

2004 Specialist Mathematics Examination 1

Suggested Solutions

Question 1 [E]

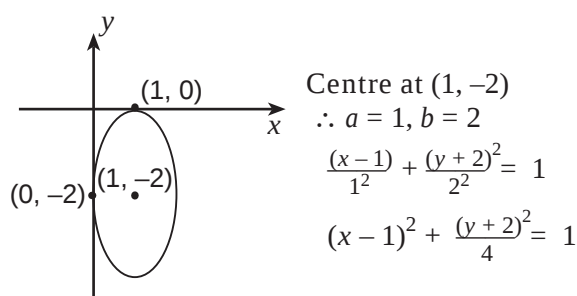
$$\begin{aligned} y &= \frac{-x^2 + 1}{2x} \\ &= \frac{-x^2}{2x} + \frac{1}{2x} \\ &= \frac{-x}{2} + \frac{1}{2x} \end{aligned}$$

Vertical asymptote at $x = 0$

As $x \rightarrow \pm \infty$, $\frac{1}{2x} \rightarrow 0$, $\therefore y \rightarrow \frac{-x}{2}$

So asymptote at $y = \frac{-x}{2}$

Question 2 [C]



Question 3 [E]

$$(A) \frac{\sin\left(\frac{\pi}{5}\right)}{\cos\left(\frac{\pi}{5}\right)} = \tan\left(\frac{\pi}{5}\right)$$

$$\begin{aligned} (B) \frac{1}{\cot\left(\frac{\pi}{5}\right)} &= \frac{1}{\frac{\cos\left(\frac{\pi}{5}\right)}{\sin\left(\frac{\pi}{5}\right)}} \\ &= \frac{\sin\left(\frac{\pi}{5}\right)}{\cos\left(\frac{\pi}{5}\right)} \\ &= \tan\left(\frac{\pi}{5}\right) \end{aligned}$$

$$\begin{aligned} (C) \cot\left(\frac{3\pi}{10}\right) &= \cot\left(\frac{\pi}{2} - \frac{\pi}{5}\right) \\ &= \frac{\cos\left(\frac{\pi}{2} - \frac{\pi}{5}\right)}{\sin\left(\frac{\pi}{2} - \frac{\pi}{5}\right)} \\ &= \frac{\cos\frac{\pi}{2} \cos\frac{\pi}{5} + \sin\frac{\pi}{2} \sin\frac{\pi}{5}}{\sin\frac{\pi}{2} \sin\frac{\pi}{5} - \cos\frac{\pi}{2} \cos\frac{\pi}{5}} \\ &= \frac{\sin\frac{\pi}{5}}{\cos\frac{\pi}{5}} \\ &= \tan\frac{\pi}{5} \end{aligned}$$

$$\begin{aligned}
 \text{(D)} \quad \frac{2 \tan\left(\frac{\pi}{10}\right)}{1 - \tan^2\left(\frac{\pi}{10}\right)} &= \tan\left(2 \times \frac{\pi}{10}\right) \\
 &= \tan \frac{\pi}{5} \\
 &= \tan\left(2\left(\frac{\pi}{5}\right)\right)
 \end{aligned}$$

$$\text{(E)} \quad 2 \tan\left(\frac{2\pi}{5}\right)$$

$$= \tan\left(\frac{4\pi}{5}\right) = -\tan\left(\frac{\pi}{5}\right) \text{ which is incorrect.}$$

Question 4 [D]

Period of graph = $\pi \therefore a = 2$

Horizontal shift of $\frac{\pi}{4}$ parallel to x axis $\therefore b = \frac{\pi}{4}$

Question 5 [E]

$$\begin{aligned}
 \text{(A)} \quad \frac{1}{z} &= \frac{1}{x - iy} \\
 &= \frac{x + iy}{x^2 + y^2} \quad \text{which is not real.}
 \end{aligned}$$

