

Year 2004

VCE

Specialist Mathematics

Trial Examination 2

Suggested Solutions

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Question 1

- i. differentiating using the Product Rule

$$\frac{d}{dx} [x \cos^{-1}(x)] = \cos^{-1}(x) + x \frac{d}{dx} (\cos^{-1}(x)) = \cos^{-1}(x) - \frac{x}{\sqrt{1-x^2}}$$

$$\text{so } \int \left(\cos^{-1}(x) - \frac{x}{\sqrt{1-x^2}} \right) dx = x \cos^{-1}(x)$$

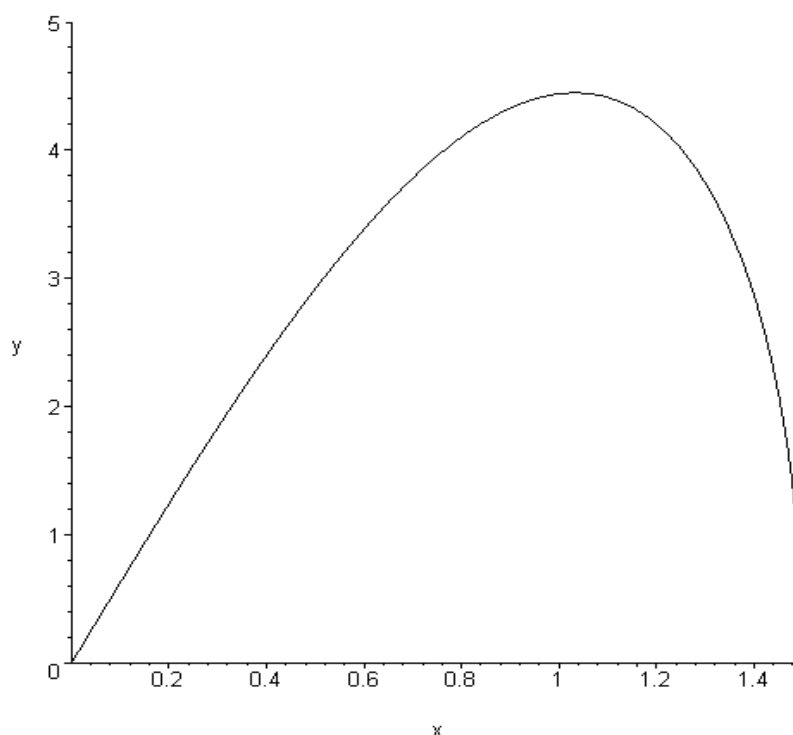
$$\int \cos^{-1}(x) dx = x \cos^{-1}(x) + \int \frac{x}{\sqrt{1-x^2}} dx$$

$$\text{Now consider } \int \frac{x}{\sqrt{1-x^2}} dx \quad \text{let } u = 1-x^2 \quad \text{so that } \frac{du}{dx} = -2x$$

$$\int \frac{x}{\sqrt{1-x^2}} dx = -\frac{1}{2} \int u^{-\frac{1}{2}} du = -u^{\frac{1}{2}} = -\sqrt{1-x^2}$$

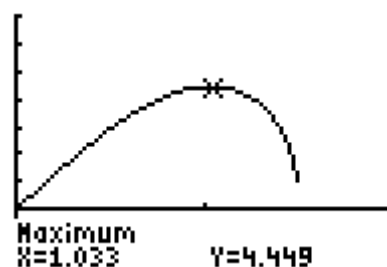
$$\text{so } \int \cos^{-1}(x) dx = x \cos^{-1}(x) - \sqrt{1-x^2} \quad \text{as required.}$$

- ii.



we require that $\left| \frac{4x^2}{9} \right| \leq 1$ or $0 \leq \frac{4x^2}{9} \leq 1$

$$0 \leq x^2 \leq \frac{9}{4} \quad 0 \leq x \leq \frac{3}{2} \quad \text{so } b = \frac{3}{2}$$

