

Year 2005

VCE

Specialist Mathematics

Trial Examination 2

Suggested Solutions

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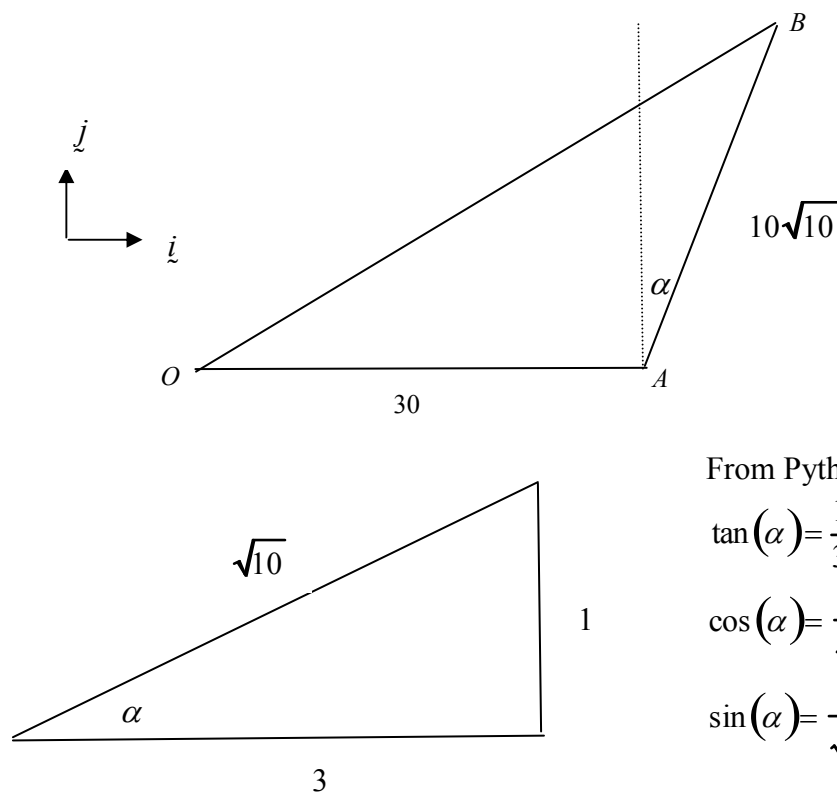
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Question 1

a.

i.



From Pythagorus

$$\tan(\alpha) = \frac{1}{3}$$

$$\cos(\alpha) = \frac{3}{\sqrt{10}}$$

$$\sin(\alpha) = \frac{1}{\sqrt{10}}$$

$$\begin{aligned}\vec{OA} &= 30\vec{i} & \vec{AB} &= 10\sqrt{10}\sin(\alpha)\vec{i} + 10\sqrt{10}\cos(\alpha)\vec{j} = 10\vec{i} + 30\vec{j} \\ \vec{OB} &= \vec{OA} + \vec{AB} = 40\vec{i} + 30\vec{j}\end{aligned}$$

1 mark

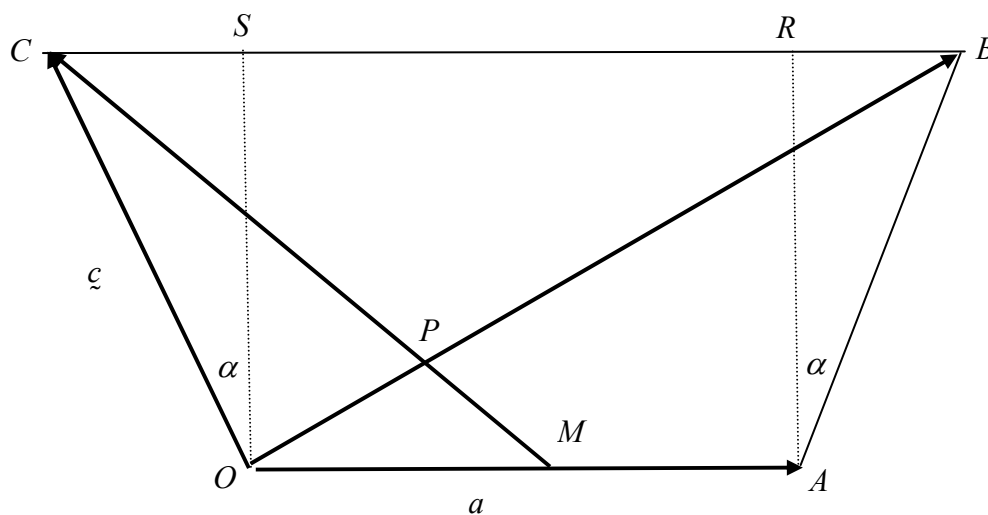
ii. Now $|\vec{OA}| = 30$ and $|\vec{OB}| = \sqrt{40^2 + 30^2} = 50$ and $\vec{OA} \cdot \vec{OB} = 30 \times 40 = 1200$

