

# MAV Specialist Mathematics Examination 2

## Answers & Solutions

### Question 1

**a**  $4x^2 + y^2 - 8y = 0$

$$4x^2 + y^2 - 8y + 16 = 16$$

$$4x^2 + (y - 4)^2 = 16 \quad [\text{M1}]$$

$$\frac{x^2}{4} + \frac{(y - 4)^2}{16} = 1 \quad [\text{A1}]$$

$$\frac{x^2}{2^2} + \frac{(y - 4)^2}{4^2} = 1$$

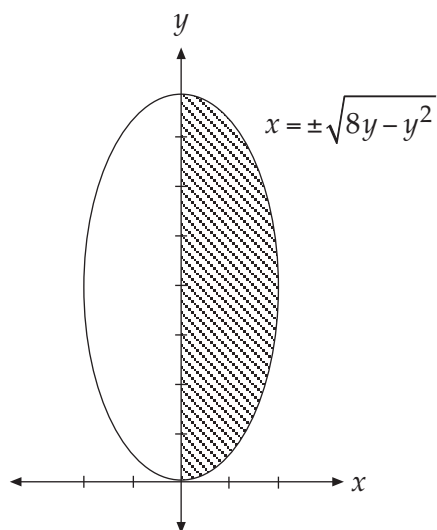
**b i**  $4x^2 = 8y - y^2$

$$x^2 = \frac{8y - y^2}{4}$$

$$x = \pm \frac{1}{2} \sqrt{8y - y^2} \quad [\text{A1}]$$

The area of the ellipse will be double the shaded area.

$$\therefore A = 2 \times \int_{y=0}^{y=8} \frac{1}{2} \sqrt{8y - y^2} dy \quad [\text{M1}]$$



$$= \int_0^8 \sqrt{8y - y^2} dy \text{ use integration function on a graphics calculator}$$

$$= 25.13 \text{ square units} \quad [\text{A1}]$$

Note: Area of an ellipse given by  $A = \pi ab$  is also acceptable.

**b ii** Volume formed by rotating about the y-axis is given by  $V = \int \pi x^2 dy$

$$V = \int_0^8 \pi \left( \frac{8y - y^2}{4} \right) dy \quad [\text{M1}]$$

$$= \frac{\pi}{4} \int_0^8 (8y - y^2) dy$$

$$= \frac{\pi}{4} \left[ 4y^2 - \frac{1}{3} y^3 \right]_0^8 \quad [\text{A1}]$$

$$= \frac{\pi}{4} \left[ \left( 256 - \frac{512}{3} \right) - 0 \right]$$

$$= \frac{64\pi}{3} \quad [\text{A1}]$$

**c i** Being released from the balloon, the tennis ball will initially also be moving upwards at 3m/s. Considering upwards as the positive direction:  $u = 3 \text{ m/s}$ ,  $a = -9.8 \text{ m/s}^2$ ,  $s = -75 \text{ m}$ .

$$s = ut + \frac{1}{2} at^2 \quad [\text{M1}]$$

$$-75 = 3t + \frac{1}{2} (-9.8)t^2$$

$$9.8t^2 - 6t - 150 = 0 \text{ using the quadratic formula}$$

$$t = 4.230, -3.618$$

$$\therefore t = 4.230, \text{ since } t > 0 \quad [\text{A1}]$$

**c ii**  $u = 3 \text{ m/s}$ ,  $a = -9.8 \text{ m/s}^2$ ,  $s = -75 \text{ m}$ ,  $v = ?$

$$v^2 - u^2 = 2as \quad [\text{M1}]$$

$$v^2 = 2as + u^2$$

$$v = 38.458 \text{ m/s} \quad [\text{A1}]$$

### Question 2

**a i**  $z^2 - 4z + 6 = 0$

$$z = \frac{4 \pm \sqrt{16 - 4(1)(6)}}{2}$$

$$= \frac{4 \pm \sqrt{-8}}{2}$$

$$= \frac{4 \pm \sqrt{8i^2}}{2}$$

$$= \frac{4 \pm 2i\sqrt{2}}{2}$$

$$= 2 \pm i\sqrt{2} \quad [\text{A1}]$$

**a ii**  $\tan \theta = \frac{\sqrt{2}}{2}$

$$\theta = 35.26^\circ \quad [\text{A1}]$$

$$z = \sqrt{6} \text{cis} 35.26^\circ, \sqrt{6} \text{cis} (-35.26^\circ) \quad [\text{A1}]$$

