# SPECIALIST MATHS EXAM 2 SOLUTIONS

**Question 1**

**a.** [0, 0.8] **A1**

**b.** $f(x) = 1 - (4x^2 + 1)^{-1}$

$f'(x) = 0 + (4x^2+1)^{-2} \times 8x$

$= \dfrac{8x}{(4x^2+1)^2}$ **A1**

$f''(x) = \dfrac{8(4x^2+1)^2 - 8x \times 2(4x^2+1) \times 8x}{(4x^2+1)^4}$

At point of greatest slope $f''(x) = 0$

$0 = \dfrac{8(4x^2+1)[(4x^2+1) - 16x^2]}{(4x^2+1)^4}$ **M1**

$0 = 4x^2 + 1 - 16x^2$

$12x^2 = 1$

$x = \dfrac{1}{\sqrt{12}} = \dfrac{\sqrt{12}}{12} = \dfrac{\sqrt{3}}{6}$

(positive root only, since domain is [0, 1]) **A1**

$y = 1 - \dfrac{1}{4\left(\dfrac{\sqrt{3}}{6}\right)^2 + 1} = \dfrac{1}{4}$ **M1**

$\therefore$ The point of greatest slope is at $\left(\dfrac{\sqrt{3}}{6}, \dfrac{1}{4}\right)$.

**c.** $y = 1 - \dfrac{1}{4x^2+1}$

Transpose to find $x^2$

$4x^2 + 1 = \dfrac{1}{1-y}$

$x^2 = \dfrac{1}{4}\left(\dfrac{1}{1-y} - 1\right)$ **A1**

$V = \pi \int_0^{0.8} x^2 dy$

$V = \dfrac{\pi}{4}\int_0^{0.8} (\dfrac{1}{1-y} - 1) dy$ **M1**

$= \dfrac{\pi}{4}\left[-\log_e(1-y) - y\right]_0^{0.8}$

$= -\dfrac{\pi}{4}\left[\log_e(1-0.8) + 0.8 - \log_e(1-0) - 0\right]$

**M1**

$= 0.636\text{ m}^3$ **H1**

**d.** $\dfrac{dV}{dt} = \dfrac{dV}{dy} \times \dfrac{dy}{dt}$

$0.012 = \dfrac{\pi}{4}\left(\dfrac{1}{1-y} - 1\right) \times \dfrac{dy}{dt}$ **M1**

$\left(\dfrac{1}{1-y} - 1\right)\dfrac{dy}{dt} = \dfrac{0.012 \times 4}{\pi}$

At point of greatest slope, $y = \dfrac{1}{4}$ **M1**

$\left(\dfrac{1}{1-\frac{1}{4}} - 1\right)\dfrac{dy}{dt} = \dfrac{0.012 \times 4}{\pi}$

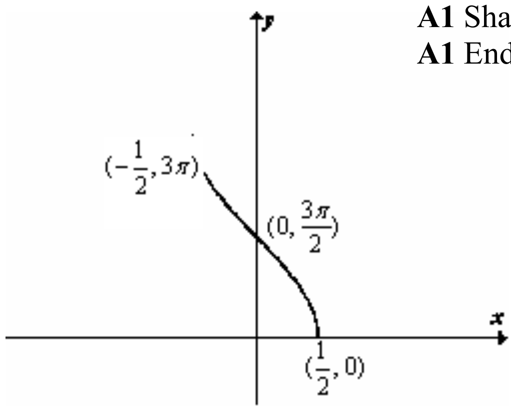
$\left(\dfrac{4}{3} - 1\right) \times \dfrac{dy}{dt} = \dfrac{0.012 \times 4}{\pi}$ **M1**

$\dfrac{dy}{dt} = \dfrac{0.012 \times 4}{\pi} \times \dfrac{3}{1}$

$\dfrac{dy}{dt} = \dfrac{0.144}{\pi}$ m/min **A1**

**Question 2**

**a.**




**A1** Shape
**A1** Endpoints

**b. i**

$f(x) = x\,\text{Cos}^{-1}(2x)$

$f'(x) = 1.\text{Cos}^{-1}(2x) + x.\dfrac{-1}{\sqrt{\left(1-(2x)^2\right)}} \times 2$ **M1**

$f'(x) = \text{Cos}^{-1}(2x) - \dfrac{2x}{\sqrt{\left(1-4x^2\right)}}$ **A1**

**ii.** $\left\{x : -\dfrac{1}{2} < x < \dfrac{1}{2}\right\}$ **or** $\left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$

