

2004 Mathematical Methods, Specialist Examination 2

Question 1

(a) (i) $v = t^2(3 - t)$

$$= 3t^2 - t^3$$

$$a = \frac{dv}{dt}$$

$$= 6t - 3t^2$$

(ii) maximum acceleration when $\frac{da}{dt} = 0$

$$\frac{da}{dt} = 6 - 6t = 0$$

$$\therefore t = 1$$

so maximum acceleration $= 6 \times 1 - 3 \times 1^2$
 $= 3 \text{ ms}^{-2}$

(b) Total distance = area under the curve

$$41 = 2 \int_0^2 3t^2 - t^3 dt = (T - 4) 4$$

$$= 2 \left[t^3 - \frac{t^4}{4} \right]_0^2 + 4T - 16$$

$$= 2(8 - 4) + 4T - 16$$

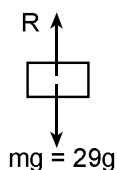
$$= 8 + 4T - 16$$

$$= 4T - 8$$

$$\text{so } T = \frac{49}{4}$$

$$= 12.25 \text{ s}$$

(c)



$$\sum \tilde{F} = m\tilde{a}$$

$$R - 29g = 29a$$

$$R = 29a + 29g$$

$$= 29(a + g)$$

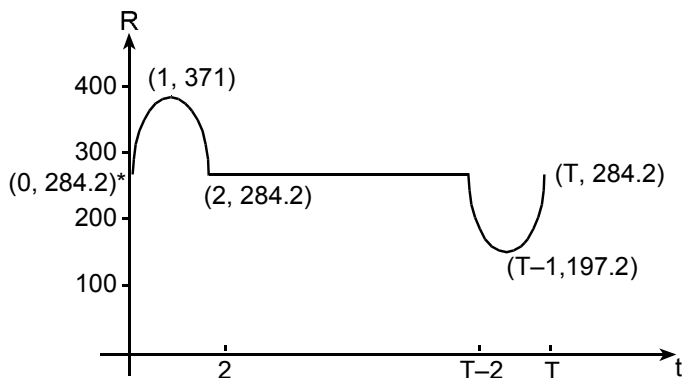
(d) maximum R when a is a maximum, i.e. $a = 3 \text{ ms}^{-2}$

$$\text{so } R = 29(3 + 9.8)$$

$$= 371.2$$

$$\text{so } R \approx 371 \text{ N (to nearest integer)}$$

(e)



(f) At $t = 1$, $|R_B| = |R_G| = 371.2$

(i)

Equation for velocity of boy $v_B = t^2(t - 3)$

$$\therefore a_B = 3t^2 - 6t$$

$$\text{At } t = 1, a_B = -3 \text{ ms}^{-2}$$

$$R_B - m_B g = m a_B$$

$$R_B = m_B(a + g) = 371.2$$

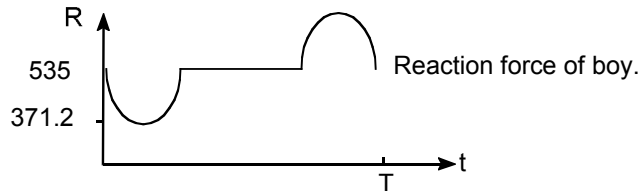
$$\therefore m_B(-3 + 9.8) = 371.2$$

$$m_B = \frac{371.2}{6.8} = 54.6 \text{ kg}$$

so mass of boy is 54.6 kg

(ii) No

The boy's reaction force will always be greater than 371.2 and the girl's reaction force will have reached a maximum of 371.2



Question 2

(a) At P , $t = 0$

$$\begin{aligned} \underline{r}(0) &= (2 \sin(0))\underline{i} + (2 + 0 - \frac{5}{3}\sin(0))\underline{j} \\ &= 0\underline{i} + 2\underline{j} \end{aligned}$$

so P is at $(0, 2)$

(b) $\underline{v}(t) = \dot{\underline{r}}(t)$

$$= \frac{4}{15} \cos\left(\frac{2t}{15}\right)\underline{i} + \left(\frac{5}{3} - \frac{5}{9} \cos\left(\frac{1}{3}t\right)\right)\underline{j}, 0 \leq t \leq \frac{15}{2}$$

