Q5b In this situation it is the force of friction that accelerates the package up the belt. If the acceleration is greater than 0.8 ms^{-2} , the package will slip. This indicates that the friction force is at its maximum value μN . Same diagram as above.

Component $\perp$ to belt: $R = 0$,

$$N - 5g \cos 30^\circ = 0 \dots\dots(1)$$

Component $\parallel$ to belt: $R = ma$,

$$\mu N - 5g \sin 30^\circ = 5 \times 0.8 \dots\dots(2)$$

Solve (1) and (2), $\mu \approx 0.67$

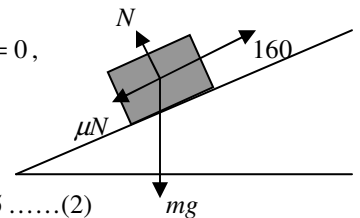
Q5c Component $\perp$ to belt: $R = 0$,

$$N - mg \cos 30^\circ = 0 \dots\dots(1)$$

Component $\parallel$ to belt: $R = ma$,

$$160 - \mu N - mg \sin 30^\circ = m \times 0.5 \dots\dots(2)$$

Solve (1) and (2), where $\mu \approx 0.67$, $m = 14.4 \text{ kg}$



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