$$\mathbf{b} \quad \mathbf{i} \quad z+(a+4) \quad [\mathbf{M1}]$$

$$\begin{array}{r} z^2 - 4z + 6 \overline{) z^3 + az^2 + bz + 6} \\ \underline{z^3 - 4z^2 + 6z} \phantom{+ 6} \\ (a+4)z^2 + (b-6)z + 6 \\ \underline{(a+4)z^2 + (-4a-16)z + (6a+24)} \\ (b+4a+10)z + (-6a-18) \end{array}$$

Since  $z^2 - 4z + 6$  is a solution, then  
remainder = 0.

$$\therefore b + 4a + 10 = 0 \quad [\text{M1}]$$

$$-6a - 18 = 0 \Rightarrow a = -3, b = 2 \quad [\text{A1}]$$

**b ii** Since  $a = -3$  then the third factor is  $z + 1$ , hence third solution is  $z = -1$ . [A1]

### Question 3

**a**  $\frac{dN}{dt} = kN$

$$\frac{dt}{dN} = \frac{1}{kN}$$

$$t = \frac{1}{k} \log_e N + c \quad [\text{M1}]$$

$$t = 0, N = 10 \Rightarrow c = -\frac{1}{k} \log_e 10$$

$$\therefore t = \frac{1}{k} \log_e N - \frac{1}{k} \log_e 10$$

$$t = \frac{1}{k} \log_e \left( \frac{N}{10} \right) \quad \text{[M1]}$$

$$tk = \log_e \left( \frac{N}{10} \right)$$

$$e^{kt} = \frac{N}{10} \quad \text{[M1]}$$

$$N = 10e^{kt}$$

**b**  $N = 10e^{kt}$

$t = 2, N = 70$

$$70 = 10e^{2k}$$

$$\log_e 7 = \log_e e^{2k} \quad [\text{M1}]$$

$$k = \frac{1}{2} \log_e 7$$

$$\mathbf{c} \quad N = 10e^{\left(\frac{1}{2}\log_e 7\right)5}$$

N = 1296 birds [A1]

$$d_{10000} = 10e^{\left(\frac{1}{2} \log_e 7\right)t}$$

$$\log_e 1000 = \left( \frac{1}{2} \log_e 7 \right) t$$

$$t = \frac{2 \log_e 1000}{\log_e 7} \quad [\text{M1}]$$

$t = 7.099$ , therefore it takes 8 years to exceed 10 000. [A1]

$$\text{e i} \quad \frac{dN}{dt} = kN(6000 - N)$$

$$\frac{dt}{dN} = \frac{1}{kN(6000 - N)}$$

$$t = \frac{1}{k} \int \frac{1}{N(6000 - N)} dN$$

$$kt = \int \frac{1}{N(6000 - N)} dN$$

$$\frac{1}{N(6000 - N)} = \frac{a}{N} + \frac{b}{6000 - N}$$

$$= \frac{a(6000 - N) + bN}{N(6000 - N)} \quad [\text{M1}]$$

$$1 = a(6000 - N) + bN$$

$$N = 0, a = \frac{1}{6000}$$

$$N = 6000, b = \frac{1}{6000}$$

$$\therefore kt = \frac{1}{6000} \int \left( \frac{1}{N} + \frac{1}{6000 - N} \right) dN$$

