

2005 Specialist Mathematics
Written Examination 1 (Facts, skills, and applications)
Suggested answers and solutions

Part 1 (Multiple-choice) Answers

1. E	2. D	3. C	4. D	5. E
6. D	7. A	8. C	9. B	10. B
11. A	12. C	13. A	14. A	15. D
16. E	17. D	18. A	19. B	20. C
21. B	22. B	23. C	24. E	25. A
26. C	27. E	28. A	29. C	30. B

Part 1 (Multiple-choice) Solutions**Question 1****[E]**

$$\frac{3(x+3)^2}{5} + \frac{(y-4)^2}{6} = 3$$

$$\frac{(x+3)^2}{5} + \frac{(y-4)^2}{18} = 1$$

Maximum y -value occurs when $x + 3 = 0$

$$(y-4)^2 = 18$$

$$y-4 = 3\sqrt{2}$$

$$y = 4 + 3\sqrt{2}$$

Question 2**[D]**

For there to be no vertical asymptote

$$x^2 + mx + n \neq 0$$

This means that $b^2 - 4ac < 0$, where $a = 1$, $b = m$ and $c = n$

$$\therefore m^2 + 4n < 0$$

$$\Rightarrow m^2 < -4n$$

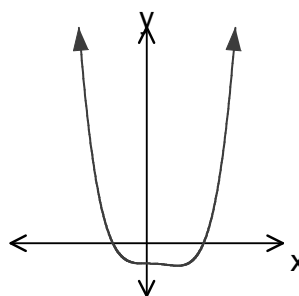
Question 3**[C]**

$$x^4 - x^3 = \csc^2 x - \cot^2 x$$

$$x^4 - x^3 = 1$$

$$x^4 - x^3 - 1 = 0$$

Use a graphics calculator to sketch the graph.



We have two points of intersection

Question 4**[A]**

The graph has the same shape as $y = \cot(x)$

$$y = \cot(x)$$

Period length is $\frac{\pi}{2}$

$$y = -\cot(2x)$$

Phase shift is $\frac{\pi}{12}$

$$y = -\cot\left(2\left(x - \frac{\pi}{12}\right)\right)$$

$$= -\cot\left(2x - \frac{\pi}{6}\right)$$

Question 5**[E]**

$$v = \tan^{-1}(\sqrt{3x})$$

$$\text{Let } u = \sqrt{3x} \text{ and } v = \tan^{-1}(u)$$

$$\frac{du}{dx} = \frac{\sqrt{3}}{2\sqrt{x}} \text{ and } \frac{dv}{du} = \frac{1}{1+u^2}$$

$$\begin{aligned} \frac{dv}{dx} &= \frac{1}{1+3x} \times \frac{\sqrt{3}}{2\sqrt{x}} \\ &= \frac{\sqrt{3}}{2\sqrt{x}(1+3x)} \end{aligned}$$

Question 6**[D]**

$$\begin{aligned} z &= \frac{3-6i}{2+i} \times \frac{2-i}{2-i} \\ &= \frac{-15i}{5} \\ &= -3i \end{aligned}$$

$$|z| = 3 \text{ and } \arg(z) = \frac{3\pi}{2}$$

Question 7**[A]**

$$u = 7\operatorname{cis}\left(\frac{\pi}{4}\right) \text{ and } v = a\operatorname{cis}(b)$$

$$uv = 7a\operatorname{cis}\left(\frac{\pi}{4} + b\right)$$

$$7a = 42 \text{ and } \frac{\pi}{4} + b = \frac{\pi}{20}$$

$$a = 6 \text{ and } b = \frac{\pi}{20} - \frac{\pi}{4} = -\frac{3\pi}{20}$$

Question 8**[C]**

$$\Delta = b^2 - 4ac$$

$$a = 1+i, b = 4i \text{ and } c = -2(1-i)$$

$$\begin{aligned} \Delta &= 16i^2 + 8(1-i)(1+i) \\ &= -16 + 16 \\ &= 0 \end{aligned}$$

Question 9**[B]**

$$u = \sqrt{2} \operatorname{cis}\left(\frac{\pi}{16}\right)$$

$$z = u^4$$

$$z = (\sqrt{2})^4 \operatorname{cis}\left(\frac{4\pi}{16}\right)$$

$$= 4\operatorname{cis}\left(\frac{\pi}{4}\right)$$

$$z^{-1} = \frac{1}{4} \operatorname{cis}\left(-\frac{\pi}{4}\right)$$

Question 10**[B]**

$$|z-1| + |z+1| = 3$$

This locus is an ellipse with foci at (1,0) and (-1,0). This sum of the distances from these points is 3.

