

SPECIALIST MATHS EXAM 1 SOLUTIONS

Exam 1 Part I.

Question 1 A

General equation hyperbola: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

From graph $a = 2$

Asymptote equations: $y = \pm \frac{b}{a}x$

From graph asymptotes are $y = \pm 2x$

$$\Rightarrow 2 = \frac{b}{2}, \quad b = 4$$

Equation hyperbola: $\frac{x^2}{2^2} - \frac{y^2}{4^2} = 1$

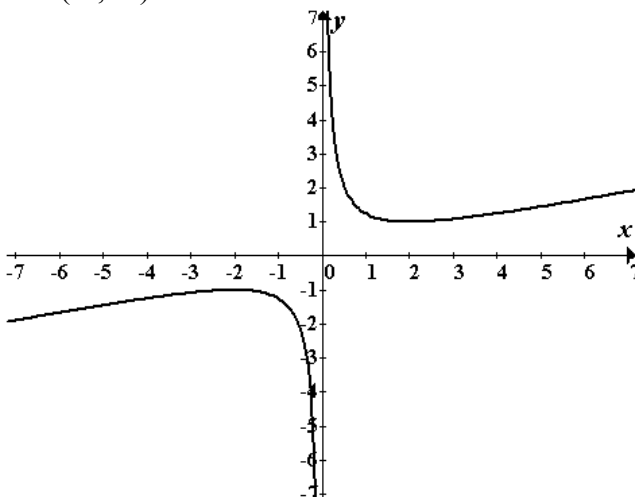
$$\Rightarrow \frac{x^2}{4} - \frac{y^2}{16} = 1$$

$$\Rightarrow 4x^2 - y^2 = 16$$

Question 2 C

Sketch $y = \frac{(x^2 + 4)}{(4x)}$

Graph shows there is a maximum turning point at $(-2, -1)$



Question 3 E

$f(x)$ has asymptotes at $x = -1, 2$

$$\Rightarrow \frac{1}{f(x)} = 0 \text{ at } x = -1, 2$$

$$\Rightarrow \frac{1}{f(-x)} = 0 \text{ at } x = -2, 1$$

$f(x) > 0$ for $x < -1 \cup x > 2$ and

$f(x) < 0$ for $-1 < x < 2$

$$\Rightarrow \frac{1}{f(-x)} > 0 \text{ for } x < -2 \cup x > 1$$

$$\text{and } \frac{1}{f(-x)} < 0 \text{ for } -2 < x < 1$$

Question 4 C

$$\tan x = a \text{ and } x \in \left[\pi, \frac{3\pi}{2} \right]$$

(3rd quadrant: $\cos x$ is negative)

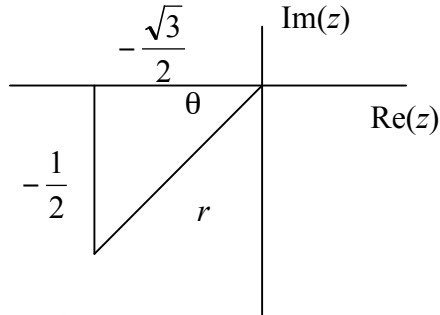
$$\sec^2 x = \tan^2 x + 1$$

$$\sec^2 x = a^2 + 1$$

$$\cos^2 x = \frac{1}{a^2 + 1}$$

$$\cos x = -\frac{1}{\sqrt{a^2 + 1}}, \text{ since } \cos x \text{ is negative in the}$$

third quadrant.