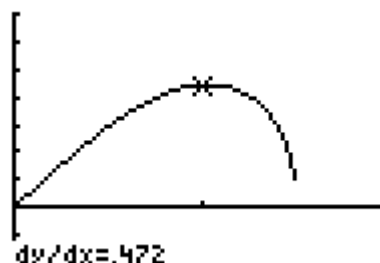


iii. $\max(1.033, 4.449)$ so the height is 4.449 m

iv. $\frac{dy}{dt} = -2 \quad \left. \frac{dy}{dx} \right|_{x=1} = 0.472$

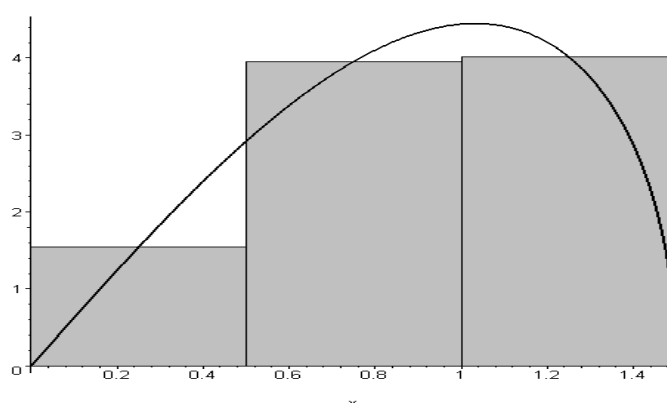
$$\frac{dx}{dt} = \frac{dx}{dy} \cdot \frac{dy}{dt} = -\frac{2}{0.472}$$

$$= -4.239 \text{ m/s}$$



v. $f(x) = 4x \cos^{-1}\left(\frac{4x^2}{9}\right) \quad a=0 \quad b=\frac{3}{2} \quad n=3 \quad h=\frac{b-a}{n}=\frac{1}{2}$

x	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{5}{4}$
f(x)	$\cos^{-1}\left(\frac{1}{36}\right)$	$3 \cos^{-1}\left(\frac{1}{4}\right)$	$5 \cos^{-1}\left(\frac{25}{36}\right)$



$$M = \frac{1}{2} \left(\cos^{-1}\left(\frac{1}{36}\right) + 3 \cos^{-1}\left(\frac{1}{4}\right) + 5 \cos^{-1}\left(\frac{25}{36}\right) \right) = 4.757$$

vi. $A = \int_0^{\frac{3}{2}} 4x \cos^{-1}\left(\frac{4x^2}{9}\right) dx$ let $u = \frac{4x^2}{9}$ $\frac{du}{dx} = \frac{8x}{9}$ so $4x dx = \frac{9}{2} du$

change terminals

when $x = \frac{3}{2}$ $u = 1$

and when $x = 0$ $u = 0$

$A = \frac{9}{2} \int_0^1 \cos^{-1}(u) du$ from i.

$A = \frac{9}{2} \left[u \cos^{-1}(u) - \sqrt{1-u^2} \right]_0^1$

$A = \frac{9}{2} \left[(1 \cos^{-1} 1 - 0) - (0 - \sqrt{1}) \right]$

$A = \frac{9}{2} = 4.5 \text{ m}^2$ (exactly)

as a check only



Question 2

a.

i $T = \{z : 3\operatorname{Re}(z) - 4\operatorname{Im}(z) = 25\}$

Let $z = x + iy$ $\operatorname{Re}(z) = x$ and $\operatorname{Im}(z) = y$

So T is the line $3x - 4y = 25$ or $y = \frac{3x}{4} - \frac{25}{4}$

This line has a gradient of $m = \frac{3}{4}$ and intersects the real axis (x -axis) at $\left(8\frac{1}{3}, 0\right)$

and the imaginary axis (y -axis) at $\left(0, -6\frac{1}{4}\right)$

Now $U = \{z : |z| = |z - 6 + 8i|\}$ Let $z = x + iy$

$$|x + iy| = |(x - 6) + i(y + 8)|$$

$$\sqrt{x^2 + y^2} = \sqrt{(x - 6)^2 + (y + 8)^2} \quad \text{squaring both sides}$$