Question 11**[A]**

$$\begin{aligned} &\int \frac{6}{\sqrt{1-4x^2}} dx \\ &= \int \frac{6}{2\sqrt{\frac{1}{4}-x^2}} dx \\ &= \int \frac{3}{\sqrt{\frac{1}{4}-x^2}} dx \\ &= 3\sin^{-1}(2x) \end{aligned}$$

Question 12**[C]**

$$\int_{\frac{\pi}{2}}^{\pi} \sin^2(2x) \sin(2x) dx$$

$$= \int_{\frac{\pi}{2}}^{\pi} (1 - \cos^2(2x)) \sin(2x) dx$$

$$\text{Let } u = \cos(2x)$$

$$\frac{du}{dx} = -2\sin(2x)$$

$$dx = \frac{du}{-2\sin(2x)}$$

For the terminals

$$u = \cos(2x)$$

$$\text{When } x = \pi, \cos(2\pi) = 1$$

$$\text{When } x = \frac{\pi}{2}, \cos(\pi) = -1$$

Substituting appropriate values we gain

$$\int_{-1}^1 (1 - u^2) \sin(2x) \times \frac{du}{-2\sin(2x)}$$

$$= -\frac{1}{2} \int_{-1}^1 (1 - u^2) du$$

Question 13**[A]**

$$V = \pi \int y^2 dx$$

$$\text{For } y = \frac{5}{x^2 + 1}$$

$$V_1 = \pi \int_0^2 \left(\frac{5}{x^2 + 1} \right)^2 dx$$

$$\text{For } v = 1$$

$$V_2 = \pi \int_0^2 1^2 dx$$

$$V = V_1 - V_2$$

$$= \pi \int_0^2 \left(\frac{5}{x^2 + 1} \right)^2 - 1 dx$$

Question 14**[A]**

Using a graphics calculator for example

fnInt((x+3)/(2sin(x)),x,4,5) we gain the result - 4.014 correct to three decimals places.**Question 15****[D]**

$$\int x\sqrt{3-x} dx$$

$$\text{Let } u = 3 - x$$

$$\Rightarrow x = 3 - u$$

$$\frac{du}{dx} = -1$$

$$dx = -du$$

Substituting appropriate values into original equation

$$\int x\sqrt{3-x} dx$$

$$= \int (3 - u)\sqrt{u} (-du)$$

$$= - \int (3 - u)\sqrt{u} du$$

$$= - \int \left(3u^{\frac{1}{2}} - u^{\frac{3}{2}} \right) du$$

$$= - \left[2u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}} \right]$$

$$= -2u^{\frac{3}{2}} + \frac{2}{5}u^{\frac{5}{2}}$$

Substituting $u = 3 - x$

$$-2(3 - x)^{\frac{3}{2}} + \frac{2}{5}(3 - x)^{\frac{5}{2}}$$

Question 16**[E]**

$$\int_0^{\frac{\pi}{8}} \sec^2(2x) e^{2\tan(2x)}$$

$$\text{Let } u = 2\tan(2x)$$

$$\frac{du}{dx} = 4\sec^2(2x)$$

$$dx = \frac{du}{4\sec^2(2x)}$$

Evaluating terminals

$$u_1 = 2\tan\left(\frac{\pi}{4}\right) = 2$$

$$u_2 = 2\tan(0) = 0$$

Substituting into the original equation to gain

$$\begin{aligned} & \int_0^2 \sec^2(2x) e^u \times \frac{du}{4\sec^2(2x)} \\ &= \frac{1}{4} \int_0^2 e^u du \\ &= \frac{1}{4} [e^u]_0^2 \\ &= \frac{1}{4} (e^2 - e^0) \\ &= \frac{1}{4} (e^2 - 1) \end{aligned}$$