**A1**

**c.** $g(x) = (1 - 4x^2)^{\frac{1}{2}}$

$g'(x) = \frac{1}{2}(1 - 4x^2)^{-\frac{1}{2}}(-8x)$ **M1**

$g'(x) = \frac{-4x}{\sqrt{1 - 4x^2}}$

$\Rightarrow \left\{x : -\frac{1}{2} < x < \frac{1}{2}\right\}$ **A1**

**d. i.** $f'(x) = \text{Cos}^{-1}(2x) - \frac{2x}{\sqrt{(1 - 4x^2)}}$

$f'(x) = \text{Cos}^{-1}(2x) + \frac{1}{2} \times \frac{-4x}{\sqrt{(1 - 4x^2)}}$

$f'(x) = \text{Cos}^{-1}(2x) + \frac{1}{2}g'(x)$ **M1**

$\text{Cos}^{-1}(2x) = f'(x) - \frac{1}{2}g'(x)$

$3\text{Cos}^{-1}(2x) = 3\left[f'(x) - \frac{1}{2}g'(x)\right]$ **A1**

**ii.** $\int_0^{\frac{1}{4}} 3\text{Cos}^{-1}(2x)\, dx$

$= 3\int_0^{\frac{1}{4}} f'(x) - \frac{1}{2}g'(x)\, dx$

$= 3\left[f(x) - \frac{1}{2}g(x)\right]_0^{\frac{1}{4}}$ **M1**

$= 3\left[x\text{Cos}^{-1}(2x) - \frac{1}{2}\sqrt{1 - 4x^2}\right]_0^{\frac{1}{4}}$

$= 3\left[\frac{1}{4}\text{Cos}^{-1}(2 \times \frac{1}{4}) - \frac{1}{2}\sqrt{1 - 4\left(\frac{1}{4}\right)^2} - 0 + \frac{1}{2}\right]$ M1

$= 3\left[\frac{1}{4}\text{Cos}^{-1}\left(\frac{1}{2}\right) - \frac{1}{2}\sqrt{\frac{3}{4}} + \frac{1}{2}\right]$

$= 3\left[\frac{1}{4} \times \frac{\pi}{3} - \frac{\sqrt{3}}{4} + \frac{1}{2}\right]$

$= 3\left[\frac{\pi}{12} - \frac{3\sqrt{3}}{12} + \frac{6}{12}\right]$

$= \frac{\pi - 3\sqrt{3} + 6}{4}$ **A1**

**Question 3**

**a.**

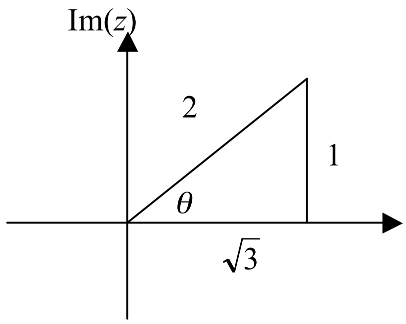



fig 1

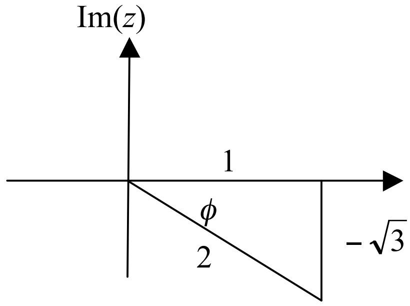



fig 2

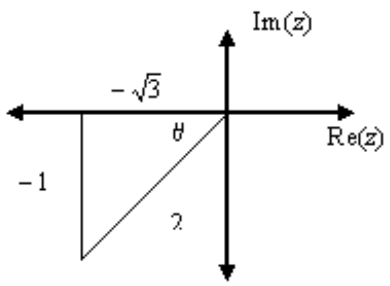
| fig 1 | fig 2 |
|---|---|
| $r = \sqrt{(\sqrt{3})^2 + 1^2} = 2$ | $r = \sqrt{(-\sqrt{3})^2 + 1^2} = 2$ |
| $\sin\theta = \frac{1}{2}$ or $\tan\theta = \frac{1}{\sqrt{3}}$ | $\cos\phi = \frac{\pi}{6}$ or $\tan\phi = \frac{\sqrt{3}}{1}$ |
| $\theta = \frac{\pi}{6}$ | $\phi = \frac{\pi}{3}$ |
| $u = 2\text{cis}\frac{\pi}{6}$ | $v = 2\text{cis}\left(-\frac{\pi}{3}\right)$ |