$$\cos(\theta) = \frac{\vec{OA} \cdot \vec{OB}}{|\vec{OA}| |\vec{OB}|} = \frac{1200}{30 \times 50} = \frac{4}{5}$$

$$\theta = \cos^{-1}\left(\frac{4}{5}\right) = 36^\circ 52'$$

2 marks

b.



i. Now $\overrightarrow{OA} = \overrightarrow{SR} = \underline{a}$ and $\overrightarrow{CS} = \overrightarrow{RB} = \frac{1}{3}\underline{a}$ and $\overrightarrow{OC} = \underline{c}$

$$\overrightarrow{OB} = \overrightarrow{OC} + \overrightarrow{CS} + \overrightarrow{SR} + \overrightarrow{RB}$$

$$\overrightarrow{OB} = \underline{c} + \frac{1}{3}\underline{a} + \underline{a} + \frac{1}{3}\underline{a}$$

$$\overrightarrow{OB} = \frac{1}{3}(5\underline{a} + 3\underline{c})$$

1 mark

ii. Now $\overrightarrow{OM} = \frac{1}{2}\underline{a}$ and $\overrightarrow{MC} = \overrightarrow{MO} + \overrightarrow{OC} = -\frac{1}{2}\underline{a} + \underline{c}$ so

$$\overrightarrow{MP} = \frac{3}{13}\overrightarrow{MC}$$

$$\overrightarrow{MP} = \frac{3}{13}\left(-\frac{1}{2}\underline{a} + \underline{c}\right)$$

$$\overrightarrow{MP} = \frac{3}{26}(2\underline{c} - \underline{a})$$

2 marks

iii. $\overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{MP} = \frac{1}{2}\underline{a} + \frac{3}{26}(2\underline{c} - \underline{a})$

$$\overrightarrow{OP} = \left(\frac{1}{2} - \frac{3}{26}\right)\underline{a} + \frac{3}{13}\underline{c}$$

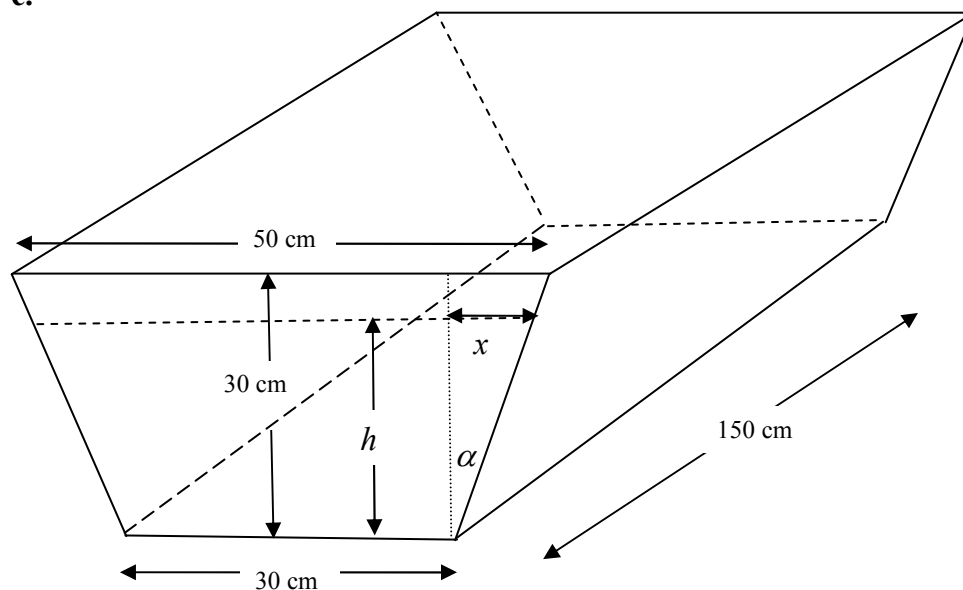
$$\overrightarrow{OP} = \frac{5}{13}\underline{a} + \frac{3}{13}\underline{c} = \frac{1}{13}(5\underline{a} + 3\underline{c})$$

$$\overrightarrow{OP} = \frac{3}{13}\overrightarrow{OB}$$

3 marks

Hence the points O, P, B are collinear and $OP : OB = \frac{3}{13}$

c.