(c) let the angle be given by θ

$$\begin{aligned} \tan \theta &= \frac{(\frac{5}{3} - \frac{5}{9} \cos(\frac{1}{3} \cdot 0))}{\frac{4}{15} \cos(\frac{2}{15} \cdot 0)} \\ &= \frac{\frac{10}{9}}{\frac{4}{15}} \\ &= 4.1667 \end{aligned}$$

$$\text{So } \theta = 76.5^\circ$$

(d) "swings" i.e., $\frac{dx}{dt} = 0$ (turning point in i direction, max i)

$$\frac{dx}{dt} = \frac{4}{15} \cos\left(\frac{2}{15}t\right) = 0$$

$$\cos\left(\frac{2}{15}t\right) = 0$$

$$\frac{2}{15}t = \frac{\pi}{2}$$

$$t = \frac{15\pi}{4}$$

$$= 11.78 \text{ s}$$

ie. $t = 11.8 \text{ s}$ (to nearest tenth sec)

(e) At J, $x = 1$ so $2 \sin(\frac{2}{15}t) = 1$

$$\sin(\frac{2}{15}t) = \frac{1}{2}$$

$$\frac{2}{15}t = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$t = \frac{5\pi}{4}, \frac{25\pi}{4}$$

Substituting into $y = 2 + \frac{5}{3}t - \frac{5}{3} \sin(\frac{1}{3}t)$

If $t = \frac{5\pi}{4}$, $y = 2 + \frac{25\pi}{12} - \frac{5}{3} \sin(\frac{5\pi}{12})$
 ≈ 6.935 which occurs before ball swings

If $t = \frac{25\pi}{4}$, $y = 2 + \frac{25\pi}{12} - \frac{5}{3} \sin(\frac{25\pi}{12})$
 $= 34.3$

So ball does not pass through J.

Question 3

(a) (i) $v = 8,000 \times 1 \times \tan^{-1}(1)$

$$= 8,000 \times \frac{\pi}{4}$$

$$= 2,000 \pi \text{ litres}$$

(ii) $10\,000 = 8000h \tan^{-1}(h)$

$$\frac{10}{8} - h \tan^{-1}(h) = 0$$

$$h = 1.3429933 \text{ (using calculator)}$$

$$\text{so } h = 1343 \text{ mm}$$

(b)

$$\frac{dh}{dt} = \frac{dh}{dv} \cdot \frac{dv}{dt}$$

$$\frac{dv}{dh} = 8000 \tan^{-1}(h) + 8000h \times \left(\frac{1}{1+h^2}\right)$$

$$= 8000 \left(\tan^{-1}(h) + \frac{h}{1+h^2} \right)$$

$$\text{so } \frac{dh}{dv} = \frac{1}{8000 \left(\tan^{-1}(h) + \frac{h}{1+h^2} \right)}$$

$$\frac{dh}{dt} = \frac{1}{8000 \left(\tan^{-1}(h) + \frac{h}{1+h^2} \right)} \times 2000$$

$$= \frac{1}{4 \left(\tan^{-1}(h) + \frac{h}{1+h^2} \right)}$$

$$= \frac{1+h^2}{4 \left((1+h^2) \tan^{-1}(h) + h \right)}$$

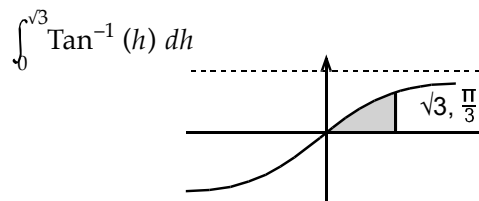
$$(i) \frac{dt}{dh} = \frac{4 \left((1+h^2) \tan^{-1}(h) + h \right)}{1+h^2}$$