Question 5
A


$$\begin{aligned}\tan \theta &= \frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}} \\ &= -\frac{1}{2} \times -\frac{2}{\sqrt{3}} \\ &= \frac{1}{\sqrt{3}} \\ \theta &= \frac{\pi}{6}\end{aligned}$$

$$\text{Hence Arg } z = -\pi + \frac{\pi}{6} = -\frac{5\pi}{6}$$

Question 6
A

General complex form of a circle is

$$|z - (\text{centre})| = \text{radius}$$

$$|z - (2 - 3i)| = 4 \quad \text{a circle, centre } (2, -3), \text{ radius } 4$$

OR Let $z = x + yi$

$$|(x + yi) - 2 + 3i| = 4$$

$$|(x - 2) + (y + 3)i| = 4$$

$$\sqrt{(x - 2)^2 + (y + 3)^2} = 4$$

$$(x - 2)^2 + (y + 3)^2 = 16$$

circle, centre $(2, -3)$ radius 4

Question 7
C

$$\bar{z} = c + di$$

$$\bar{z} - z = (c + di) - (c - di) = 2di$$

Imaginary part is $2d$ NOT $2di$

Question 8
E

If $z = 2 - i$ is a root, then $(z - 2 + i)$ is a factor

Polynomial division:

$$\begin{array}{r} z - 1 - i \\ z - 2 + i \overline{) z^2 - 3z + (3+i)} \\ \underline{-(z^2 - 2z + iz)} \\ -z - iz + (3+i) \\ \underline{-(-z + 2 - i)} \\ -iz - 2 + (3 + i) + i \\ \underline{-(-iz + 2i - i^2)} \\ 0 \end{array}$$

Hence $z - 1 - i$ is a factor,

So $z = 1 + i$ is a root.

Question 9
E

$$\tan\left(3x - \frac{\pi}{2}\right) = -\frac{1}{\sqrt{3}}$$

$$3x - \frac{\pi}{2} = n\pi - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \quad n = 0, 1, 2, 3, 4, 5$$

$$3x = n\pi - \frac{\pi}{6} + \frac{\pi}{2}$$

$$3x = n\pi + \frac{\pi}{3}$$

$$x = \frac{3n\pi + \pi}{9}$$

$$x = \frac{\pi}{9}, \frac{4\pi}{9}, \frac{7\pi}{9}, \frac{10\pi}{9}, \frac{13\pi}{9}, \frac{16\pi}{9}$$

$$\Rightarrow x = \frac{13\pi}{9}$$

Question 10
B

$$\begin{aligned}\frac{1}{\sin x} \times \frac{\cos 2x}{1} &= \frac{1}{\sin x} \times \frac{\cos 2x}{\sin 2x} \times \frac{\sin 2x}{1} \\ &= \frac{1}{\sin x} \times \cot 2x \times \frac{2 \sin x \cos x}{1} \\ &= \cot 2x \times \frac{2 \cos x}{1} \\ &= 2 \cos x \cot 2x\end{aligned}$$

Question 11 **D**

Let $f(x) = \cos x$

Range $f(x)$ is $[-1, 1]$

$\Rightarrow$ Domain: $\tan^{-1}(f(x))$ is $[-1, 1]$

Range: $\tan^{-1}(f(x))$ is $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

Question 12 **A**

$$\frac{dy}{dx} = \frac{2 \cos 2x}{\sin 2x}$$

Applying the quotient rule

$$\begin{aligned} \frac{d^2 y}{d^2 x} &= \frac{2(-2 \sin 2x) \sin 2x - 2 \cos 2x(2 \cos 2x)}{(\sin 2x)^2} \\ &= \frac{-4 \sin^2 2x - 4 \cos^2 2x}{(\sin 2x)^2} \\ &= \frac{-4(\sin^2 2x + \cos^2 2x)}{(\sin 2x)^2} \\ &= \frac{-4}{(\sin 2x)^2} \\ &= -4 \operatorname{cosec}^2 2x \end{aligned}$$

Question 13 **B**

By partial fractions

$$\begin{aligned} \frac{1}{(1-3x)(1+3x)} &= \frac{1}{2(1-3x)} + \frac{1}{2(1+3x)} \\ \int \frac{1}{1-9x^2} dx &= \int \frac{1}{2(1+3x)} + \frac{1}{2(1-3x)} dx \\ &= \frac{1}{6} \int \frac{3}{(1+3x)} - \frac{3}{(1-3x)} dx \\ &= \frac{1}{6} [\log_e(1+3x) - \log_e(1-3x)] \\ &= \frac{1}{6} \log_e \left(\frac{1+3x}{1-3x} \right) \end{aligned}$$