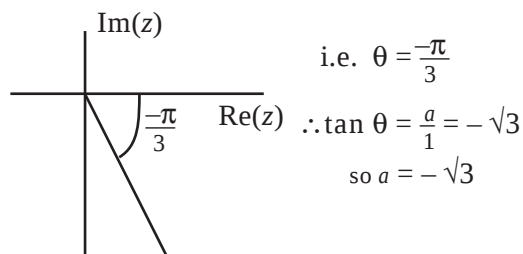
$$\begin{aligned}
 \text{(B)} \quad \frac{1}{\bar{z}} &= \frac{1}{x + iy} \\
 &= \frac{x - iy}{x^2 + y^2} \quad \text{which is not real.}
 \end{aligned}$$

$$\begin{aligned}
 \text{(C)} \quad \frac{1}{z - \bar{z}} &= \frac{1}{(x + iy) - (x - iy)} \\
 &= \frac{1}{2iy} \\
 &= \frac{-1}{2y} i \quad \text{which is not real.}
 \end{aligned}$$

$$\begin{aligned}
 \text{(D)} \quad \frac{1}{z} - \frac{1}{\bar{z}} &= \frac{1}{x + iy} - \frac{1}{x - iy} \\
 &= \frac{x - iy}{x^2 + y^2} - \frac{x + iy}{x^2 + y^2} \\
 &= \frac{-2yi}{x^2 + y^2} \quad \text{which is not real.}
 \end{aligned}$$

$$\begin{aligned}
 \text{(E)} \quad \frac{1}{z} + \frac{1}{\bar{z}} &= \frac{1}{x + iy} + \frac{1}{x - iy} \\
 &= \frac{x - iy}{x^2 + y^2} + \frac{x + iy}{x^2 + y^2} \\
 &= \frac{2x}{x^2 + y^2} \quad \text{which is real.}
 \end{aligned}$$

Question 6 [C]



Question 7 [B]

If $p(z)$ is of degree 4 and has real coefficients, then the Complex Conjugate Rule applies. So $p(z)$ must either have one pair of complex conjugate roots and two real roots or two pairs of complex conjugate roots or four real roots. So $p(z)$ cannot have an *odd* number of non-real roots.

Question 8 [A]

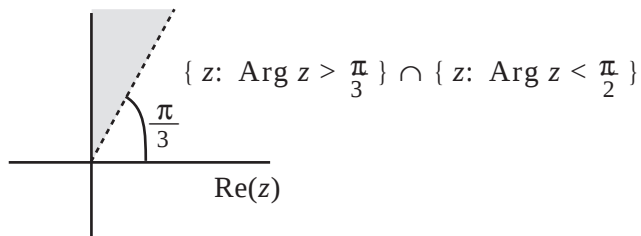
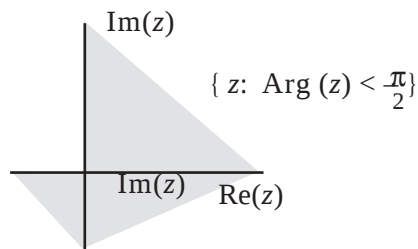
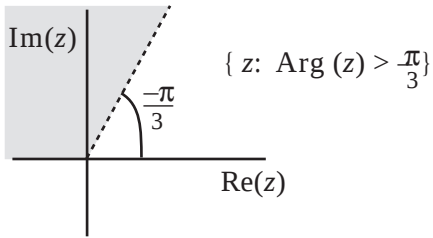
If $|W| = 1.5$ then $w = 1.5 \operatorname{cis} \theta$

so $W^{-1} = (1.5)^{-1} \operatorname{cis} (-\theta)$

$$= \frac{2}{3} \operatorname{cis} (-\theta)$$

So P is best representation.

Question 9 [C]



Question 10 [D]

$$\int \frac{1}{x^2 + 16} dx = \frac{1}{4} \int \frac{4}{x^2 + 16} dx$$

$$= \frac{1}{4} \operatorname{Tan}^{-1}\left(\frac{x}{4}\right)$$

Question 11 [B]

$$\int_0^a \left(\sin^2\left(\frac{3x}{2}\right) - \cos^2\left(\frac{3x}{2}\right) \right) dx = - \int_0^a \cos\left(2\left(\frac{3x}{2}\right)\right) dx$$

$$= - \int_0^a \cos(3x) dx$$

$$= \left[-\frac{1}{3} \sin(3x) \right]_0^a$$

$$= -\frac{1}{3} \sin(3a)$$

Question 12 [A]

$$\begin{aligned}\int_0^{\frac{\pi}{3}} \cos^2(x) - \sin^3(x) \, dx &= \int_0^{\frac{\pi}{3}} \cos^2(x) \cdot \sin^2(x) \cdot \sin(x) \, dx \\ &= \int_0^{\frac{\pi}{3}} \cos^2(x) (1 - \cos^2(x)) \sin(x) \, dx\end{aligned}$$

$$\text{Let } u = \cos(x)$$

$$\frac{du}{dx} = -\sin(x)$$

$$-du = \sin(x) \, dx$$

$$\text{When } x = 0, \, u = \cos(0) = 1$$

$$\text{When } x = \frac{\pi}{3}, \, u = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

$$\begin{aligned}&= \int_1^{\frac{1}{2}} u^2 (1 - u^2) \cdot -1 \, du \\ &= \int_{\frac{1}{2}}^1 u^2 (1 - u^2) \, du\end{aligned}$$

