

2004 Specialist Mathematics Written Examination 1 (facts, skills and applications) Suggested answers and solutions

Part I (Multiple-choice) Answers

- | | | | | |
|-------|-------|-------|-------|-------|
| 1. C | 2. A | 3. B | 4. C | 5. E |
| 6. D | 7. A | 8. D | 9. D | 10. B |
| 11. A | 12. E | 13. E | 14. A | 15. C |
| 16. E | 17. B | 18. C | 19. A | 20. C |
| 21. E | 22. E | 23. B | 24. D | 25. C |
| 26. B | 27. A | 28. D | 29. D | 30. B |

- 1 General equation for an ellipse: [C]

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$a = 4 \text{ and } b = 9$$

$$\therefore \frac{x^2}{16} + \frac{y^2}{81} = 1$$

- 2 $\cos\left(\frac{7\pi}{6}\right) = \frac{\sqrt{3}}{2}$ [A]

$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$$

$$\text{Note: } \frac{-\pi}{2} \leq \sin^{-1}(x) \leq \frac{\pi}{2}$$

- 3 $\sec^2(x) - 1 = \tan^2(x)$ [B]

$$\cos(x) = -\frac{2}{3}$$

$$\Rightarrow \sec^2(x) = \frac{9}{4}$$

$$\therefore \tan^2(x) = \frac{9}{4} - 1$$

$$= \frac{5}{4}$$

$$\tan(x) = \pm \frac{\sqrt{5}}{2}$$

$$x \text{ is in the 2}^{\text{nd}} \text{ quadrant } \therefore \tan(x) = -\frac{\sqrt{5}}{2}$$

- 4 $z^4 - 81 = 0$ [C]

$$(z^2 - 9)(z^2 + 9) = 0$$

$$(z - 3)(z + 3)(z - 3i)(z + 3i) = 0$$

$$z = \pm 3, \pm 3i$$

- 5 $z = 2\text{cis}\left(\frac{5\pi}{6}\right)$ and $w = 5\text{cis}\left(\frac{\pi}{3}\right)$ [E]

$$zw = 10\text{cis}\left(\frac{5\pi}{6} + \frac{\pi}{3}\right)$$

$$= 10\text{cis}\left(\frac{5\pi}{6} + \frac{2\pi}{6}\right)$$

$$= 10\text{cis}\left(\frac{7\pi}{6}\right)$$

$$\arg(zw) = \frac{7\pi}{6}$$

$$-\pi < \text{Arg}(zw) \leq \pi$$

$$\therefore \text{Arg}(zw) = -\frac{5\pi}{6}$$

6 $z = 2 + 3i$ [D]

$$\bar{z} = 2 - 3i$$

$$\bar{z}w = (2 - 3i)(2 - i)$$

$$= 4 - 2i - 6i + 3i^2$$

$$= 1 - 8i$$

7 $z = -\sqrt{3} + i$ [A]