Question 14 **B**

Stationary points at $x = -3, -1, 2$

$x = -3$, gradient changes positive, zero, negative

$\Rightarrow$ max turning pt

$x = -1$ gradient changes negative, zero, positive

$\Rightarrow$ min turning pt

$x = 2$ gradient changes positive, zero, positive

$\Rightarrow$ inflexion

Question 15 **C**

$$\begin{aligned} \int_0^m \sin^2 x dx &= \int_0^m \frac{1}{2} (1 - \cos 2x) dx \\ &= \frac{1}{2} \left[x - \frac{1}{2} \sin 2x \right]_0^m \\ &= \frac{m}{2} - \frac{\sin 2m}{4} \end{aligned}$$

Question 16 **D**

$$f'(x) = \frac{dy}{dx} = \frac{x^2}{x-1}$$

Using $f(x+h) \approx hf'(x) + f(x)$

When $x = 2$, $f(x) = 4$

$$\Rightarrow f'(2) = \frac{dy}{dx} = \frac{2^2}{2-1} = 4, \quad h = 0.1$$

$$f(2+0.1) \approx 0.1f'(2) + f(2)$$

$$f(2.1) \approx 0.1 \times 4 + 4 = 4.4$$

$$f(2.1+0.1) \approx 0.1f'(2.1) + f(2.1) \quad \text{and}$$

$$f'(2.1) = \frac{2.1^2}{2.1-1} = 4.0091$$

$$f'(2.2) \approx 0.1 \times 4.0091 + 4.4 = 4.8009$$

Question 17 **D**

Volume of water in tank after t minutes is $100 + 3t$ litres

Concentration of sugar after t minutes is

$$\frac{Q}{100 + 3t} \text{ kg/litre}$$

Rate of inflow per minute is 8×0.5

$$= 4 \text{ kg/minute}$$

Rate of outflow per minute is

$$5 \times \frac{Q}{100 + 3t} = \frac{5Q}{100 + 3t} \text{ kg/minute}$$

$$\frac{dQ}{dt} = \text{rate of inflow} - \text{rate of outflow}$$

$$= 4 - \frac{5Q}{100 + 3t} \text{ kg/minute}$$

Question 18
A

Vector resolute of $\mathbf{a}$ parallel to $\mathbf{b}$ is given by

$$(\hat{\mathbf{a}} \cdot \hat{\mathbf{b}}) \hat{\mathbf{b}}$$

$$|\mathbf{b}| = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{6}$$

$$\hat{\mathbf{b}} = \frac{1}{\sqrt{6}}(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$$

$$\hat{\mathbf{a}} \cdot \hat{\mathbf{b}} = \frac{1}{\sqrt{6}}(2 \times 1 - 3 \times -2 + 1 \times 1)$$

$$= \frac{9}{\sqrt{6}}$$

$$(\hat{\mathbf{a}} \cdot \hat{\mathbf{b}}) \hat{\mathbf{b}} = \frac{9}{\sqrt{6}} \times \frac{1}{\sqrt{6}}(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$$

$$= \frac{3}{2}(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$$

Question 19
D

$$\vec{OA} = \vec{OB} + \vec{BA}$$

$$= \vec{OB} - \vec{AB}$$

$$= (2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) - (\mathbf{i} - 3\mathbf{j} - 2\mathbf{k})$$

$$= \mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$$

$$|\vec{OA}| = \sqrt{1^2 + 6^2 + (-2)^2} = \sqrt{41}$$

Question 20
B

$$\mathbf{r}'(t) = 2 \sin(3t)\mathbf{i} - \mathbf{j}$$

$$\mathbf{r}(t) = \int (2 \sin(3t)\mathbf{i} - \mathbf{j}) dt$$

$$\mathbf{r}(t) = -\frac{2}{3} \cos(3t)\mathbf{i} - t\mathbf{j} + \mathbf{c}$$

Since when $t = 0$, $\mathbf{r}(t) = -\frac{2}{3}\mathbf{i} + \mathbf{j}$

$$-\frac{2}{3}\mathbf{i} + \mathbf{j} = -\frac{2}{3}\mathbf{i} + \mathbf{c}$$

$$\mathbf{c} = \mathbf{j}$$

Hence $\mathbf{r}(t) = -\frac{2}{3} \cos(3t)\mathbf{i} - t\mathbf{j} + \mathbf{j}$