**b.** $uv = (\sqrt{3} + i)(1 - \sqrt{3}i)$

$= \sqrt{3} - 3i + i - \sqrt{3}i^2$

$= 2\sqrt{3} - 2i$ **A1**

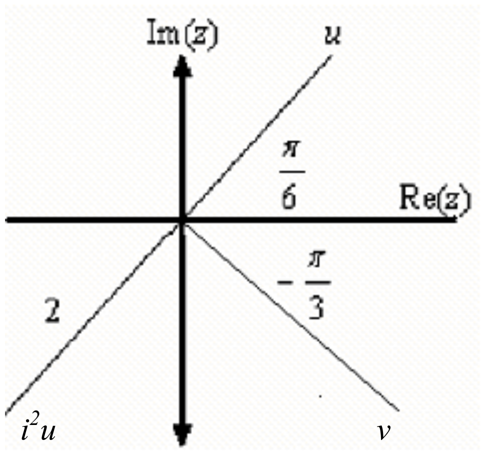
**c.**




| | |
|---|---|
| $i^2u = -1(\sqrt{3}+i)$ | $r = \sqrt{(-\sqrt{3})^2 + (-1)^2} = 2$ |
| $i^2u = -\sqrt{3} - i$ | $\tan\theta = \dfrac{1}{\sqrt{3}}$ |
| $i^2u = 2\text{cis}\left(-\dfrac{5\pi}{6}\right)$ | $\theta = \dfrac{\pi}{6}$ |

**A1**

**d.**




**e.** $i^2u = 2\text{cis}\left(-\dfrac{5\pi}{6}\right)$ from **c.**

$$i^3v = i^2 i \times (1-\sqrt{3}i)$$
$$= -i(1-\sqrt{3}i)$$
$$= -\sqrt{3} - i$$
$$= 2\text{cis}\left(-\frac{5\pi}{6}\right)$$ **A1**

Multiplication by $i$ represents an anticlockwise rotation of $90^0$ or $\dfrac{\pi}{2}$. Since $u$ and $v$ are perpendicular, if $u$ is rotated by $180^0$ ($i^2$) and $v$ rotated by $270^0$ ($i^3$) then the two complex numbers will coincide and hence will be equal.

**A1**

**Question 4**

**a.** $\dfrac{dN}{dt} = kN$

$$t = \frac{1}{k}\int \frac{1}{N}\,dN + c$$
$$t = \frac{1}{k}\log_e N + \text{c}$$

When $t = 0$, $N = 700$ **M1**

$$c = -\frac{1}{k}\log_e 700$$
$$t = \frac{1}{k}\left(\log_e N - \log_e 700\right)$$
$$t = \frac{1}{k}\log_e\left(\frac{N}{700}\right)$$
$$e^{kt} = \frac{N}{700}$$
$$N = 700e^{kt}$$ **A1**

**b.** When $t = 2$, $N = 550$

$$550 = 700e^{2k}$$
$$k = \frac{1}{2}\log_e\left(\frac{550}{700}\right)$$
$$k = -0.12$$ **A1**

**c.** $700e^{-0.12t} < 50$

$$t > -\frac{1}{0.12}\log_e\left(\frac{50}{700}\right) = 22 \text{ years}$$ **A1**

**d.** $\dfrac{dN}{dt} = P + mN$

$$t = \frac{1}{m}\int \frac{m}{P+mN}\,dN$$
$$t = \frac{1}{m}\log_e\left(P+mN\right) + c$$

When $t = 3$, $N = 488$

$$c = 3 - \frac{1}{m}\log_e\left(P + 488m\right)$$ **A1**

$$t = \frac{1}{m}\log_e\left(P+mN\right) + 3 - \frac{1}{m}\log_e\left(P+488m\right)$$
$$t - 3 = \frac{1}{m}\log_e\left(\frac{P+mN}{P+488m}\right)$$ **M1**

$$e^{m(t-3)} = \frac{P+mN}{P+488m}$$
$$N = \frac{(P+488m)e^{m(t-3)} - P}{m}$$ **A1**