$$|z| = \sqrt{3+1} = 2$$

$$\theta = \tan^{-1} \frac{1}{\sqrt{3}} = \frac{\pi}{6}$$

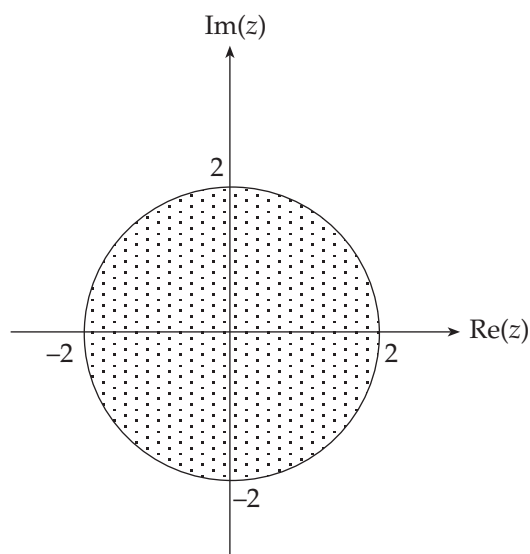
$$z = 2\text{cis} \frac{5\pi}{6}$$

$$z^4 = 2^4 \text{cis} \frac{20\pi}{6}$$

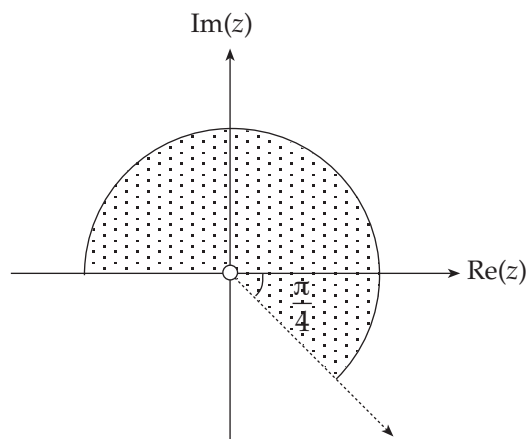
$$= 16\text{cis} \left(\frac{10\pi}{3} \right)$$

$$= 16\text{cis} \left(\frac{-2\pi}{3} \right)$$

8 [D]



$$\{z : z\bar{z} \leq 2\} \text{ or } \{z : |z| \leq 2\}$$



$$\left\{ z : \text{Arg}(z) \geq -\frac{\pi}{4} \right\}$$

$$\therefore \{z : |z| \leq 2\} \cap \left\{ z : \text{Arg}(z) \geq -\frac{\pi}{4} \right\}$$

9 $\frac{3x-1}{x^2+x-6} = \frac{A}{x+3} + \frac{B}{x-2}$ [D]

$$= \frac{A(x-2) + B(x+3)}{(x+3)(x-2)}$$

$$\Rightarrow 3x-1 = A(x-2) + B(x+3)$$

Let $x = 2$

$$5 = 5B$$

$$\Rightarrow B = 1$$

Let $x = -3$

$$-10 = -5A$$

$$\Rightarrow A = 2$$

$$\therefore A = 2 \text{ and } B = 1$$

10 $\frac{d(x \tan^{-1} x)}{dx} = \frac{x}{x^2+1} + \tan^{-1}(x)$ [B]

$$\Rightarrow \tan^{-1}(x) = \frac{d(x \tan^{-1} x)}{dx} - \frac{x}{x^2+1}$$

$$\Rightarrow \int \tan^{-1}(x) dx = x \tan^{-1}(x) - \int \frac{x}{x^2+1} dx$$

$$= x \tan^{-1}(x) - \frac{1}{2} \ln(x^2+1)$$

11 $\int_1^3 (2x+1)\sqrt{2x-1}dx$ [A]

Let $u = 2x - 1$

$\Rightarrow 2x = u + 1$

and $\frac{du}{dx} = 2$

$\Rightarrow dx = \frac{du}{2}$

For terminals: $x = 3$ $u = 6 - 1$

$u = 5$

$x = 1$ $u = 2 - 1$

$u = 1$

$\frac{1}{2} \int_1^5 (u+2)\sqrt{u}du$

12 $y = x$ $y = 2 - x^2$ [E]

$x = 2 - x^2$

$x^2 + x - 2 = 0$

$(x+2)(x-1) = 0$

$x = 1, -2$

$y = 1, -2$

$V = \pi \int y_1^2 dx - \pi \int y_2^2 dx$

where $y_1 = 2 - x^2$ and $y_2 = x$

$V = \pi \int_0^1 (2 - x^2)^2 dx - \pi \int_0^1 (x^2) dx$

$= \pi \int_0^1 (2 - x^2)^2 - x^2 dx$

13 $\int_0^1 x - \frac{-2}{\sqrt{4-x^2}} dx$ [E]