$$\mathbf{r}(t) = -\frac{2}{3} \cos(3t)\mathbf{i} + (1-t)\mathbf{j}$$

Question 21
D

$$x = \sin t, \quad y = \cos 2t = 1 - 2 \sin^2 t$$

$$y = 1 - 2x^2$$

since $0 \leq t \leq \pi$, $0 \leq \sin t \leq 1$

hence, since $x = \sin t$, $0 \leq x \leq 1$

Question 22
C

$$v = \frac{dx}{dt} = 3.2 - \cos\left(\frac{t}{4}\right)$$

Since $-1 \leq \cos\left(\frac{t}{4}\right) \leq 1$,

$$v_{\max} = 3.2 + 1 = 4.2 \text{ m/s}$$

Question 23
C

$$u = 12 \text{ m/s}, \quad a = -3 \text{ m/s}^2, \quad v = 0 \text{ m/s}$$

$$v^2 - u^2 = 2as$$

$$0 - 12^2 = 2(-3)s$$

$$-144 = -6s$$

$$s = 24 \text{ metres}$$

Question 24 **E**

$$\frac{d\left(\frac{1}{2}v^2\right)}{dx} = 4 - e^{2x}$$

$$\frac{1}{2}v^2 = \int(4 - e^{2x})dx$$

$$\frac{1}{2}v^2 = 4x - \frac{1}{2}e^{2x} + c$$

$x = 0, v = -2$, hence

$$\frac{1}{2}(-2)^2 = 4(0) - \frac{1}{2}e^{2 \times 0} + c$$

$$2 = 0 - \frac{1}{2} + c$$

$$c = \frac{5}{2}$$

Hence $v^2 = 8x - e^{2x} + 5$

$$v = \pm\sqrt{8x - e^{2x} + 5}$$

$$v = -\sqrt{8x - e^{2x} + 5},$$

since $x = 0, v = -2$

Question 25 **D**

Considering upwards as positive:

$u = 18 \text{ m/s}, a = -9.8 \text{ m/s}^2, s = -520 \text{ m}$

$$s = ut + \frac{1}{2}at^2$$

$$-520 = 18t - 4.9t^2$$

$$49t^2 - 180t - 5200 = 0$$

Quadratic formula gives $t = 12.3$ and -8.6

Since $t \geq 0, t = 12.3$ seconds

Alternatively, consider upward and downward motion separately:

Upward motion (with upwards positive)

$u = 18 \text{ m/s}, v = 0, a = -9.8 \text{ m/s}^2$

$$v^2 - u^2 = 2as$$

$$s = \frac{v^2 - u^2}{2a}$$

$$= \frac{-18^2}{-19.6}$$

$= 16.53$ metres (further distance the stone rises)

$$v = u + at$$

$$t_1 = \frac{v - u}{a}$$

$$= \frac{-18}{-9.8}$$

$= 1.84$ seconds to reach maximum height

Downward motion (with downwards positive)

$u = 0 \text{ m/s}, s = 520 + 16.53 = 536.53 \text{ m},$

$a = 9.8 \text{ m/s}^2$

$$s = ut + \frac{1}{2}at^2$$

$$t^2 = \frac{2s}{a} = 109.50$$

$t_2 = 10.46$ seconds to fall from the peak to the ground.

$t_1 + t_2 = 12.3$ seconds

Question 26 **B**

Since body is in equilibrium

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = \mathbf{0}$$

Therefore $\mathbf{F}_3 = -(\mathbf{F}_1 + \mathbf{F}_2)$

$$\begin{aligned} \mathbf{F}_1 + \mathbf{F}_2 &= (2\mathbf{i} + 3\mathbf{j} - \mathbf{k}) + (2\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}) \\ &= 4\mathbf{i} - 3\mathbf{k} \end{aligned}$$

Hence $\mathbf{F}_3 = -4\mathbf{i} + 3\mathbf{k}$

Question 27 **C**

Points A, B and C will be collinear if $\vec{AB}$ and $\vec{BC}$ are parallel and have a point in common, ie B.