Question 13 [D]

$$\begin{aligned}\int \frac{2}{(3-x)^2} - \frac{1}{3-x} \, dx &= \int 2(3-x)^{-2} - \frac{1}{3-x} \, dx \\ &= 2(3-x)^{-1} + \log_e(3-x) \\ &= \frac{2}{(3-x)} + \log_e(3-x)\end{aligned}$$

Question 14 [D]

$$\text{For } x = \frac{\pi}{2}, \, f(x) > 0$$

$$x = \frac{\pi}{2}, \, f(x) = 0$$

$$\frac{\pi}{2} < x, \, f(x) < 0 \quad \text{So local maximum at } x = \frac{\pi}{2}$$

$$\text{For } \frac{\pi}{2} < x < \pi, \, f(x) < 0$$

$$x = \pi, \, f(x) = 0$$

$$\pi < x < \frac{3\pi}{2}, \, f(x) < 0 \quad \text{So stationary point of inflection at } x = \pi$$

$$\text{For } \pi < x < \frac{3\pi}{2}, \, f(x) < 0$$

$$x = \frac{3\pi}{2}, \, f(x) = 0$$

$$x < \frac{3\pi}{2}, \, f(x) > 0 \quad \text{So local minimum at } x = \frac{3\pi}{2}$$

Question 15 [C]

$$\text{When } x = 3.5, \, y = 1.803$$

$$\text{When } x = 4.5, \, y = 3.354$$

$$\text{Area} \approx 1.803 \times 1 + 3.354 \times 1$$

$$= 5.157$$

$$= 5.16 \text{ (to two decimal places)}$$

Question 16 [D]

$$\begin{aligned}
 y &= \sqrt{x^2 - 9} \\
 \therefore y^2 &= x^2 - 9 \\
 \therefore x^2 &= y^2 + 9 \\
 V &= \pi \int x^2 dy \\
 &= \pi \int_0^4 25 dy - \pi \int_0^4 y^2 + 9 dy \\
 &= \pi \int_0^4 (25 - y^2 - 9) dy \\
 &= \pi \int_0^4 \sqrt{16 - y^2} dy
 \end{aligned}$$

Question 17 [A]

$$\begin{aligned}
 \underline{\underline{P}} &= 2 \underline{\underline{i}} + x \underline{\underline{j}} + 3 \underline{\underline{k}} \\
 |\underline{\underline{P}}| &= 4 \quad \therefore 4 + x^2 + 9 = 4 \\
 x^2 + 13 &= 16 \\
 x^2 &= 3 \\
 x &= \pm \sqrt{3}
 \end{aligned}$$

$$\begin{aligned}
 \underline{\underline{q}} \text{ is parallel to } \underline{\underline{p}} \quad \text{so } \underline{\underline{q}} &= -2 \underline{\underline{p}} \\
 \text{so } y &= -2x
 \end{aligned}$$

$$\text{If } x = \sqrt{3}, y = -2\sqrt{3}$$

$$\text{If } x = -\sqrt{3}, y = 2\sqrt{3} \text{ (not an alternative)}$$

Question 18 [D]

$$\begin{aligned}
 &\underline{\underline{u}} \cdot (\underline{\underline{u}} - 2\underline{\underline{v}}) \\
 &= (3\underline{\underline{i}} - 4\underline{\underline{j}} + 5\underline{\underline{k}}) \cdot ((3\underline{\underline{i}} - 4\underline{\underline{j}} + 5\underline{\underline{k}}) - 2(2\underline{\underline{i}} + 3\underline{\underline{j}} - \underline{\underline{k}})) \\
 &= (3\underline{\underline{i}} - 4\underline{\underline{j}} + 5\underline{\underline{k}}) \cdot (-\underline{\underline{i}} - 10\underline{\underline{j}} + 7\underline{\underline{k}}) \\
 &= -3 + 40 + 35 \\
 &= 72
 \end{aligned}$$