$$6000kt = \log_e N - \log_e (6000 - N) + c \quad [\text{A1}]$$

$$6000kt - c = \log_e \left( \frac{N}{6000 - N} \right)$$

$$e^{6000kt} \cdot e^{-c} = \frac{N}{6000 - N}$$

$$Ae^{6000kt} = \frac{N}{6000 - N}, \text{ where } A = e^{-c}$$

$$N = (6000 - N)Ae^{6000kt} \quad [\text{M1}]$$

$$N = 6000Ae^{6000kt} - NAe^{6000kt}$$

$$N + NAe^{6000kt} = 6000Ae^{6000kt}$$

$$N(1 + Ae^{6000kt}) = 6000Ae^{6000kt} \quad [\text{A1}]$$

$$N = \frac{6000Ae^{6000kt}}{1 + Ae^{6000kt}}$$

$$\text{e ii} \quad t = 0, N = 10 \Rightarrow 10 = \frac{6000A}{1 + A}$$

$$10 + 10A = 6000A$$

$$10 = 5990A$$

$$A = \frac{1}{599} \quad [\text{A1}]$$

#### Question 4

$$\text{a} \quad \overrightarrow{OA} = 60\mathbf{j}$$

$$\overrightarrow{OB} = 50\mathbf{i} + 90\mathbf{j} + 5\mathbf{k}$$

$$\overrightarrow{OC} = 80\mathbf{i} + 65\mathbf{j} + 17\mathbf{k}$$

$$\overrightarrow{OD} = 15\mathbf{i} + 45\mathbf{j} + 20\mathbf{k} \quad [\text{A2}]$$

$$\text{b i} \quad \overrightarrow{BC} = 30\mathbf{i} - 25\mathbf{j} + 12\mathbf{k}$$

$$|\overrightarrow{BC}| = \sqrt{30^2 + 25^2 + 12^2}$$

$$= 41 \text{ metres} \quad [\text{A1}]$$

$$\text{b ii} \quad \cos \theta = \frac{\overrightarrow{AB} \cdot \overrightarrow{BC}}{|\overrightarrow{AB}| |\overrightarrow{BC}|} \quad [\text{M1}]$$

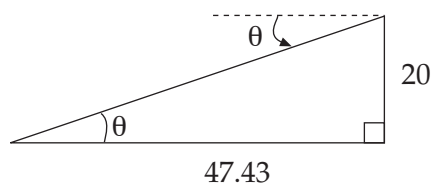
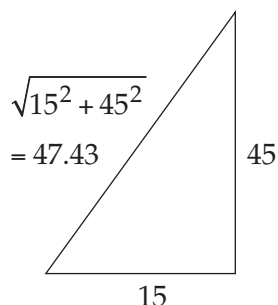
$$\cos \theta = \frac{50(30) + 30(-25) + 5(12)}{\sqrt{50^2 + 30^2 + 5^2} \cdot \sqrt{30^2 + 25^2 + 12^2}}$$

$$\cos \theta = 0.3388$$

$$\theta = 70^\circ \quad [\text{A1}]$$

c  $\overrightarrow{OD} = 15\mathbf{i} + 45\mathbf{j} + 20\mathbf{k}$

$$\sqrt{15^2 + 45^2} = 47.43 \quad [\text{A1}]$$



$$\tan \theta = \frac{20}{47.43}$$

$$\theta = 23^\circ$$

- d Want scalar resolute of  $\overrightarrow{AB}$  in the direction of  $\overrightarrow{BC}$ .

$$\overrightarrow{AB} \cdot \overrightarrow{BC}$$

$$= \left( 50\mathbf{i} + 30\mathbf{j} + 5\mathbf{k} \right) \cdot \frac{(30\mathbf{i} - 25\mathbf{j} + 12\mathbf{k})}{\sqrt{30^2 + 25^2 + 12^2}}$$

$$= 19.83\text{m}$$

$$\approx 20\text{m}$$

[A1]

[M1]

[A1]

[M1]

[A1]

### Question 5

a  $[1, \infty)$  [A1]

b  $y = 4\cos^{-1}\left(\frac{1}{\sqrt{x}}\right)$  Let  $u = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$  [M1]

$$y = 4\cos^{-1}u \quad \frac{du}{dx} = -\frac{1}{2}x^{-\frac{3}{2}}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$f'(x) = \frac{-4}{\sqrt{1-u^2}} \times -\frac{1}{2}x^{-\frac{3}{2}} \quad [\text{M1}]$$

$$= \frac{2}{\sqrt{1-\frac{1}{x}}} \times \frac{1}{x\sqrt{x}} \quad [\text{A1}]$$

$$= \frac{2}{x\sqrt{x-1}}$$

c  $\int_2^4 \frac{1}{x\sqrt{x-1}} dx$

$$= \left[ 2\cos^{-1}\left(\frac{1}{\sqrt{x}}\right) \right]_2^4 \quad [\text{A1}]$$

$$= 2\left(\cos^{-1}\frac{1}{2} - \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)\right)$$

$$= 2\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \quad [\text{A1}]$$

$$= \frac{\pi}{6}$$

d i  $A = f(2.5) + f(3.5)$  [M1]