Hence $\vec{AB} = k \vec{BC}$

Question 28 **D**

Since lift is accelerating upwards, the resultant force is upwards, hence

$$N - mg = ma \Rightarrow N = ma + mg$$

$$N = 55(3) + 55(9.8) = 704 \text{ Newtons}$$

Question 29 **E**

Assuming the 3kg mass moves down,

$$2 \text{ kg mass: } T - 2g = 2a \quad (\text{I})$$

$$3 \text{ kg mass: } 3g - T = 3a \quad (\text{II})$$

$$(\text{I}) + (\text{II}) \quad g = 5a$$

$$a = \frac{g}{5}$$

From (I), $T = 2a + 2g$

$$= 2\left(\frac{g}{5}\right) + 2g$$

$$= \frac{2g}{5} + \frac{10g}{5}$$

$$= \frac{12g}{5}$$

Question 30 **C**

$$3 \text{ kg mass: } T - 3g \sin 60 = 3a \quad (\text{I})$$

$$5 \text{ kg mass: } 5g - T = 5a \quad (\text{II})$$

Adding equations (I) and (II)

$$5g - 3g \frac{\sqrt{3}}{2} = 8a$$

$$\frac{g(10 - 3\sqrt{3})}{2} = 8a$$

$$a = \frac{g(10 - 3\sqrt{3})}{16}$$

EXAM 1 Part II
Question 1

$$\text{Let } u = 2x + 1 \Rightarrow 2x = u - 1$$

$$\Rightarrow \frac{du}{dx} = 2 \quad x = \frac{u-1}{2}$$

M1

$$\int x\sqrt{2x+1} dx = \frac{1}{2} \int \left(\frac{u-1}{2}\right) u^{\frac{1}{2}} \frac{du}{dx} dx$$

$$= \frac{1}{4} \int (u-1) u^{\frac{1}{2}} du$$

$$= \frac{1}{4} \int u^{\frac{3}{2}} - u^{\frac{1}{2}} du$$

$$= \frac{1}{4} \left(\frac{2}{5} u^{\frac{5}{2}} - \frac{2}{3} u^{\frac{3}{2}} \right) + c$$

c is a constant

M1

$$= \frac{1}{10} u^{\frac{5}{2}} - \frac{1}{6} u^{\frac{3}{2}} + c$$

$$= \frac{1}{10} (2x+1)^{\frac{5}{2}} - \frac{1}{6} (2x+1)^{\frac{3}{2}} + c \quad \text{A1}$$

Question 2

$$y = (\cos x)^{-1}$$

Applying the chain rule

$$\frac{dy}{dx} = -1(\cos x)^{-2}(-\sin x) = \frac{\sin x}{\cos^2 x}$$

A1

$$\text{At } x = \frac{\pi}{4}, \quad \frac{dy}{dx} = \frac{\sin(\frac{\pi}{4})}{\cos^2(\frac{\pi}{4})}$$

$$= \frac{\frac{\sqrt{2}}{2}}{\left(\frac{\sqrt{2}}{2}\right)^2}$$

$$= \frac{\sqrt{2}}{\frac{2}{2}} = \sqrt{2}$$

$$\Rightarrow a = \sqrt{2}$$

A1

Equation of tangent $y = \sqrt{2}x + b$

$$\text{At } \left(\frac{\pi}{4}, \sec \frac{\pi}{4}\right), \quad \sec \frac{\pi}{4} = \sqrt{2} \times \frac{\pi}{4} + b$$

M1

$$b = \sqrt{2} - \sqrt{2} \times \frac{\pi}{4} = \sqrt{2} \left(1 - \frac{\pi}{4}\right)$$