$$x^2 + y^2 = (x - 6)^2 + (y + 8)^2 \quad \text{expanding}$$

$$x^2 + y^2 = x^2 - 12x + 36 + y^2 + 16y + 64$$

$$12x - 16y = 100$$

$$3x - 4y = 25$$

so $T = U$

ii. $S = \{z : |z - 3 + 4i| = 5\}$ Let $z = x + iy$

$$|(x - 3) + i(y + 4)| = 5$$

$$\sqrt{(x - 3)^2 + (y + 4)^2} = 5$$

$$(x - 3)^2 + (y + 4)^2 = 25$$

S is the circle with centre $(3, -4)$ and radius 5

$$R = \{z : (z - 3 + 4i)(\bar{z} - 3 - 4i) = 25\}$$

$$= \{z : (z - c)(\bar{z} - \bar{c}) = 25\}$$

with $c = 3 - 4i$ so that $\bar{c} = 3 + 4i$ and $c\bar{c} = 9 - 16i^2 = 25$

Now $(z - c)(\bar{z} - \bar{c}) = 25$ expanding becomes

$$z\bar{z} - z\bar{c} - \bar{z}c + c\bar{c} = 25 \quad \text{with } z = x + iy \quad \bar{z} = x - iy \quad \text{and } z\bar{z} = x^2 + y^2$$

$$x^2 + y^2 - (x + iy)(3 + 4i) - (x - iy)(3 - 4i) + 25 = 25$$

$$x^2 + y^2 - [3x + 3iy + 4ix + 4i^2y] - [3x - 3iy - 4ix + 4i^2y] = 0$$

$$x^2 + y^2 - 6x + 8y = 0 \quad \text{completing the squares}$$

$$x^2 - 6x + 9 + y^2 + 8y + 16 = 25$$

$$(x - 3)^2 + (y + 4)^2 = 25 \quad \text{is the circle with centre } (3, -4) \text{ and radius } 5$$

so $S = R$

iii. Let $z_A = 7 - i$ substituting z_A into $S = \{z : |z - 3 + 4i| = 5\}$

$$|(7 - i) - (3 - 4i)| = |4 + 3i| = 5 \quad \text{so } z_A \text{ lies on } S$$

$$z_A = 7 - i \text{ substituting } z_A \text{ into } T = \{z : 3\operatorname{Re}(z) - 4\operatorname{Im}(z) = 25\}$$

$$3 \times 7 - 4 \times (-1) = 21 + 4 = 25 \quad \text{so } z_A \text{ lies on } T$$

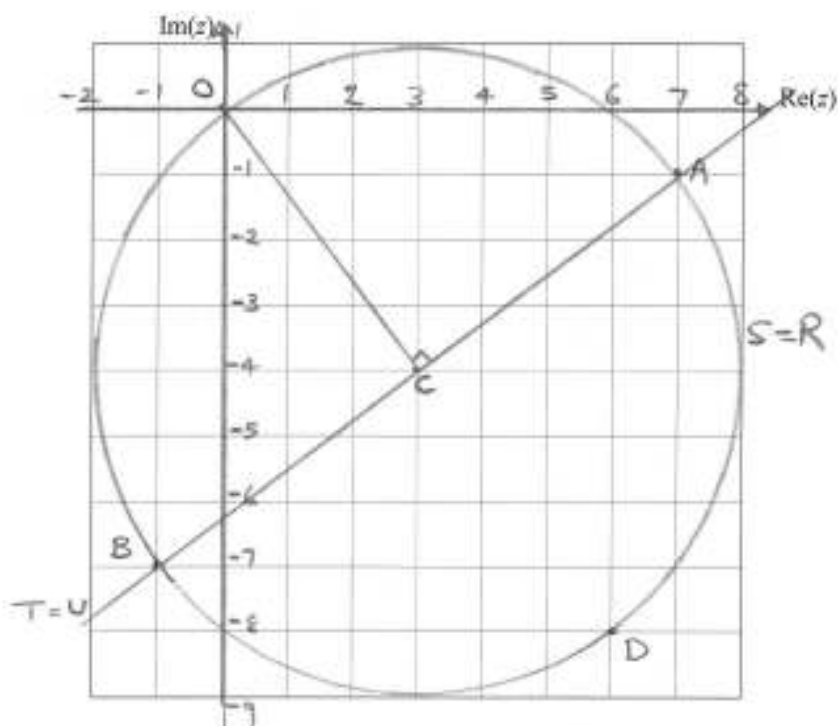
Let $z_B = -1 - 7i$ substituting z_B into $S = \{z : |z - 3 + 4i| = 5\}$