Question 17**[D]**

$$\frac{dy}{dx} = f(x) = e^{\frac{1}{2}x}$$

$$x_0 = 2, y_0 = 1, \text{ and } h = 0.1$$

$$x_1 = 2.1 \quad y_1 = 1 + 0.1e^{-2} = 1.01353$$

$$x_2 = 2.2 \quad y_2 = 1.01353 + 0.1e^{-2.1} = 1.02578$$

$$y_2 \approx 1.0258$$

Question 18**[A]**

$$\frac{dS}{dt} = 10$$

$$S = \pi r^2$$

$$\frac{dS}{dt} = 2\pi r$$

$$\begin{aligned} \frac{dr}{dt} &= \frac{dr}{ds} \times \frac{ds}{dt} \\ &= \frac{1}{2\pi r} \times 10 \\ &= \frac{5}{\pi r} \end{aligned}$$

Question 19**[B]**

$$\frac{dy}{dx} = y^2 + 1$$

$$\frac{dx}{dy} = \frac{1}{y^2 + 1}$$

$$x = \tan^{-1} y + c$$

$$\text{At } y = 1, x = 0$$

$$0 = \tan^{-1} 1 + c$$

$$0 = \frac{\pi}{4} + c$$

$$c = -\frac{\pi}{4}$$

$$x = \tan^{-1}(y) - \frac{\pi}{4}$$

$$x + \frac{\pi}{4} = \tan^{-1}(y)$$

$$y = \tan\left(x + \frac{\pi}{4}\right)$$

Question 20**[C]**

$$a = v \frac{dv}{dx}$$

$$\frac{dv}{dx} = \frac{1}{(1+x^2)^{\frac{3}{2}}} \times \frac{2x}{1} = \frac{2x}{(1+x^2)^{\frac{3}{2}}}$$

$$a = \frac{2}{(1+x^2)^{\frac{3}{2}}} \times \frac{2x}{(1+x^2)^{\frac{3}{2}}} \\ = \frac{4x}{(1+x^2)^2}$$

Alternatively

$$a = \frac{d\left(\frac{1}{2}v^2\right)}{dx}$$

$$v = \frac{2}{\sqrt{1+x^2}}$$

$$v^2 = \frac{4}{1+x^2}$$

$$\frac{v^2}{2} = \frac{2}{1+x^2}$$

$$\frac{d\left(\frac{1}{2}v^2\right)}{dx} = \frac{4x}{(1+x^2)^2}$$

Question 21**[B]**

$$\frac{dv}{dt} = \frac{3}{v^2+9}$$

$$dt = \frac{v^2+9}{3} dv$$

$$t = \int_2^1 \frac{v^2+9}{3} dv$$

The terminals are reversed because the curve is below the x -axis for the interval $[1, 2]$

Question 22**[B]**For $PQRS$ to be a parallelogram

$$\vec{PO} = \vec{SR}$$

$$\vec{PQ} = \vec{q} - \vec{p} \\ = \hat{i} + y\hat{j} - 3\hat{k} \\ = \hat{i} + v\hat{j} + 3\hat{k}$$

$$\vec{SR} = \vec{r} - \vec{s} \\ = 5\hat{j} + 2x\hat{j} + \hat{k} - y\hat{j} + 2\hat{k} \\ = (5+v)\hat{j} + 2x\hat{j} + 3\hat{k}$$

$$5+v = 1$$

$$\Rightarrow v = 4 \\ 2x = y$$

$$\Rightarrow x = 2$$

Question 23**[C]**

Let the point $(2, 2, -1)$ be represented by the vector $\mathbf{a} = 2\hat{i} + 2\hat{j} - \hat{k}$ and the point $(-4, 0, -3)$ be represented by the vector $\mathbf{b} = -4\hat{i} - 3\hat{k}$

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

$$\mathbf{a} \times \mathbf{b} = -8 + 0 + 3$$

$$|\mathbf{a}| = \sqrt{4 + 4 + 1} = 3$$

$$|\mathbf{b}| = \sqrt{16 + 9} = 5$$

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

$$= \frac{-5}{3 \times 5}$$

$$= -\frac{1}{3}$$

Question 24
[E]

$(x + 1)^2 + y^2 + 1$ is a circle with centre $(-1, 0)$ and radius 1

$$\text{Let } x + 1 = \cos t$$

$$\Rightarrow x = \cos t - 1$$

This possibility is not evident in the answers.

$$\text{Let } x + 1 = \sin t$$

$$\Rightarrow x = \sin t - 1$$

$$\text{and } y = \cos t$$

$$\mathbf{r}(t) = (\sin(t) - 1)\mathbf{i} + \cos(t)\mathbf{j}$$

Question 25
[A]

$$\mathbf{r}(t) = (3t^2 - 2)\mathbf{j} + (7 - 5t)\mathbf{j} + 4\mathbf{k}$$

$$\dot{\mathbf{r}}(t) = 6t\mathbf{j} - 5\mathbf{j}$$

$$= 6t\mathbf{j} + 5\mathbf{j}$$

Question 26
[C]

