

Specialist Mathematics

Written examination 2



2005 Trial Examination

SOLUTIONS

Question 1

$$\begin{aligned} \text{a. i. } \sin\left(\frac{5\pi}{12}\right) &= \sin\left(\frac{\pi}{4} + \frac{\pi}{6}\right) = \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right) \\ &= \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \times \frac{1}{2} = \frac{\sqrt{2}}{2} \left(\frac{\sqrt{3}}{2} + \frac{1}{2} \right) = \frac{\sqrt{2}(\sqrt{3}+1)}{4} \end{aligned}$$

$$\begin{aligned} \text{ii. } \cos\left(\frac{5\pi}{12}\right) &= \cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right) = \cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) - \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right) \\ &= \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \times \frac{1}{2} = \frac{\sqrt{2}}{2} \left(\frac{\sqrt{3}}{2} - \frac{1}{2} \right) = \frac{\sqrt{2}(\sqrt{3}-1)}{4} \end{aligned}$$

$$\begin{aligned} \text{b. } u - v &= \left[\cos\left(\frac{5\pi}{12}\right) + i \sin\left(\frac{5\pi}{12}\right) \right] - \left[\cos\left(\frac{5\pi}{12}\right) - i \sin\left(\frac{5\pi}{12}\right) \right] \\ &= 0 + 2i \sin\left(\frac{5\pi}{12}\right) \end{aligned}$$

$$\therefore \text{Arg}(u - v) = \frac{\pi}{2} \quad (\text{as } 2\sin\left(\frac{5\pi}{12}\right) > 0)$$

$$\begin{aligned} u + v &= \left[\cos\left(\frac{5\pi}{12}\right) + i \sin\left(\frac{5\pi}{12}\right) \right] + \left[\cos\left(\frac{5\pi}{12}\right) - i \sin\left(\frac{5\pi}{12}\right) \right] \\ &= 2\cos\left(\frac{5\pi}{12}\right) + 0i \end{aligned}$$

$$\therefore \text{Arg}(u + v) = 0 \quad (\text{as } 2\cos\left(\frac{5\pi}{12}\right) > 0)$$

$$\text{c. Note that } \text{Arg}(u) = 5\pi/12 \text{ and } \text{Arg}(v) = \text{Arg}(\bar{u}) = -5\pi/12.$$

$$\therefore \text{Arg}(uv) = \text{Arg}(u) + \text{Arg}(v) = \frac{5\pi}{12} + \left(-\frac{5\pi}{12}\right) = 0$$

$$\therefore \text{Arg}\left(\frac{u}{v}\right) = \text{Arg}(u) - \text{Arg}(v) = \frac{5\pi}{12} - \left(-\frac{5\pi}{12}\right) = \frac{5\pi}{6}$$

$$\text{d. Note that } u + v = 2\cos(5\pi/12) = 2 \times \frac{\sqrt{2}(\sqrt{3}-1)}{4} = \frac{\sqrt{2}(\sqrt{3}-1)}{2} \text{ and } uv = 1.$$

Now, u and v are roots of a quadratic polynomial of the form

$$P(z) = (z - u)(z - v) = z^2 - (u + v)z + uv$$

$$\therefore P(z) = z^2 - \left(\frac{\sqrt{2}(\sqrt{3}-1)}{2} \right)z + 1$$

Question 2

a. $x = 0 \Rightarrow f(0) = \sin(0) = 0 \therefore x = 0$ is a solution to the equation $f(x) = x$.

$x = 1 \Rightarrow f(1) = \sin\left(\frac{\pi \times 1}{2}\right) = 1 \therefore x = 1$ is a solution to the equation $f(x) = x$.

b. Domain of $f = [0, 1]$ and range of $f = [0, 1]$.

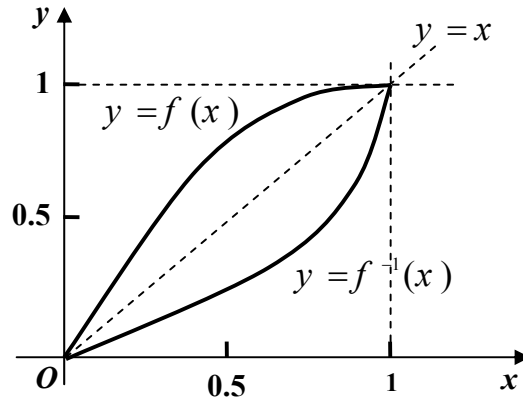
c. Yes, f^{-1} , the inverse function of f , exist because f is a one-to-one function.