A1

$\therefore$ Equation of tangent is

$$y = \sqrt{2}x + \sqrt{2} \left(1 - \frac{\pi}{4}\right)$$

Question 3

$$\frac{dx}{dy} = \frac{1}{2y-1}$$

$$x = \frac{1}{2} \int \frac{2}{2y-1} dy + c$$

$$x = \frac{1}{2} \log_e(2y-1) + c \quad \text{A1}$$

$$y(0) = 1 \Rightarrow 0 = \frac{1}{2} \log_e(2-1) + c$$

$$\therefore c = 0 \quad \text{A1}$$

$$x = \frac{1}{2} \log_e(2y-1)$$

$$e^{2x} = 2y-1$$

$$y = \frac{1}{2}(e^{2x} + 1) \quad \text{A1}$$

Question 4

a. i. $\vec{OM} = p\vec{OB}$

$$= p(\vec{OB} + \vec{BC}) \quad \text{M1}$$

$$= p(\mathbf{a} + \mathbf{c})$$

ii. $\vec{OM} = \vec{OC} + \vec{CM}$ M1

$$= \vec{OC} + q\vec{CA}$$

$$= \mathbf{c} + q(\mathbf{a} - \mathbf{c}) \quad \text{M1}$$

$$= qa + (1-q)c$$

b. Equating the two equations for $\vec{OM}$

$$p(\mathbf{a} + \mathbf{c}) = qa + (1-q)c$$

$$pa + pc - qa - c + qc = 0$$

$$(p-q)a + (p+q-1)c = 0 \quad \text{M1}$$

Since $\mathbf{a}$ and $\mathbf{c}$ are adjacent sides of a parallelogram, they cannot be parallel.

Hence the above expression can only be true if:

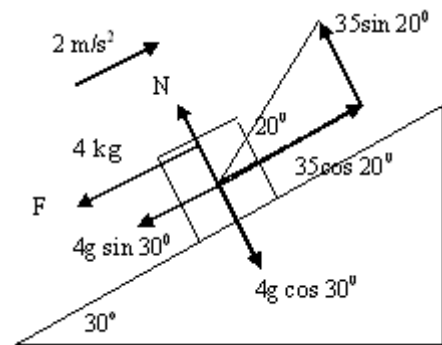
$$p-q=0 \text{ and } p+q-1=0$$

Hence $p=q$ and $2p-1=0$ A1

$$\Rightarrow p=q=\frac{1}{2}$$

Therefore, from $\vec{OM} = p\vec{OB}$, the diagonals of a parallelogram bisect each other.

Question 5



a. From $R = ma$

(force up plane) - (force down plane) = ma

$$35 \cos 20 - (F + 4g \sin 30) = 4(2) \quad \text{M1}$$

$$F = 35 \cos 20 - 4g \sin 30 - 8$$

$$= 5.29 \text{ newtons}$$

Considering forces perpendicular to the plane, where the forces balance:

$$N + 35 \sin 20 = 4g \cos 30$$

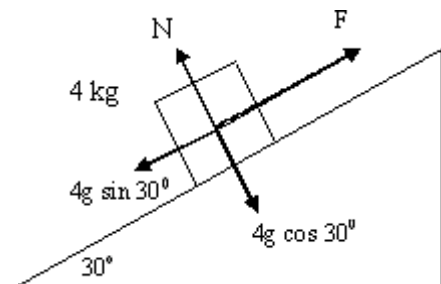
$$N = 4g \cos 30 - 35 \sin 20$$

$$= 21.98$$

$$F = \mu N$$

$$\mu = \frac{F}{N} = \frac{5.29}{21.98} = 0.24 \quad \text{A1}$$

b.



$$N = 4g \cos 30 = 4g \frac{\sqrt{3}}{2}$$

$$F = \mu N, \text{ where from part a. } \mu = 0.24$$

$$F = 0.24 \times 4g \frac{\sqrt{3}}{2} \quad \text{M1}$$

$$4g \sin 30 - F = ma$$

$$4g \sin 30 - 0.24 \times 2g \sqrt{3} = 4a$$

$$\Rightarrow a = 2.86 \text{ m/s}^2 \quad \text{A1}$$