i. From the diagram, the cross-section and by similar triangles

$$\tan(\alpha) = \frac{x}{h} = \frac{1}{3} \quad \text{so that} \quad x = \frac{h}{3}$$

The cross-sectional area (the area of the trapezium)

$$A = \frac{1}{2} (30 + (2x + 30))h = (30 + x)h$$

$$A = \left(30 + \frac{h}{3} \right)h \quad \text{the volume at height } h \text{ is given by}$$

$$V = Ah = \frac{1}{3}(90 + h)150h = 50h(90 + h) = 50(90h + h^2)$$

$$\text{Now } \frac{dV}{dh} = 50(90 + 2h) \quad \text{and we are given that } \frac{dV}{dt} = -80\sqrt{h}$$

$$\text{By the Chain rule } \frac{dh}{dt} = \frac{dh}{dV} \cdot \frac{dV}{dt} = \frac{-80\sqrt{h}}{50(90 + 2h)} = \frac{-4\sqrt{h}}{5(45 + h)}$$

2 marks

ii. Inverting gives $\frac{dt}{dh} = \frac{5(45+h)}{-4\sqrt{h}} = -\frac{5}{4}\left(45h^{-\frac{1}{2}} + h^{\frac{1}{2}}\right)$ integrating wrt h

$$t = -\frac{5}{4} \int_0^{25} \left(45h^{-\frac{1}{2}} + h^{\frac{1}{2}}\right) dh$$

$$t = -\frac{5}{4} \left[90h^{\frac{1}{2}} + \frac{2}{3}h^{\frac{3}{2}} \right]_0^{25} = -\frac{5}{4} \left[90\sqrt{25} + \frac{2}{3}(25)^{\frac{3}{2}} - 0 \right]$$

$$t = 666\frac{2}{3} \text{ minutes}$$

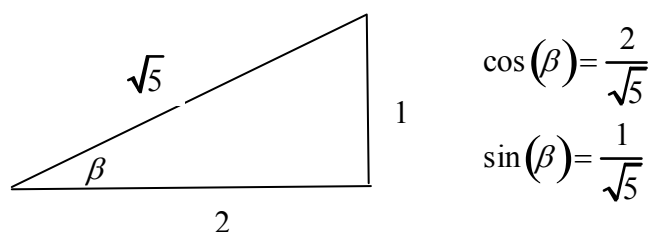
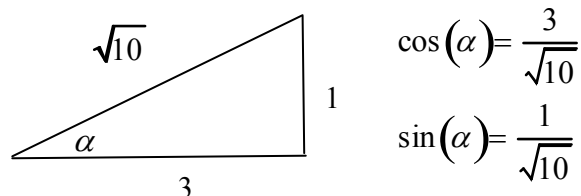
$$t = 11\frac{1}{9} \text{ hours}$$

3 marks

Question 2

Let $\alpha = \tan^{-1}\left(\frac{1}{3}\right)$ $\beta = \tan^{-1}\left(\frac{1}{2}\right)$ $u = 3+i$ and $v = 2+i$

From Pythagorus Theorem



a. $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$

$$\cos(\alpha + \beta) = \frac{3}{\sqrt{10}} \cdot \frac{2}{\sqrt{5}} - \frac{1}{\sqrt{10}} \cdot \frac{1}{\sqrt{5}} = \frac{6-1}{\sqrt{50}} = \frac{5}{\sqrt{25 \cdot 2}} = \frac{\sqrt{2}}{2}$$