Question 19 [C]

$$(A) |\underline{\underline{b}}|^2 + |\underline{\underline{c}}|^2 = |\underline{\underline{a}}|^2 \quad (\text{Pythagoras Rule}) \therefore \text{True}$$

$$\text{Now } \underline{\underline{b}} + \underline{\underline{c}} = \underline{\underline{a}} \quad \therefore \underline{\underline{a}} - \underline{\underline{c}} = \underline{\underline{b}}$$

$$\begin{aligned}
 (B) \text{ So } \underline{\underline{b}} \cdot (\underline{\underline{a}} - \underline{\underline{c}}) &= \underline{\underline{b}} \cdot \underline{\underline{b}} \\
 &= |\underline{\underline{b}}|^2 \quad \text{True}
 \end{aligned}$$

$$\begin{aligned}
 (C) \underline{\underline{b}} \cdot (\underline{\underline{a}} - \underline{\underline{b}}) &= \underline{\underline{b}} \cdot \underline{\underline{c}} \\
 &= |\underline{\underline{b}}| |\underline{\underline{c}}| \cos(90^\circ) \\
 &= 0 \\
 &\neq |\underline{\underline{b}}| |\underline{\underline{c}}| \quad \text{Not true}
 \end{aligned}$$

$$(D) \underline{\underline{a}} \cdot \underline{\underline{b}} = |\underline{\underline{a}}| |\underline{\underline{b}}| \cos(\theta) \quad (\text{Dot Product Rule}) \quad \text{True}$$

$$\begin{aligned}
 (E) \underline{\underline{a}} \cdot \underline{\underline{c}} &= |\underline{\underline{a}}| |\underline{\underline{c}}| \cos(90^\circ - \theta) \\
 &= |\underline{\underline{a}}| |\underline{\underline{c}}| \sin(\theta) \quad \text{True}
 \end{aligned}$$

Question 20 [E]

$$\vec{r}(t) = 2 \sin(t) \vec{i} + \cos(t) \vec{j}$$

$$\text{so } x = 2 \sin(t) + \sin(t) = \frac{x}{2}$$

$$\text{and } y = \cos(t)$$

$$\sin^2(t) + \cos^2(t) = 1$$

$$\therefore \left(\frac{x}{2}\right)^2 + y^2 = 1$$

$$\frac{x^2}{4} + y^2 = 1$$

$$0 \leq t \leq \pi \quad \text{so when } t = 0, x = 0$$

$$t = \frac{\pi}{2}, x = 2$$

$$t = \pi, x = 0$$

$$\text{so } 0 \leq x \leq 2$$

Question 21 [B]

$$\vec{r}(t) = \int 3 \sin(2t) \vec{i} + 4 \vec{j} \, dt$$

$$= \left(-\frac{3}{2} \cos(2t) + c\right) \vec{i} + (4t + d) \vec{j}$$

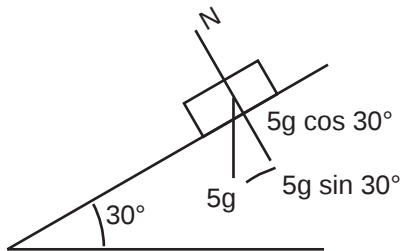
$$\vec{r}(0) = \frac{3}{2} \vec{i} \quad \text{so } -\frac{3}{2} \cos(0) + c = \frac{3}{2}$$

$$\therefore c = 3$$

$$\text{and } d = 0$$

$$\text{so } \vec{r}(t) = \left(-\frac{3}{2} \cos(2t) + 3\right) \vec{i} + 4t \vec{j}$$

Question 22 [B]

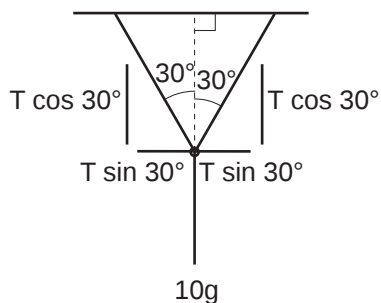


$$5g \cos 30^\circ = 5a$$

$$5g \cdot \frac{1}{2} = 5a$$

$$a = \frac{g}{2}$$

Question 23 [D]



$$2T \cos(30^\circ) = 10g$$

$$2T \cdot \frac{\sqrt{3}}{2} = 10g$$

$$T = \frac{10g}{\sqrt{3}}$$

$$= \frac{10\sqrt{3}}{3} g$$

Question 24 [A]

$$\Sigma \vec{F} = 5\vec{a}$$

$$(2\vec{i} + \vec{j}) + (\vec{i} + 10\vec{j}) + (3\vec{i} - 3\vec{j}) = 5\vec{a}$$