$= \int_0^1 x + \frac{-2}{\sqrt{4-x^2}} dx$

$= \left[\frac{x^2}{2} + 2\text{Sin}^{-1}\left(\frac{x}{2}\right) \right]_0^1$

$= \left(\frac{1}{2} + 2\text{Sin}^{-1}\left(\frac{1}{2}\right) \right) - (0+0)$

$= \frac{1}{2} + 2 \times \frac{\pi}{6}$

$= \frac{1}{2} + \frac{\pi}{3}$

14 Using graphics calculator. [A]

$\int_0^1 (e^{x^2} + e^{-x^2})^2 dx = 4.9626$

15 Midpoint at $x = \frac{1}{2}$ [C]

Approximate area $= \cos\left(2 \times \frac{1}{2}\right)$
 $= \cos(1)$

16 Inflow: $\frac{dx_1}{dt} = 0.1 \times 2$ [E]

$= 0.2 \text{ kg/min}$

Outflow: $\frac{dx_2}{dt} = \frac{x}{100} \times 2$

$= \frac{x}{50}$

$\frac{dx}{dt} = \text{inflow} - \text{outflow}$

$\frac{dx}{dt} = 0.2 - \frac{x}{50}$

17 $\frac{d^2y}{dx^2} = \cos(2x)$ [B]

$$\frac{dy}{dx} = \frac{1}{2} \sin(2x) + c$$

$$\frac{dy}{dx} = 0 \text{ at } x = 0$$

$$\therefore c = 0$$

$$\frac{dy}{dx} = \frac{1}{2} \sin(2x)$$

$$y = -\frac{1}{4} \cos(2x) + d$$

at $y = 0$ and $x = 0$

$$d = \frac{1}{4}$$

$$y = -\frac{1}{4} \cos(2x) + \frac{1}{4}$$

18 $\underline{a} \cdot \underline{b} = |\underline{a}| |\underline{b}| \cos \theta$ [C]

$$\Rightarrow \cos \theta = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}| |\underline{b}|}$$

$$\underline{a} \cdot \underline{b} = 2 - 3 - 6 = -7$$

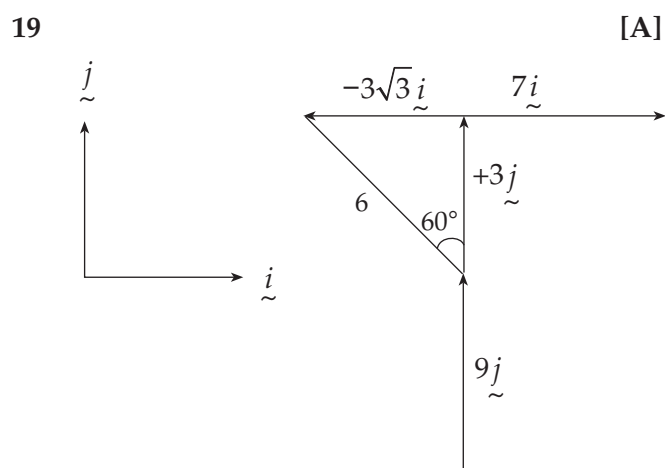
$$|\underline{a}| = \sqrt{4 + 1 + 9} = \sqrt{14}$$

$$|\underline{b}| = \sqrt{1 + 9 + 4} = \sqrt{14}$$

$$\cos \theta = \frac{-7}{\sqrt{14} \times \sqrt{14}} = -\frac{1}{2}$$

$$\theta = \cos^{-1}\left(-\frac{1}{2}\right)$$

$$= \frac{2\pi}{3}$$



$$(9+3)\underline{j} + (7-3\sqrt{3})\underline{i}$$

Position vector: $(7-3\sqrt{3})\underline{i} + 12\underline{j}$

20 $\overrightarrow{OT} \cdot \overrightarrow{TB} = |\overrightarrow{OT}| |\overrightarrow{TB}| \cos \angle OTB$ [C]

$$\angle OTB = 90^\circ$$

$$\cos 90^\circ = 0$$

$$\therefore \overrightarrow{OT} \cdot \overrightarrow{TB} = 0$$

21 $\underline{a} = 2\underline{i} + \underline{j} + 5\underline{k}$ [E]

$$4\underline{a} = 8\underline{i} + 4\underline{j} + 20\underline{k}$$

$$\underline{b} = -4\underline{i} + \underline{j}$$

$$4\underline{a} + \underline{b} = 4\underline{i} + 5\underline{j} + 20\underline{k}$$