2 marks

b. $v = 2+i$

$$v^2 = (2+i)^2 = 4 + 4i + i^2 = 3 + 4i$$

$$\operatorname{Im}(v^2) = 4 \quad |v^2| = \sqrt{9+16} = \sqrt{25} = 5$$

$$\sin(2\beta) = \frac{\operatorname{Im}(v^2)}{|v^2|} = \frac{4}{5}$$

$$\sin(2\beta) = 2\sin(\beta)\cos(\beta) = 2 \times \frac{1}{\sqrt{5}} \times \frac{2}{\sqrt{5}} = \frac{4}{5}$$

2 marks

c. $uv = (3+i)(2+i) = 6 + 2i + 3i + i^2 = 5 + 5i$

$$\operatorname{Arg}(uv) = \tan^{-1}\left(\frac{5}{5}\right) = \tan^{-1}(1) = \frac{\pi}{4} \quad \text{and}$$

$$\operatorname{Arg}(uv) = \operatorname{Arg}(u) + \operatorname{Arg}(v) = \alpha + \beta \quad \text{so} \quad \alpha + \beta = \frac{\pi}{4}$$

2 marks

d. $u = 3+i$ and $\bar{u} = 3-i$ $u + \bar{u} = 6$ $u\bar{u} = 9 - i^2 = 10$

Since a and b are real numbers by the conjugate root theorem

$$z^2 - 6z + 10 \text{ is a factor}$$

$$z^3 + az^2 + bz + 20 = 0$$

$$(z^2 - 6z + 10)(z + 2)$$

$$\text{expanding gives coefficient of } z^2: a = 2 - 6 = -4 \quad z: b = 10 - 12 = -2$$

$$a = -4 \quad b = -2 \quad \text{the roots are } z = 3 \pm i \text{ and } z = -2$$