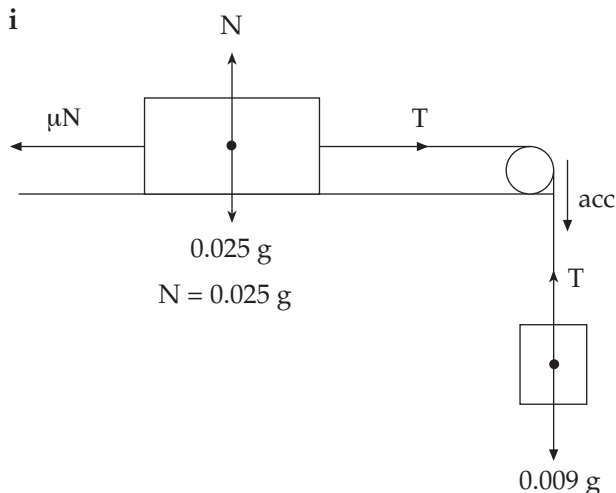
$$= 0.3266 + 0.1807$$

$$\approx 0.507 \quad [\text{A1}]$$

ii  $\frac{\pi}{6} \approx 0.507$  Inaccuracy results

from the steep slope  
approaching  $x = 2$ .

[A1]

**Question 6****a i**

$$T - \mu N = m(0.025)$$

$$T - 0.2(0.025g) = 0.025a \quad (1)$$

$$0.009g - T = 0.009a \quad (2) \quad [\text{M1}]$$

$$(1) + (2)$$

$$0.0392 = 0.034a$$

$$a = 1.153 \text{ m/s}^2 \quad [\text{A1}]$$

ii substituting  $a = 1.153 \text{ m/s}^2$  into (2) gives

$$T = 0.078 \text{ Newtons} \quad [\text{A1}]$$

$$\text{b } T - \mu N = 0.025a \quad (1)$$

$$0.009g - T = 0.009a \quad (2)$$

$$(1) + (2)$$

$$0.009g - 0.025\mu g = 0.034a \quad [\text{A1}]$$

$$(0.009 - 0.025\mu)g = 0.034a$$

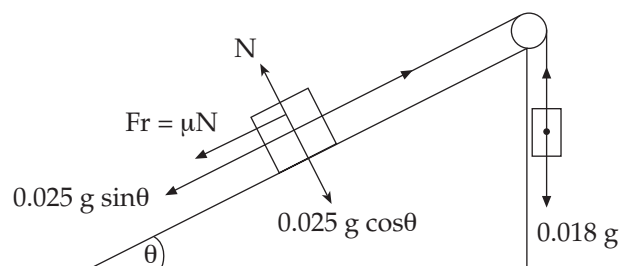
$$a = \frac{(0.009 - 0.025\mu)g}{0.034}$$

since acceleration is constant

$$s = ut + \frac{1}{2}at^2 \quad [\text{M1}]$$

$$0.5 = 0(2) + \frac{1}{2} \left[ \frac{(0.009 - 0.025\mu)g}{0.034} \right] 4$$

$$\mu = 0.325 \quad [\text{A1}]$$

**c**

$$N = 0.025g \cos \theta$$

$$T = 0.018g \quad (1)$$

$$T = 0.025g \sin \theta + \mu N$$

$$T = 0.025g \sin \theta + 0.40 \times 0.025g \cos \theta \quad (2) \quad [\text{M1}]$$

equating (1) and (2)

$$0.018g = 0.025g \sin \theta + 0.01g \cos \theta \quad [\text{A1}]$$

$$0.018 = 0.025 \sin \theta + 0.01 \cos \theta$$

$$18 = 25 \sin \theta + 10 \cos \theta$$

using a graphics calculator to solve:

$$\theta = 20.15^\circ \quad [\text{A1}]$$