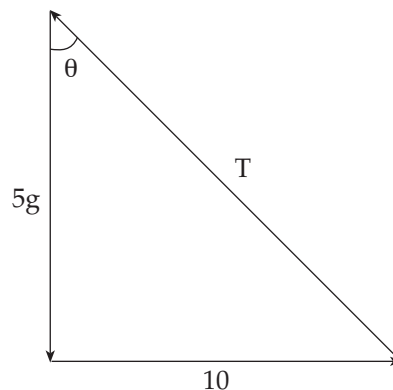
$$|4\underline{a} + \underline{b}| = \sqrt{16 + 25 + 400}$$

$$= \sqrt{441}$$

$$= 21$$

- 22 $\vec{a} = 6\vec{i} - 2\vec{j} + 6\vec{k}$ [E]
 $\vec{b} = -6\vec{i} - 2\vec{j} + \vec{k}$
 $\hat{a} = \frac{1}{2\sqrt{19}}(6\vec{i} - 2\vec{j} + 6\vec{k})$
 $\vec{b} \cdot \hat{a} = \frac{1}{2\sqrt{19}}(-36 + 4 + 6)$
 $= \frac{-26}{2\sqrt{19}}$
 $= \frac{-13}{\sqrt{19}}$
 $(\vec{b} \cdot \hat{a})\hat{a} = \frac{-13}{\sqrt{19}} \times \frac{1}{2\sqrt{19}}(6\vec{i} - 2\vec{j} + 6\vec{k})$
 $= \frac{-13}{19}(3\vec{i} - \vec{j} + 3\vec{k})$
- 23 $x = t - 1$ [B]
 $y = 5(t - 1)^2$
 $y = 5(x)^2$
we know that $t \geq 0$
 $\Rightarrow x \geq -1$
 $\therefore y = 5x^2$ where $x \geq -1$
- 24 $\vec{r}(t) = 4\cos 2t\vec{i} + 3e^t\vec{j}$ [D]
 $\vec{r}(0) = 4\vec{i} + 3\vec{j}$
 $|\vec{r}(0)| = \sqrt{16 + 9}$
 $= \sqrt{25}$
 $= 5$
- 25 $\vec{F}_1 = 2\vec{i} + \vec{j}$ $\vec{F}_2 = 8\vec{i} - 5\vec{k}$ [C]
 $\vec{F}_T = 10\vec{i} + \vec{j} - 5\vec{k}$
 $\vec{F}_T = m\vec{a}$
 $= 4\vec{a} = 10\vec{i} + \vec{j} - 5\vec{k}$
 $\vec{a} = 2.5\vec{i} + 0.25\vec{j} - 1.25\vec{k}$

- 26 [B]



$$T \sin \theta = 10$$

$$T \cos \theta = 5g$$

$$\tan \theta = \frac{10}{5g}$$

$$\theta = \tan^{-1}\left(\frac{2}{g}\right)$$

$$\theta \approx 0.2013$$

- 27 $a = \frac{d\left(\frac{1}{2}v^2\right)}{dx} = x + 6$ [A]

$$\frac{1}{2}v^2 = \frac{1}{2}x^2 + 6x + c$$

$$v^2 = x^2 + 12x + d$$

$$2c = d$$

$$\text{at } x = 1, v = 7$$

$$49 = 1 + 12 + d$$

$$d = 36$$

$$\therefore v^2 = x^2 + 12x + 36$$

$$v = \pm(x + 6)$$

- 28 Distance travelled is area enclosed by graph over the interval: [D]

$$A = \frac{1}{2} \times 6 \times 4 + \frac{1}{2} \times 6 \times 4 + 2 \times 6$$

$$= 12 + 12 + 12$$

$$= 36$$

29 We know: $T - \mu m_1 g = m_1 a$ — 1 [D]

$T = m_2 g$ — 2

If $m_1 a > 0$ then the block will move.