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = e^{-0.1t}\mathbf{j} + (6t)\mathbf{j}$$

$$\mathbf{v} = 10e^{-0.1t}\mathbf{i} + 3t^2\mathbf{i} + \mathbf{c}$$

$$\text{At } t = 0, \mathbf{v} = 0$$

$$0 = -10\mathbf{j} + \mathbf{c}$$

$$\mathbf{c} = 10\mathbf{j}$$

$$\begin{aligned}\mathbf{v} &= -10e^{-0.1t}\mathbf{j} + 3t^2\mathbf{j} + 10\mathbf{j} \\ &= 10(1 - e^{-0.1t})\mathbf{j} + 3t^2\mathbf{j}\end{aligned}$$

Question 27
[E]

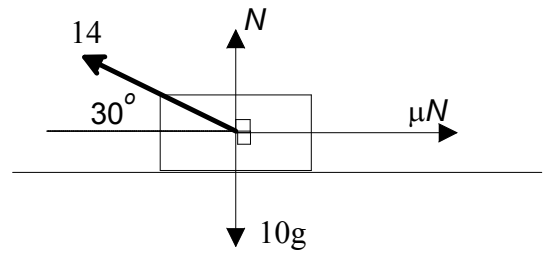
$$F = ma$$

$$= 5(20 - 10\cos(2t))$$

$$\text{Max value occurs when } \cos(2t) = -1$$

$$F = 5(20 + 10)$$

$$= 150$$

Question 28
[A]


$$7\sqrt{3} = \mu N$$

$$\mu = \frac{7\sqrt{3}}{N}$$

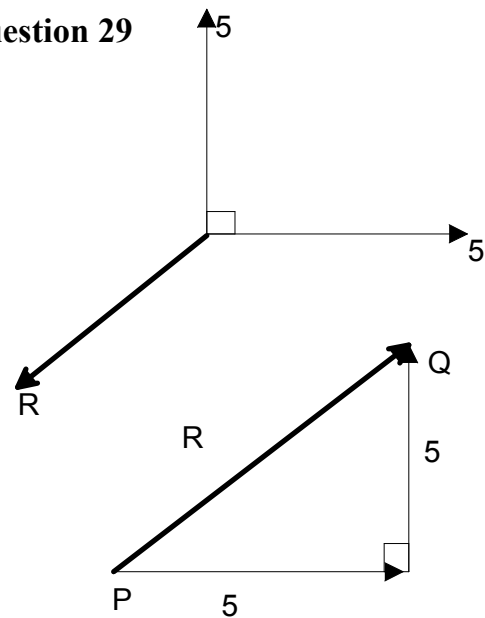
$$7 + N = 10g$$

$$N = 10g - 7$$

$$= 98 - 7 = 91$$

$$\mu = \frac{7\sqrt{3}}{91}$$

$$= \frac{\sqrt{3}}{13}$$

Question 29
[C]


$$\vec{Q} + \vec{P} = 5\sqrt{2} \text{ North East}$$

$\vec{R}$ must act with equal and opposite force, which is $5\sqrt{2}$ Southwest

Question 30**[B]**

$$F_r = 200a = 1000 \text{ } \checkmark 200g$$

$$a = \frac{1000}{200} \checkmark \frac{200g}{200}$$

$$= 5 \checkmark 9.8$$

$$= -4.8$$

Part II (Short-answer) Solutions**Question 1****a**Equation of line: $y \checkmark x_1 = m(x \checkmark x_1)$