$$= 4 \left(\tan^{-1}(h) + \frac{h}{1+h^2} \right)$$

$$\text{so } t = \int 4 \left(\tan^{-1}(h) + \frac{h}{1+h^2} \right) dh$$

$$\text{so } t = 4 \int_0^{\sqrt{3}} \tan^{-1}(h) + \frac{h}{1+h^2} dh$$

$$(ii) \quad t = 4 \left(\int_0^{\sqrt{3}} \tan^{-1}(h) \, dh + \int_0^{\sqrt{3}} \frac{h}{1+h^2} \, dh \right)$$



$$= \sqrt{3} \times \frac{\pi}{3} - \int_0^{\frac{\pi}{3}} \tan^{-1}(h) \, dh$$

$$= \frac{\pi}{\sqrt{3}} - \int_0^{\frac{\pi}{3}} \frac{\sin(h)}{\cos(h)} \, dh$$

let $u = \cos(h)$ so $du = -\sin(h) \, dh$

$$= \frac{\pi}{\sqrt{3}} + \int_0^{0.5} \frac{1}{u} \, du$$

$$= \frac{\pi}{\sqrt{3}} + [\log_e u]^{0.5},$$

$$= \frac{\pi}{\sqrt{3}} + \log_e 0.5$$

$$\int_0^{\sqrt{3}} \frac{h}{1+h^2} \, dh$$

$$\text{let } w = 1 + h^2$$

$$dw = 2h \, dh$$

$$0.5dw = h \, dh$$

$$= \frac{1}{2} \int_1^4 \frac{1}{w} \, dw$$

$$= \left[\frac{1}{2} \log_e w \right]_1^4$$

$$= \frac{1}{2} \log_e 4$$

$$= \log_e 2$$

$$\text{So } t = 4 \left(\frac{\pi}{\sqrt{3}} + \log_e \frac{1}{2} + \log_e 2 \right)$$

$$= 4 \left(\frac{\pi}{\sqrt{3}} + \log_e 1 \right)$$

$$= \frac{4\pi}{\sqrt{3}} \text{ minutes}$$

Question 4

(a) (i) Since a, b, c are real numbers the complex conjugate theorem applies so the third root is $-1 + 2i$

$$(ii) \quad z^3 + az^2 + bz + c = (z - 4)(z + 1 + 2i)(z + 1 - 2i)$$

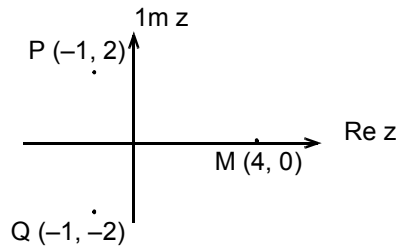
$$= (z - 4)((z + 1)^2 - 4i^2)$$

$$= (z - 4)(z^2 + 2z + 5)$$

$$= z^3 - 2z^2 - 3z - 20$$

$$\text{So } a = -2, b = -3, c = -20$$

(b)