$$6\vec{i} + 8\vec{j} = 5\vec{a}$$

$$|\vec{a}| = \frac{\sqrt{6^2 + 8^2}}{5}$$

$$= \frac{10}{5}$$

$$= 2$$

Question 25 [B]

$$u = -21, t = 10$$

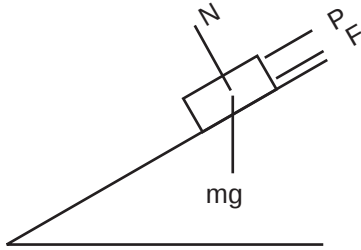
$$a = 9.8$$

$$s = ut + \frac{1}{2}at^2$$

$$\begin{aligned}\therefore h &= -21 \times 10 + \frac{1}{2} \times 9.8 \times 10^2 \\ &= 280 \text{ m}\end{aligned}$$

Question 26 [A]

Body is on a point of sliding down, so Friction Force (F) must be *up* the plane



Question 27 [B]

$$v = \frac{dx}{dt} = 2.5 - \frac{9}{2} \sin\left(\frac{t}{2}\right)$$

$$\text{minimum when } \frac{dv}{dt} = 0$$

$$\frac{dv}{dt} = -\frac{9}{4} \cos\left(\frac{t}{2}\right) = 0$$

$$\therefore \cos\left(\frac{t}{2}\right) = 0$$

$$t = \pi$$

$$\text{so } v = 2.5 - \frac{9}{2} \sin\left(\frac{\pi}{2}\right)$$

$$= 2.5 - 4.5$$

$$= -2$$

Question 28 [C]

$$y = \sin(2x)$$

$$\frac{dy}{dx} = 2 \cos(2x)$$

$$\frac{d^2y}{dx^2} = -4 \sin(2x)$$

$$(A) \frac{d^2y}{dx^2} + 4 \frac{dy}{dx} + y = 4 \cos(2x)$$

$$-4 \sin(2x) + 8 \cos(2x) + \sin(2x) \neq 4 \cos 2x$$

$$(B) \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} - 4y = 4 \cos(2x)$$

$$-4 \sin(2x) + 4 \cos(2x) - \sin(2x) \neq 4 \cos(2x)$$

$$(C) \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} + 4y = 4 \cos(2x)$$

$$-4 \sin(2x) + 4 \cos(2x) + 4 \sin(2x) = 4 \cos 2x$$

$$(D) \frac{d^2y}{dx^2} - 2 \frac{dy}{dx} + 4y = 4 \cos(2x)$$

$$-4 \sin(2x) - 4 \cos(2x) - \sin(2x) \neq 4 \cos(2x)$$

$$(E) \frac{d^2y}{dx^2} - 2 \frac{dy}{dx} + 4y = 4 \cos(2x)$$

$$-4 \sin(2x) - 4 \cos(2x) + 4 \sin(2x) \neq 4 \cos(2x)$$

Question 29 [E]

$$\frac{dy}{dt} \propto y - T_s \quad T_s = 4$$

$$\text{so } \frac{dy}{dt} = -k(y - 4) \quad t = 0, y = 20$$

Question 30 [C]