$$m = \frac{2}{3}$$

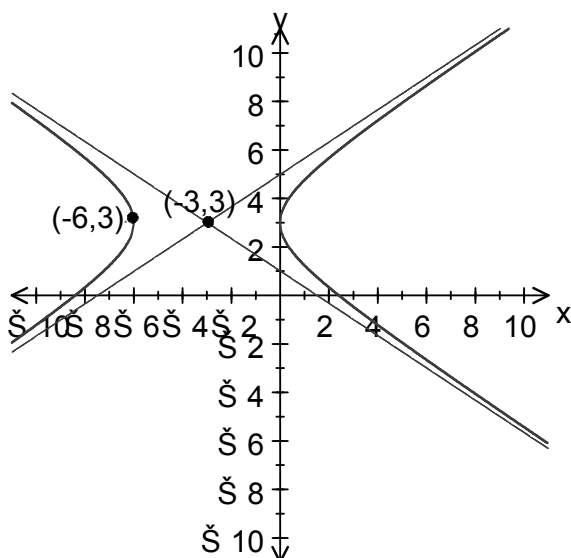
Point on straight line: $(c, 3)$

$$y \checkmark 3 = \frac{2}{3}(x \checkmark c)$$

$$y = \frac{2}{3}x \checkmark \frac{2}{3}c + 3$$

$$\checkmark \frac{2}{3}c + 3 = 5$$

$$c = -3$$

b**Question 2**

$$y = e^{2x} \cos(x)$$

$$\frac{dy}{dx} = 2e^{2x} \cos(x) \checkmark e^{2x} \sin(x)$$

$$= e^{2x} (2\cos(x) \checkmark \sin(x))$$

$$\frac{d^2y}{dx^2} = 2e^{2x} (2\cos(x) \checkmark \sin(x)) + e^{2x} (-2\sin(x) \checkmark \cos(x))$$

$$= e^{2x} (4\cos(x) \checkmark 2\sin(x) \checkmark 2\sin(x) \checkmark \cos(x))$$

$$= e^{2x} (3\cos(x) \checkmark 4\sin(x))$$

$$\frac{d^2y}{dx^2} + k \frac{dy}{dx} + y = -2e^{2x} \sin(x)$$

$$LHS = e^{2x} (3\cos(x) \checkmark 4\sin(x)) + ke^{2x} (2\cos(x) \checkmark \sin(x)) + e^{2x} \cos(x)$$

$$= e^{2x} (4\cos(x) + 2k\cos(x) \checkmark 4\sin(x) \checkmark k\sin(x))$$

$$\cos(x): 3 + 2k + 1 = 0$$

$$k = -2$$

$$\sin(x): \checkmark 4 \checkmark k = -2$$

$$k = -2$$

Question 3**a**Let $\underline{a} = 3\hat{i} - 2\hat{j} + \hat{k}$ and $\underline{b} = 2\hat{i} + x\hat{j} + 2\hat{k}$

We know that the two vectors are perpendicular.

$$\begin{aligned}\underline{a} \cdot \underline{b} &= 6 - 2x + 2 = 0 \\ 2x &= 8 \\ x &= 4\end{aligned}$$

b

$$\begin{aligned}\underline{u} &= \underline{a} + \underline{b} \\ &= 3\hat{i} - 2\hat{j} + \hat{k} + 2\hat{i} + 4\hat{j} + 2\hat{k} \\ &= 5\hat{i} + 2\hat{j} + 3\hat{k}\end{aligned}$$

Question 4**a**

$$\begin{aligned}v &= (t - 4)\tan\left(\frac{\pi t}{48}\right) \\ \text{at } t &= 12 \\ v &= (12 - 4)\tan\left(\frac{12\pi}{48}\right) \\ &= 8\tan\left(\frac{\pi}{4}\right) \\ &= 8\end{aligned}$$

b

Cyclist B passes Cyclist A when they are the same distance from the starting point.

At $t = 12$

Cyclist A: Dist = 72 m

Cyclist B:

$$\text{Dist} = \int_4^{12} (t - 4)\tan\left(\frac{\pi t}{48}\right) dt = 22.8922$$

Cyclist B remains $72 - 22.89222 = 49.1078$

Cyclist B is traveling 2 m/s faster than Cyclist A

He will now catch Cyclist A in 24.55389 seconds

Therefore it takes him $12 + 24.55389$ seconds to catch Cyclist A, which is 36.6 seconds to the nearest tenth of a second.**Question 5**

$$\begin{aligned}135 - \mu N &= 0.5(12 + 18) \\ N &= 30g\end{aligned}$$

$$\begin{aligned}135 - \mu 30g &= 15 \\ 120 &= \mu 30g \\ \mu &= \frac{120}{30g} \\ &= 0.4081 \\ &\approx 0.41\end{aligned}$$

b

$$\begin{aligned}T - 0.41 \times 18g &= 0.5 \times 18 \\ T &= 9 + 0.41 \times 18g \\ &= 81.324\end{aligned}$$