2 marks

e. $Q(z) = z^3 - (2+i)z^2 + 5z - 10 - 5i = 0$

$$Q(2+i) = (2+i)^3 - (2+i)(2+i)^2 + 5(2+i) - 10 - 5i$$

$$Q(2+i) = (2+i)^3 - (2+i)^3 + 10 + 5i - 10 - 5i = 0 \quad \text{shown}$$

So $z = 2+i$ is a root $(z - 2 - i)$ is a factor

$$Q(z) = z^3 - (2+i)z^2 + 5z - 10 - 5i = 0$$

$$Q(z) = (z - 2 - i)(z^2 + 5) = (z - 2 - i)(z + \sqrt{5}i)(z - \sqrt{5}i) = 0$$

$$\text{the roots are } z = 2+i \text{ and } z = \pm\sqrt{5}i$$

2 marks

Question 3

a. $u = 10$, $v = 5$ and $a = -2.5$ $t = ?$ $s = ?$

using constant acceleration formulae

$$s = \left(\frac{u + v}{2} \right) t$$

$$s = \left(\frac{5 + 10}{2} \right) 2$$

$$s = 15 \text{ metres}$$

$$v = u + at$$

$$5 = 10 - 2.5t$$

$$2.5t = 5$$

$$t = 2 \text{ sec}$$

2 marks

b.i. from Newton's Law $800\ddot{x} = -40v^2$

$$\ddot{x} = v \frac{dv}{dx} = -\frac{v^2}{20}$$

$$\frac{dv}{dx} = -\frac{v}{20}$$

1 mark

ii. $\int \frac{dv}{v} = -\frac{1}{20} \int dx$

$$\log_e(v) = -\frac{x}{20} + C_1$$

to find C_1 use $v = 10$ when $x = 0$

$$C_1 = \log_e(10)$$

$$\log_e(v) = -\frac{x}{20} + \log_e(10)$$

$$\log_e(v) - \log_e(10) = -\frac{x}{20}$$

$$\log_e\left(\frac{v}{10}\right) = -\frac{x}{20}$$

$$x = -20 \log_e\left(\frac{v}{10}\right) \quad \text{Now when } v = 5$$

$$x = -20 \log_e\left(\frac{5}{10}\right) = 20 \log_e(2)$$

3 marks

iii. $\ddot{x} = \frac{dv}{dt} = -\frac{v^2}{20}$

$$\int \frac{dv}{v^2} = -\frac{1}{20} \int dt = \frac{-t}{20} + C_2$$

$$-\frac{1}{v} = \frac{-t}{20} + C_2$$

to find C_2 use $v = 10$ when $t = 0$

$$C_2 = -\frac{1}{10}$$

$$-\frac{1}{v} = -\frac{t}{20} - \frac{1}{10} = \frac{-(t+2)}{20}$$

Now when $v = 5$

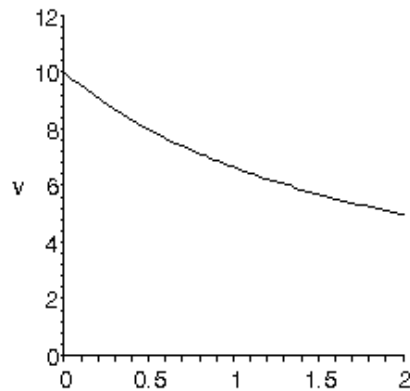
$$5 = \frac{20}{t+2}$$

$$t+2 = \frac{20}{5} = 4$$

$$t = 2 \text{ sec}$$

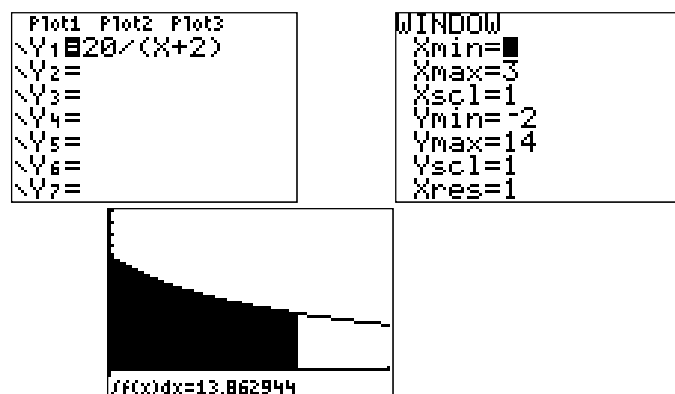
3 marks

iv. $v = v(t) = \frac{20}{t+2}$ for $0 \leq t \leq 2$



As a check $x = 20 \log_e(2) \approx 13.863$

2 marks



Question 4

a. $\vec{r}(t) = 2 \cot(t)\vec{i} + (1 - \cos(2t))\vec{j}$ for $t \geq 0$

$x(t) = 2 \cot(t)$ to eliminate t

$y(t) = 1 - \cos(2t) = 1 - (1 - 2 \sin^2(t)) = 2 \sin^2(t)$

$x^2 + 4 = 4 \cot^2(t) + 4 = 4 \operatorname{cosec}^2(t) = \frac{4}{\sin^2(t)} = \frac{4}{\frac{1}{2}y}$

so $y = \frac{8}{x^2 + 4}$

2 marks

b i. $f(x) = \frac{8}{x^2 + 4} = 8(x^2 + 4)^{-1}$ for stationary points

$f'(x) = -\frac{16x}{(x^2 + 4)^2} = 0$ when $x = 0$ so $f'(0) = 0$ $f(0) = 2$

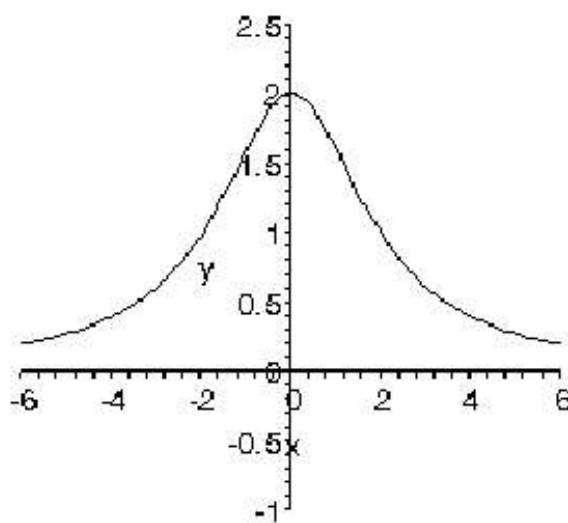
$f''(x) = -\frac{16(x^2 + 4)^2 - 4x(x^2 + 4)16x}{(x^2 + 4)^4} = \frac{16(3x^2 - 4)}{(x^2 + 4)^3} = 0$ $f''(0) = -1 < 0$