From 1 and 2:

$$m_2 g - \mu m_1 g = m_1 a$$

If $m_1 a > 0$

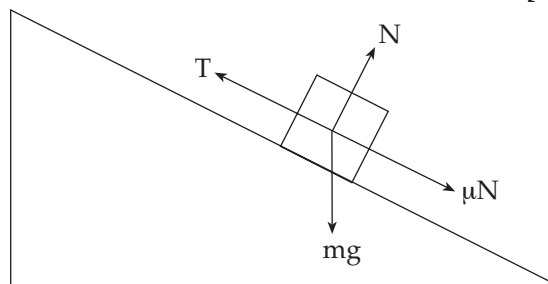
$$\text{then } m_2 g - \mu m_1 g > 0$$

$$m_2 g > \mu m_1 g$$

$$m_2 > \mu m_1$$

30

[B]



Short answer solutions

Question 1

a i $f(x) = \frac{-1}{1+x^2}$

Let $u = 1+x^2$

$$\frac{du}{dx} = 2x$$

$$\Rightarrow y = -u^{-1}$$

$$\frac{dy}{du} = u^{-2}$$

(M1)

$$f'(x) = \frac{dy}{dx} = 2x(1+x^2)^{-2}$$

$$= \frac{2x}{(1+x^2)^2}$$

(A1)

a ii $f'(1) = \frac{2}{(1+1)^2}$

$$= \frac{1}{2}$$

(A1)

b $y_2 = y_1 + hf'(x_1)$

$$y_1 = f(1) = -\frac{1}{2}$$

$$h = 0.01$$

$$f'(x_1) = f'(1) = \frac{1}{2}$$

$$y_2 = -\frac{1}{2} + 0.01 \times \frac{1}{2}$$

$$= -0.495$$

(A1)*

* must include working

Question 2

$$\int_1^{\sqrt{2}} \frac{1}{\sqrt{4-x^2}} dx$$

$$= \left[\sin^{-1} \left(\frac{x}{2} \right) \right]_1^{\sqrt{2}}$$

(M1)

$$= \sin^{-1} \left(\frac{\sqrt{2}}{2} \right) - \sin^{-1} \left(\frac{1}{2} \right)$$

$$= \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}$$

(A1)

Question 3

a $\underline{\dot{r}}(t) = 4\cos t \underline{i} + 4\sin t \underline{j} + 3t \underline{k}$

$$\underline{\dot{r}}(t) = -4\sin t \underline{i} + 4\cos t \underline{j} + 3 \underline{k} \quad (\text{M1})$$

$$\underline{\dot{r}}(0) = 4 \underline{j} + 3 \underline{k}$$

$$|\underline{\dot{r}}(0)| = \sqrt{16+9} \quad (\text{M1})$$

$$= 5 \quad (\text{A1})$$

b $\underline{\dot{r}}(t) = -4\sin t \underline{i} + 4\cos t \underline{j} + 3 \underline{k}$

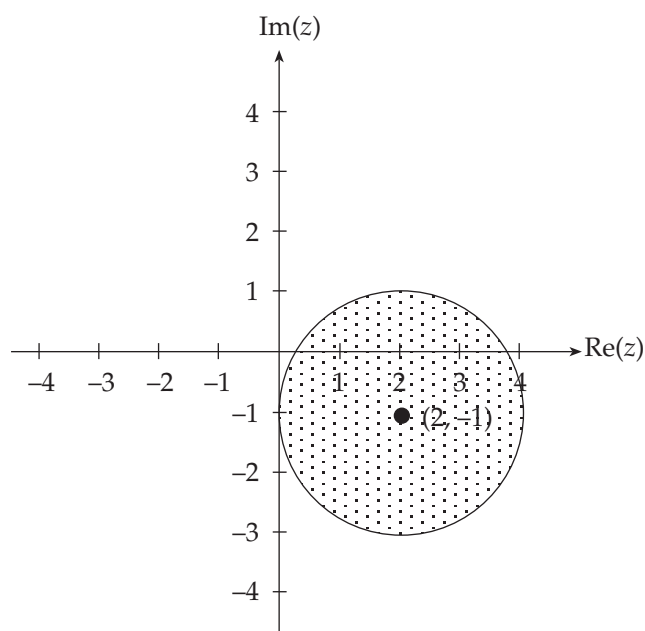
$$\underline{\ddot{r}}(t) = -4\cos(t) \underline{i} - 4\sin(t) \underline{j} + 0 \underline{k} \quad (\text{M1})$$