d. For f , $y = \sin\left(\frac{\pi x}{2}\right)$, therefore, for f^{-1} ,

$$x = \sin\left(\frac{\pi y}{2}\right) \Rightarrow \frac{\pi y}{2} = \sin^{-1} x \Rightarrow y = \frac{2}{\pi} \sin^{-1} x \Rightarrow f^{-1}(x) = \frac{2}{\pi} \sin^{-1} x$$

Domain of $f^{-1} = \text{range of } f = [0, 1]$.

e.



f. The area between the graphs of f and f^{-1} is twice the area between the graph of f and the line $y = x$. Therefore,

$$\begin{aligned} \text{Area} &= 2 \int_0^1 [\sin(\pi x / 2) - x] dx = 2 \left[-\frac{2}{\pi} \cos(\pi x / 2) - \frac{x^2}{2} \right]_0^1 \\ &= 2 \left[-\frac{2}{\pi} \cos(\pi / 2) - \frac{1^2}{2} \right] - 2 \left[-\frac{2}{\pi} \cos(0) - \frac{0^2}{2} \right] \\ &= 2 \times -\frac{1}{2} - 2 \times -\frac{2}{\pi} = \frac{4}{\pi} - 1 \text{ square units.} \end{aligned}$$

- g. The required volume is the same as the volume of the solid resulting from rotating the area between the graph of f , the x -axis and the line $x = 1$, about the y -axis. Therefore,

$$\begin{aligned} \text{Volume} &= \pi \int_0^1 \sin^2(\pi x / 2) dx = \frac{\pi}{2} \int_0^1 [1 - \cos(2 \times \pi x / 2)] dx \\ &= \frac{\pi}{2} \int_0^1 [1 - \cos(\pi x)] dx = \frac{\pi}{2} \left[x - \frac{1}{\pi} \sin(\pi x) \right]_0^1 \\ &= \frac{\pi}{2} [(1 - 0) - (0)] = \frac{\pi}{2} \text{ cubic units} \end{aligned}$$

Question 3

a. $V = V_0 + R_{in} \times t - R_{out} \times t \Rightarrow V = V_0 + (R_{in} - R_{out})t$

b. $C_{out} = \frac{Q}{V} \Rightarrow C_{out} = \frac{Q}{V_0 + (R_{in} - R_{out})t}$

c. $\frac{dQ}{dt} = \left(\frac{dQ}{dt} \right)_{in} - \left(\frac{dQ}{dt} \right)_{out} = R_{in} \times C_{in} - R_{out} \times C_{out}$

$$\frac{dQ}{dt} = R_{in} \times C_{in} - R_{out} \times \frac{Q}{V_0 + (R_{in} - R_{out})t} \Rightarrow \frac{dQ}{dt} = R_{in} C_{in} - \frac{R_{out} Q}{V_0 + (R_{in} - R_{out})t}$$

The initial condition is $Q = Q_0, t = 0$.

d. $R_{out} = R_{in} \Rightarrow \frac{dQ}{dt} = R_{in} C_{in} - \frac{R_{in}}{V_0} Q \Rightarrow \frac{dt}{dQ} = \frac{1}{R_{in} C_{in} - (R_{in} / V_0) Q}$

Integrating, we get

$$t = \int \frac{1}{R_{in} C_{in} - (R_{in} / V_0) Q} dQ = -\frac{R_{in}}{V_0} \log_e [R_{in} C_{in} - (R_{in} / V_0) Q] + c$$

To determine c , $Q = Q_0, t = 0 \Rightarrow 0 = -\frac{R_{in}}{V_0} \log_e [R_{in} C_{in} - (R_{in} / V_0) Q_0] + c$

$$\therefore c = \frac{R_{in}}{V_0} \log_e [R_{in} C_{in} - (R_{in} / V_0) Q_0]$$

$$\begin{aligned} t &= -\frac{R_{in}}{V_0} \log_e \left(\frac{R_{in} C_{in} - (R_{in} / V_0) Q}{R_{in} C_{in} - (R_{in} / V_0) Q_0} \right) \\ &\Rightarrow -V_0 t / R_{in} = \log_e \left(\frac{R_{in} C_{in} - (R_{in} / V_0) Q}{R_{in} C_{in} - (R_{in} / V_0) Q_0} \right) \end{aligned}$$