When $3x^2 - 4 = 0$ $x = \pm \frac{2}{\sqrt{3}} = \pm \frac{2\sqrt{3}}{3}$ $f\left(\frac{2\sqrt{3}}{3}\right) = \frac{3}{2}$ $f\left(-\frac{2\sqrt{3}}{3}\right) = \frac{3}{2}$

$(0, 2)$ is a maximum $\left(\frac{2\sqrt{3}}{3}, \frac{3}{2}\right)$ and $\left(-\frac{2\sqrt{3}}{3}, \frac{3}{2}\right)$ are inflexion points

4 marks

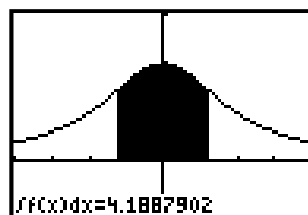
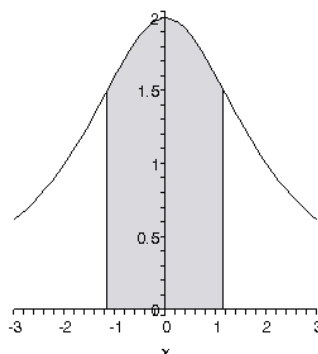
ii.



1 mark

iii. The area of the door way is

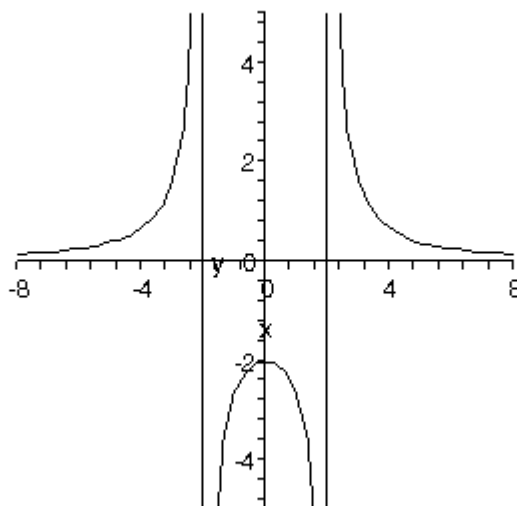
$$\begin{aligned}
 \int_{-\frac{2\sqrt{3}}{3}}^{\frac{2\sqrt{3}}{3}} \frac{8}{x^2 + 4} dx &= 2 \int_0^{\frac{2\sqrt{3}}{3}} \frac{8}{x^2 + 4} dx \\
 &= \left[8 \tan^{-1}\left(\frac{x}{2}\right) \right]_0^{\frac{2\sqrt{3}}{3}} \\
 &= 8 \tan^{-1}\left(\frac{\sqrt{3}}{3}\right) - 8 \tan^{-1}(0) \\
 &= \frac{4\pi}{3}
 \end{aligned}$$



As a check on the TI-83

3 marks

- c. i. the domain is $R \setminus \{-2, 2\}$ and the graph has a maximum turning point at $(0, -2)$ range is $(-\infty, -2] \cup (0, \infty)$ the graph has vertical asymptotes at $x = \pm 2$ and a horizontal asymptote at $y = 0$ (the x -axis)



2 marks

ii. The volume formed is given by $V = \int_6^8 [g(x)]^2 dx$

by partial fractions

$$g(x) = \frac{8}{x^2 - 4} = \frac{A}{x+2} + \frac{B}{x-2} = \frac{A(x-2) + B(x+2)}{x^2 - 4} = \frac{x(A+B) + 2(B-A)}{x^2 - 4}$$

$$A+B=0 \text{ and } B-A=4 \text{ so that } A=-2 \text{ and } B=2$$

$$g(x) = \frac{8}{x^2 - 4} = \frac{2}{x-2} - \frac{2}{x+2}$$

$$V = \pi \int_6^8 \left(\frac{4}{(x-2)^2} - \frac{8}{(x-2)(x+2)} + \frac{4}{(x+2)^2} \right) dx$$

$$V = \pi \int_6^8 \left(\frac{4}{(x-2)^2} - \frac{2}{x-2} + \frac{2}{x+2} + \frac{4}{(x+2)^2} \right) dx$$

$$V = \pi \left[\frac{-4}{x-2} + 2 \log_e \left(\frac{x+2}{x-2} \right) - \frac{4}{x+2} \right]_6^8$$

$$V = \pi \left[-\frac{4}{6} + 2 \log_e \left(\frac{10}{6} \right) - \frac{4}{10} + \frac{4}{4} - 2 \log_e \left(\frac{8}{4} \right) + \frac{4}{8} \right]$$