$$\underline{\dot{r}}(t) \cdot \underline{\ddot{r}}(t) = 16\sin(t)\cos(t) - 16\sin(t)\cos(t) + 0 \quad (\text{M1})$$

$$\underline{\dot{r}}(t) \cdot \underline{\ddot{r}}(t) = 0 \text{ for all } t, \text{ and } \underline{\dot{r}}(t) \neq 0 \text{ and } \underline{\ddot{r}}(t) \neq 0$$

$$\therefore \underline{\dot{r}}(t) \text{ is perpendicular to } \underline{\ddot{r}}(t) \quad (\text{A1})$$

Question 4

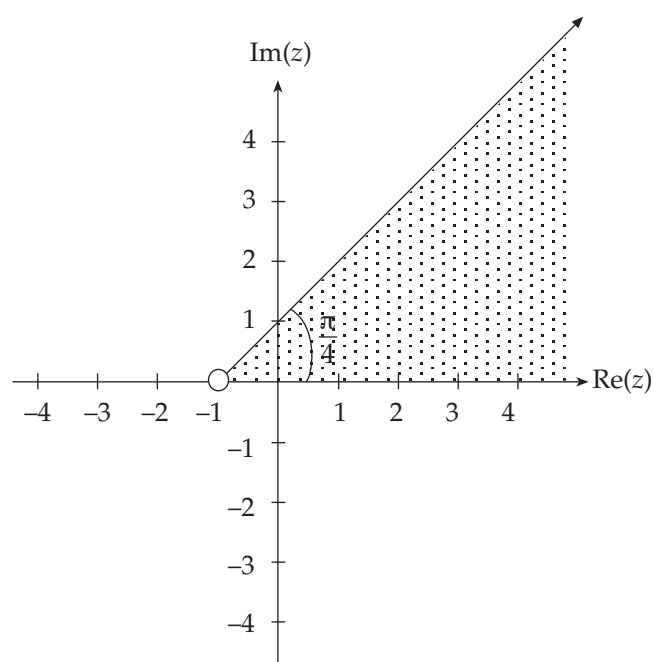


Circle of radius 2, centre at (2, -1)

(A1)

Correct shading

(A1)



Both rays starting at (-1, 0)

(A1)

Correct shading

(A1)

Question 5

a $(\cos \theta + i \sin \theta)^3$

Using De Moivre's Theorem

$$(\cos \theta + i \sin \theta)^3 = \cos 3\theta + i \sin 3\theta \quad \textbf{(A1)}$$

Using Binomial expansion

$$\begin{aligned} (\cos \theta + i \sin \theta)^3 &= \cos^3 \theta + 3\cos^2 \theta i \sin \theta + 3\cos \theta i^2 \sin^2 \theta + i^3 \sin^3 \theta \\ &= \cos^3 \theta - 3\cos \theta \sin^2 \theta + i(3\cos^2 \theta \sin \theta - \sin^3 \theta) \end{aligned} \quad \textbf{(M1)}$$

Equating real parts

$$\cos 3\theta = \cos^3 \theta - 3\cos \theta \sin^2 \theta \quad \textbf{(A1)}$$

b Equating imaginary parts

$$\sin 3\theta = 3\cos^2 \theta \sin \theta - \sin^3 \theta \quad \textbf{(A1)}$$