$$\begin{aligned} \Rightarrow \frac{R_{in}C_{in} - (R_{in}/V_0)Q}{R_{in}C_{in} - (R_{in}/V_0)Q_0} &= e^{-V_0 t / R_{in}} \\ \Rightarrow R_{in}C_{in} - (R_{in}/V_0)Q &= [R_{in}C_{in} - (R_{in}/V_0)Q_0]e^{-V_0 t / R_{in}} \\ \Rightarrow (R_{in}/V_0)Q &= R_{in}C_{in} - [R_{in}C_{in} - (R_{in}/V_0)Q_0]e^{-V_0 t / R_{in}} \\ \therefore Q &= V_0 C_{in} - (V_0 C_{in} - Q_0)e^{-V_0 t / R_{in}} \end{aligned}$$

e. $Q_0 = 200, V_0 = 10, C_{in} = 3$ and $R_{out} = R_{in} = 5$

i. $\therefore Q = 10 \times 3 - (10 \times 3 - 200)e^{-10t/5} = 30 + 170e^{-2t}$

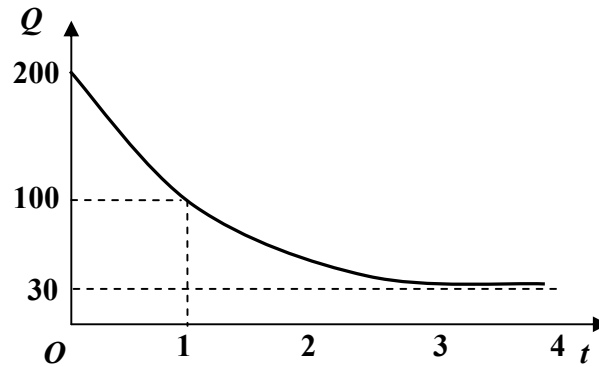
$$Q = 100/2 = 50 \Rightarrow 50 = 30 + 170e^{-2t} \Rightarrow 20 = 170e^{-2t}$$

$$\Rightarrow 2 = 17e^{-2t} \Rightarrow e^{2t} = 17/2 \Rightarrow 2t = \log_e(17/2) \Rightarrow t = \frac{1}{2} \log_e \left(\frac{17}{2} \right)$$

ii. Note that $Q = 30 + 170e^{-2t}$

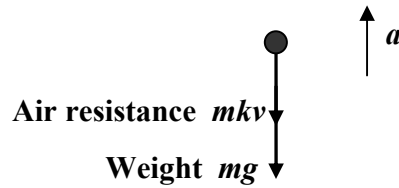
$$t = 0 \Rightarrow Q = 200, \quad t = \frac{1}{2} \log_e \left(\frac{17}{2} \right) \approx 1.07 \Rightarrow Q = 100 \text{ \& the}$$

horizontal asymptote is $Q = 30$.



Question 4

a.



b. The equation of motion is $ma = -mg - mkv \Rightarrow a = -(g + kv)$

c. By choosing $a = \frac{dv}{dt}$, we have $\frac{dv}{dt} = -(g + kv) \Rightarrow \frac{dt}{dv} = -\frac{1}{g + kv}$

$$t = -\int_u^0 \frac{1}{g + kv} dv = -\frac{1}{k} [\log_e(g + kv)]_u^0 = -\frac{1}{k} [\log_e(g + 0) - \log_e(g + ku)]$$

$$\text{should be } u \Rightarrow t = \frac{1}{k} \log_e \left(\frac{g + ku}{g} \right)$$

d. By choosing $a = v \frac{dv}{dx}$, we have

$$v \frac{dv}{dx} = -(g + kv) \Rightarrow \frac{dv}{dx} = -\frac{g + kv}{v} \Rightarrow \frac{dx}{dv} = -\frac{v}{g + kv}$$

e. Integrating, we have

$$\int_0^h dx = -\int_u^0 \frac{v}{g + kv} dv = -\frac{1}{k} \int_u^0 \frac{kv}{g + kv} dv = -\frac{1}{k} \int_u^0 \frac{-g + (g + kv)}{g + kv} dv$$

$$\Rightarrow [x]_0^h = -\frac{1}{k} \int_u^0 \left(\frac{-g}{g + kv} + 1 \right) dv = -\frac{1}{k} \left[\frac{-g}{k} \log_e(g + kv) + v \right]_u^0$$