$$a = 16x$$

$$\frac{d}{dx} \left(\frac{1}{2} v^2 \right) = 16x$$

$$\frac{1}{2} v^2 = \int 16x \, dx$$

$$\frac{1}{2} v^2 = 8x^2 + c$$

$$v = -5, x = 0 \quad \therefore \frac{25}{2} = c$$

$$\text{so } \frac{1}{2} v^2 = 8x^2 + \frac{25}{2}$$

$$v^2 = 16x^2 + 25$$

$$v^2 = \pm \sqrt{16x^2 + 25}$$

$$\text{But } v = -5 \text{ at } x = 0 \quad \text{so } v = -\sqrt{16x^2 + 25}$$

2004 Mathematical Methods, Specialist Examination 1, Part II

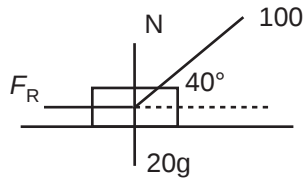
Question 1

$$\begin{aligned} \text{(a)} \quad \frac{d}{dt} (\sin^{-1}(\sqrt{2x})) &= \frac{1}{\sqrt{1 - (\sqrt{2x})^2}} \times \frac{1}{2} (2n)^{-0.5} \cdot 2 \\ &= \frac{1}{\sqrt{1 - 2x}} \times \frac{1}{\sqrt{2x}} \\ &= \frac{1}{\sqrt{2x(1 - 2x)}} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \int_{.125}^{.25} \frac{1}{\sqrt{2x(1 - 2x)}} dx &= [\sin^{-1}(\sqrt{2x})]_{0.125}^{0.25} \\ &= \sin^{-1}(\sqrt{0.5}) - \sin^{-1}(\sqrt{0.25}) \\ &= \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) - \sin^{-1}\left(\frac{1}{2}\right) \\ &= \frac{\pi}{4} - \frac{\pi}{6} \\ &= \frac{\pi}{12} \end{aligned}$$

Question 2

(a)



$$\text{(b)} \quad \Sigma \vec{F} = m\vec{a}$$

$$(100 \cos(40^\circ) - F_R)\hat{i} + (N + 100 \sin(40^\circ) - 20g)\hat{j} = 20(a\hat{i} + 0\hat{j})$$

$$\begin{aligned} F_R &= \mu N = 0.34(20g - 100 \sin(40^\circ)) \\ &= 44.7852 \end{aligned}$$

$$\text{So } 100 \cos(40^\circ) - 44.7852 = 20a$$

$$a = 1.59 \text{ m/s}^2$$

Question 3

$$f(x) = \int 15x \sqrt{2-x} dx$$

$$\text{let } u = 2 - x$$

$$\frac{du}{dx} = -1$$

$$\text{So } -du = dx$$

$$\text{and } x = 2 - u$$

$$\begin{aligned} \therefore f(x) &= \int -15(2-u)\sqrt{u} du \\ &= \int 15u^{1.5} - 30u^{0.5} du \\ &= 0.4 \times 15u^{2.5} - \frac{60}{3}u^{0.5} + c \\ &= 6u^{2.5} - 20u^{1.5} + c \\ &= u^{1.5}(6u - 20) + c \\ &= (2-x)^{1.5}(6(2-x) - 20) + c \\ &= (2-x)^{1.5}(-6x - 8) + c \end{aligned}$$

$$f(2) = 0 \quad \text{so } c = 0$$

$$\therefore f(x) = (2-x)^{1.5}(-6x-8) = (ax+b)(2-x)^{1.5}$$

so $a = -6$ and $b = -8$

Question 4

$$\begin{aligned} \text{(a) } \hat{a} \cdot \hat{b} &= (6\hat{i} + 2\hat{j}) \cdot \frac{(4\hat{i} - 3\hat{j})}{\sqrt{16+9}} \\ &= \frac{1}{5} (24 - 6) \\ &= \frac{18}{5} \end{aligned}$$

$$\begin{aligned} \text{(b) } \hat{a} \cdot \hat{b} &= |\vec{OP}| \\ |\vec{PA}|^2 &= |\vec{OP}|^2 - |\vec{OP}|^2 \\ &= 6^2 + 2^2 - \left(\frac{18}{5}\right)^2 \\ &= 27.04 \end{aligned}$$

$$\therefore |\vec{PA}| = 5.2$$

Question 5

$$W = \frac{-\sqrt{3}}{2} - \frac{1}{2}i$$

$$\text{(a) } r = \sqrt{\frac{3}{4} + \frac{1}{4}} = 1$$

$$\tan \theta = \frac{\frac{-1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} \quad (\text{third quadrant})$$

$$\text{so } \theta = -\frac{5\pi}{6}$$

$$W = \text{cis} \left(-\frac{5\pi}{6} \right)$$

$$\text{(b) } W^k = \text{cis} \left(k \times \frac{-5\pi}{6} \right) = 1$$

$$\begin{aligned} \text{cis} \left(\frac{-5k\pi}{6} \right) &= 1 = \cos \left(\frac{-5k\pi}{6} \right) + i \sin \left(\frac{-5k\pi}{6} \right) \\ &= 1 + 0i \end{aligned}$$

$$\text{so } \cos \left(\frac{-5k\pi}{6} \right) = 1$$

$$\therefore \frac{-5k\pi}{6} = \dots, -10\pi, -8\pi, -6\pi, -4\pi, -2\pi, 0, 2\pi, 4\pi, \dots$$

$$k = \dots \frac{-60\pi}{-5\pi}, \frac{-48\pi}{-5\pi}, \frac{-24\pi}{-5\pi}, \dots$$

But k is a positive integer, so $k = 12$ is the least positive integer.