(c) $\vec{MQ} = \vec{MO} + \vec{OQ}$

$$= -\vec{OM} + \vec{OQ}$$

$$= -(4\tilde{i} + 0\tilde{j}) + (-\tilde{i} - 2\tilde{j})$$

$$= -5\tilde{i} - 2\tilde{j}$$

(d) (i) $\vec{DQ} = \vec{DO} + \vec{OQ}$

$$= (-d\tilde{i}) + (-\tilde{i} - 2\tilde{j})$$

$$= (-1-d)\tilde{i} - 2\tilde{j}$$

(ii) Angle DQM = 90° since it is an angle in a semi circle

$$\therefore \vec{DQ} \cdot \vec{QM} = 0$$

$$((-1-d)\tilde{i} - 2\tilde{j}) \cdot (5\tilde{i} + 2\tilde{j}) = 0$$

$$-5 - 5d - 4 = 0$$

$$d = -\frac{9}{5}$$

(e) Q is at (-1, -2) so $-1 - 2i$

$$\text{let } w = a + ib$$

$$(a + ib)(-1 - 2i) + (a - ib)(-1 + 2i) = 0 + 0i$$

$$-a + 2b - ib - 2ai - a + 2b + ib + 2ai = 0 + 0i$$

$$(-2a + 4b) + 0i + 0 + 0i$$

$$\text{so } a = 2b$$

A possible w is $2 + i$

(f) T is a circle with centre at (b, k) and radius r

$$\text{i.e. } (x - h)^2 + (y - k)^2 = r^2$$

$$r = |OC| = \sqrt{h^2 + k^2}$$

$$\text{and } r = |QC| = \sqrt{(h + 1)^2 + (k + 2)^2}$$

$$\text{so } h^2 + k^2 = (h + 1)^2 + (k + 2)^2$$

$$2h + 4k + 5 = 0$$

$$\text{let } h = 1 \therefore k = \frac{-7}{4} \text{ and } r = \sqrt{1 + \frac{49}{16}} = \frac{\sqrt{65}}{4}$$

So a possible T is $\{ (x,y) : (x-1)^2 + (y + \frac{7}{4})^2 \leq \frac{65}{16} \}$. Others are possible.

Question 5

$$(a) \quad 1 - x^2 \neq 0 \quad \therefore x \neq \pm 1$$

$$1 - x^2 > 0 \quad \text{so } x^2 < 1 \\ \text{so domain } f = [0,1)$$

$$(b) \quad x = 0.5$$

$$\therefore f(0.5) = 2(0.5)^{0.5} (1 - 0.5^2)^{0.25} + \frac{1}{(1 - 0.5^2)^{0.25}}$$

$$\text{i.e. } D = 2.39$$

$$(c) \quad f'(x) = 2x^{0.5} - \frac{1}{4}(1 - x^2)^{-0.75}, -2x + x^{-0.5}(1 - x^2)^{0.25} + \frac{1}{4}(1 - x^2)^{-1.25} \bullet -2x$$

$$= -x^{1.5}(1 - x^2)^{-0.75} + x^{-0.5}(1 - x^2)^{0.25} + \frac{x}{2}(1 - x^2)^{-1.25}$$

$$f'(0.5) = 1.23557$$

$$\tan \theta = 1.23557$$

$$\text{so } \theta = 51.01^\circ$$

$$\text{so angle slope makes with surface} = (90 - \theta)^\circ \\ = 39^\circ \text{ (to nearest degree)}$$

$$(d) \quad (i) \quad (f(x))^2 = \left(2x^{0.5}(1 - x^2)^{0.25} + \frac{1}{(1 - x^2)^{0.25}} \right)^2 \\ = \left(2x^{0.5}(1 - x^2)^{0.25} \right)^2 + 2 \left(2x^{0.5}(1 - x^2)^{0.25} \right) \left(\frac{1}{(1 - x^2)^{0.25}} \right) + \left(\frac{1}{(1 - x^2)^{0.25}} \right)^2 \\ = 4x\sqrt{1 - x^2} + 4\sqrt{x} + \frac{1}{\sqrt{1 - x^2}}$$

So third term is $4\sqrt{x}$

$$(ii) \quad V = \pi \int y^2 dx$$

$$= \pi \int_0^{0.5} 4x\sqrt{1 - x^2} + 4\sqrt{x} + \frac{1}{\sqrt{1 - x^2}} dx$$

$$\int 4x\sqrt{1 - x^2} dx$$

$$\text{let } u = 1 - x^2$$

$$\frac{du}{dx} = -2x$$

$$-2 du = 4x dx$$

$$= -2 \int \sqrt{u} \, du$$

$$= -2 \cdot \frac{2}{3} u^{1.5}$$

$$= -\frac{4}{3} (1 - x^2)^{1.5}$$

$$\int 4 \sqrt{x} \, dx$$

$$= 4 \cdot \frac{2}{3} x^{1.5}$$

$$= \frac{8}{3} x^{1.5}$$

$$\int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1}(x)$$

$$\text{So } V = \pi \left[-\frac{4}{3} (1 - x^2)^{1.5} + \frac{8}{3} x^{1.5} + \sin^{-1}(x) \right]_0^{0.5}$$

$$= \pi (1.9337)$$

$$= 6.07 \, \text{m}^2$$

$$= 6.1 \, \text{m}^3 \text{ (to two significant figures)}$$