$$\Rightarrow h = -\frac{1}{k} \left[\frac{-g}{k} \log_e(g + 0) + 0 \right] + \frac{1}{k} \left[\frac{-g}{k} \log_e(g + ku) + u \right]$$

$$\Rightarrow h = \frac{g}{k^2} \log_e(g) - \frac{g}{k^2} \log_e(g + ku) + \frac{u}{k} = \frac{1}{k^2} \left[ku - g \log_e \left(\frac{g + ku}{g} \right) \right]$$

$$\text{f. } h = 33, u = 30, g = 9.8 \Rightarrow \frac{1}{k^2} \left[30k - 9.8 \log_e \left(\frac{9.8 + 30k}{9.8} \right) \right] = 33.$$

Use the graphics calculator to sketch the two graphs

$$y = \frac{1}{x^2} \left[30x - 9.8 \log_e \left(\frac{9.8 + 30x}{9.8} \right) \right] \text{ and } y = 33 \text{ for } 0 \leq x \leq 1. \text{ Then find}$$

the point of intersection. This gives $x = 0.20$ or $k = 0.20$.

Question 5

- a. $\vec{r}(0) = a \cos(0)\vec{i} + b \sin(0)\vec{j} = a\vec{i}$. The particle returns to its initial position after one period, i.e. after $\frac{2\pi}{\pi/3} = 6$ seconds.

b. $\therefore \vec{r}(t) = a \cos\left(\frac{\pi t}{3}\right)\vec{i} + b \sin\left(\frac{\pi t}{3}\right)\vec{j}$
 $\therefore \vec{v}(t) = \frac{d}{dt}(\vec{r}(t)) = -\frac{a\pi}{3} \sin\left(\frac{\pi t}{3}\right)\vec{i} + \frac{b\pi}{3} \cos\left(\frac{\pi t}{3}\right)\vec{j}$

Since $\vec{v}$ is perpendicular to $\vec{r}$, then

$$\begin{aligned} \vec{r} \cdot \vec{v} &= 0 \Rightarrow -\frac{\pi a^2}{3} \sin\left(\frac{\pi t}{3}\right) \cos\left(\frac{\pi t}{3}\right) + \frac{\pi b^2}{3} \sin\left(\frac{\pi t}{3}\right) \cos\left(\frac{\pi t}{3}\right) = 0 \\ \Rightarrow \frac{\pi(b^2 - a^2)}{3} \sin\left(\frac{\pi t}{3}\right) \cos\left(\frac{\pi t}{3}\right) &= 0 \Rightarrow \frac{\pi(b^2 - a^2)}{6} \sin\left(\frac{2\pi t}{3}\right) = 0 \\ \Rightarrow \sin\left(\frac{2\pi t}{3}\right) &= 0 \Rightarrow \frac{2\pi t}{3} = n\pi, n = 0, 1, 2, \dots \Rightarrow t = 3n/2, n = 0, 1, 2, \dots \end{aligned}$$

c. $\vec{a}(t) = \frac{d}{dt}(\vec{v}(t)) = -\frac{a\pi^2}{9} \cos\left(\frac{\pi t}{3}\right)\vec{i} - \frac{b\pi^2}{9} \sin\left(\frac{\pi t}{3}\right)\vec{j}$

$$\begin{aligned} |\vec{a}(t)| &= \frac{\pi^2}{9} \sqrt{a^2 \cos^2\left(\frac{\pi t}{3}\right) + b^2 \sin^2\left(\frac{\pi t}{3}\right)} = \frac{\pi^2}{9} \sqrt{a^2 \cos^2\left(\frac{\pi t}{3}\right) + b^2 \left[1 - \cos^2\left(\frac{\pi t}{3}\right)\right]} \\ &= \frac{\pi^2}{9} \sqrt{(a^2 - b^2) \cos^2\left(\frac{\pi t}{3}\right) + b^2} \end{aligned}$$

Therefore, the magnitude of the acceleration is maximum when

$$\cos\left(\frac{\pi t}{3}\right) = 1 \Rightarrow \frac{\pi t}{3} = 2n\pi, n = 0, 1, 2, \dots \Rightarrow t = 6n, n = 0, 1, 2, \dots$$

$$\max |\vec{a}(t)| = \frac{\pi^2}{9} \sqrt{a^2 - b^2 + b^2} = \frac{\pi^2 a}{9}.$$