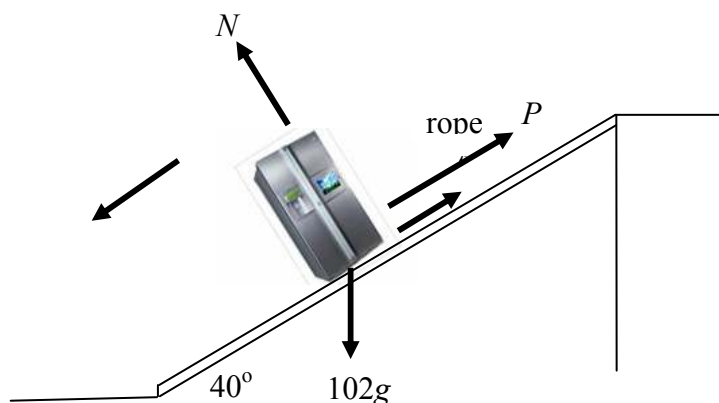
$$V = \pi \left(\frac{13}{30} + 2 \log_e \left(\frac{5}{6} \right) \right)$$

4 marks

Question 5

a.

i.



1 mark

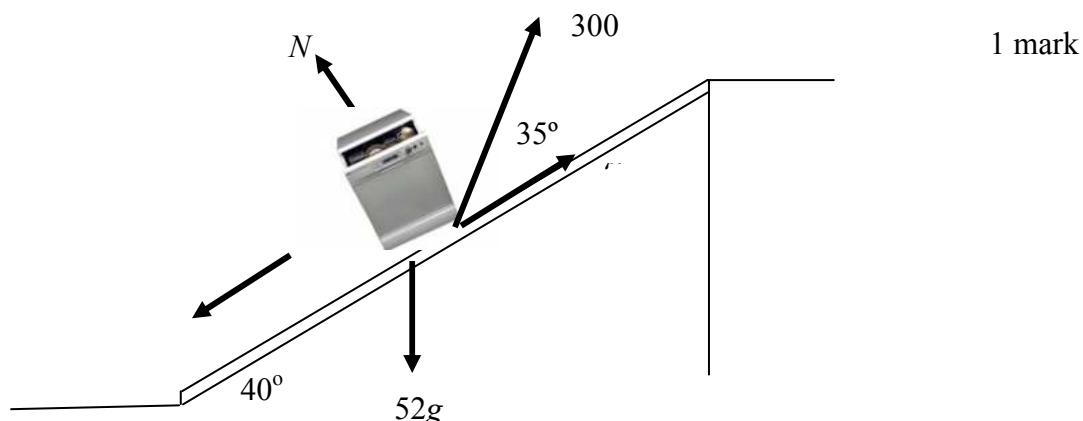
ii. Given that $\mu = 0.25$ find P

The frictional force is up, since the motion is down

Resolving the forces, using Newtons 2nd law of motionperpendicular to the plane (1) $N - 102g \cos(40^\circ) = 0$ parallel to the plane (2) $102g \sin(40^\circ) - \mu N - P = 0$ from (1) $N = 102g \cos(40^\circ)$ from (2) $P = 102g \sin(40^\circ) - \mu N = 102g \sin(40^\circ) - 0.25 \times 102g \cos(40^\circ)$ $P = 102g (\sin(40^\circ) - 0.25 \cos(40^\circ)) = 451.10$ newtons

3 marks

b.
i.



- ii. The frictional force is up, since the motion is down
Resolving the forces, using Newtons 2nd law of motion
Perpendicular to the plane (1) $N + 300 \sin(35^\circ) - 52g \cos(40^\circ) = 0$
parallel to the plane (2) $52g \sin(40^\circ) - \mu N - 300 \cos(35^\circ) = 52 \times 0.5$
from (1) $N = 52g \cos(40^\circ) - 300 \sin(35^\circ) = 218.303$
from (2) $\mu N = 52g \sin(40^\circ) - 300 \cos(35^\circ) - 52 \times 0.5 = 55.819$ so that
$$\mu = \frac{55.819}{218.303} = 0.256$$

4 marks

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