**e. i.** If $P = 60$ and $m = -0.05$

$$N = \frac{(P+488m)e^{m(t-3)} - P}{m}$$

$$N = \frac{35.6e^{-0.05(t-3)} - 60}{-0.05}$$

When $t = 8$ $N = 645$ penguins **A1**

**ii.** As $t \to \infty$, $N \to \frac{-60}{-0.05} = 1200$ penguins

**A1**

**iii.** $\frac{(P+488m)e^{m(t-3)} - P}{m} < 800$

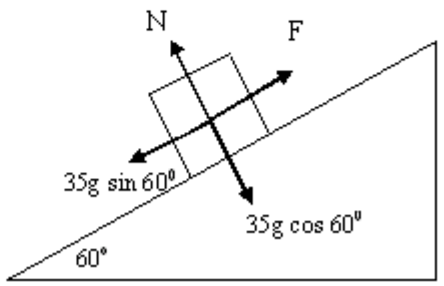
As $t \to \infty$ $e^{m(t-3)} \to 0$ since $m < 0$ **A1**

$\Rightarrow \frac{-P}{m} < 800$

$-P > 800m$ since $m < 0$

$\therefore P < -800m$ **A1**

**Question 5**




**a.** $N = mg\cos\theta$

$= 35 \times 9.8\cos 60$

$= 171.5$ newtons **A1**

**b. i** $u = 0$ m/s, $s = 5$ metres, $v = 8$ m/s

$v^2 - u^2 = 2as$

$a = \frac{v^2 - u^2}{2a}$

$= \frac{64}{10}$

$= 6.4$ m/s$^2$ **A1**

**b. ii** From R = ma

$mg\sin\theta - F = ma$, where $F = \mu N$

$mg\sin\theta - \mu mg\cos\theta = ma$ **M1**

$g\sin\theta - \mu g\cos\theta = a$

$\mu = \frac{g\sin\theta - a}{g\cos\theta}$

$= \frac{9.8\sin 60 - 6.4}{9.8\cos 60}$

$= 0.43$ **A1**

**c. i** $p = mv$

$= 35 \times 8$

$= 280$ kg m/s **A1**

**ii** $(35 + 3)v = 280$

$v = \frac{140}{19}$ m/s **A1**

**d.** $u = \frac{140}{19}$ m/s, $t = 1.4$ seconds, $s = 16$ metres

$s = ut + \frac{1}{2}at^2$ **M1**

$16 = \frac{140}{19} \times 1.4 + \frac{1}{2}a \times 1.4^2$

$a = 5.8$ m/s$^2$ **A1**

$\mu = \frac{g\sin\theta - a}{g\cos\theta}$

$= \frac{9.8\sin 60 - 5.8}{9.8\cos 60}$

$= 0.55$ **M1**

**e.** $u = \frac{140}{19}$ m/s, $t = 1.4$ seconds, $s = 16$ metres

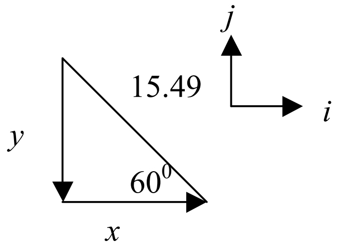
$s = \frac{(u+v)t}{2}$

$v = \frac{2s}{t} - u$

$v = \frac{2 \times 16}{1.4} - \frac{140}{19}$

$= 15.5$ m/s **A1**

**f.**




$\sin 60 = \dfrac{y}{15.49}$ $\quad y = 13.41$ **M1**

$\cos 60 = \dfrac{x}{15.49}$ $\quad x = 7.74$

$\underset{\sim}{v} = 7.74\underset{\sim}{i} - 13.41\underset{\sim}{j}$ **A1**

**g.** $\underset{\sim}{v} = -gt\underset{\sim}{j} + \underset{\sim}{c}$ **M1**

t = 0, $\underset{\sim}{v} = 7.74\underset{\sim}{i} - 13.41\underset{\sim}{j}$

$\underset{\sim}{c} = 7.74\underset{\sim}{i} - 13.41\underset{\sim}{j}$

$\underset{\sim}{v} = 7.74\underset{\sim}{i} - (13.41 + gt)\underset{\sim}{j}$ **A1**

**h. i** speed $= \left|\underset{\sim}{v}\right| = \sqrt{7.74^2 + (13.41 + 9.8 \times 1.5)^2}$

$= 29.2$ m/s **A1**

**ii** The horizontal component of the velocity is constant at 7.74 m/s.

$v = \dfrac{d}{t}$

d = vt

= 7.74 × 1.5

= 11.6 metres **A1**