$$|(-1 - 7i) - (3 - 4i)| = |-4 - 3i| = 5 \quad \text{so } z_B \text{ lies on } S$$

$$z_B = -1 - 7i \text{ substituting } z_B \text{ into } T = \{z : 3\operatorname{Re}(z) - 4\operatorname{Im}(z) = 25\}$$

$$3 \times (-1) - 4 \times (-7) = -3 + 28 = 25 \quad \text{so } z_B \text{ lies on } T$$

iv.



- v. $O(0,0)$ $A(7,-1)$ $B(-1,-7)$ $C(3,-4)$
 $\vec{OA} = 7\hat{i} - \hat{j}$ $\vec{OB} = -\hat{i} - 7\hat{j}$ $\vec{OC} = 3\hat{i} - 4\hat{j}$
 $\vec{AC} = \vec{OC} - \vec{OA} = -4\hat{i} - 3\hat{j}$
 $\vec{AC} \cdot \vec{OC} = -12 + 12 = 0$ so $\vec{OC}$ is perpendicular to $\vec{AC}$
 $\vec{AB} = \vec{OB} - \vec{OA}$
 $= -8\hat{i} - 6\hat{j}$
 $= 2(-4\hat{i} - 3\hat{j})$
 $\vec{AB} = 2\vec{AC}$
 so A, B, C are collinear

Now $\left| \vec{AB} \right| = 10$ it is the diameter of the circle
and $\left| \vec{AC} \right| = 5$ it is the radius of the circle.

vi. It is the set of points equidistant from both O and D

b. $\underline{r}(t) = (3 + 5 \cos(2t))\underline{i} + (-4 + 5 \sin(2t))\underline{j} \quad t \geq 0$

i. $x = 3 + 5 \cos(2t) \quad y = -4 + 5 \sin(2t)$

$$\cos(2t) = \frac{x-3}{5} \quad \sin(2t) = \frac{y+4}{5}$$

$$\sin^2(2t) + \cos^2(2t) = 1$$

$$\frac{(y+4)^2}{25} + \frac{(x-3)^2}{25} = 1$$

$$\Rightarrow (x-3)^2 + (y+4)^2 = 25$$

so P moves on S

ii. $\dot{\underline{r}}(t) = -10 \sin(2t)\underline{i} + 10 \cos(2t)\underline{j}$

$$\left| \dot{\underline{r}}(t) \right| = \sqrt{100 \sin^2(2t) + 100 \cos^2(2t)}$$

$$\left| \dot{\underline{r}}(t) \right| = \sqrt{100(\sin^2(2t) + \cos^2(2t))}$$

$$\left| \dot{\underline{r}}(t) \right| = \sqrt{100}$$

$$\left| \dot{\underline{r}}(t) \right| = 10$$

momentum $\underline{p} = m \left| \dot{\underline{r}}(t) \right|$

$$= 2 \times 10$$

$$= 20 \text{ kg m/s}$$

Question 3

a. no air resistance

i. $s = -100$ $a = -9.8$ $u = 0$ $t = ?$

$$s = ut + \frac{1}{2}at^2$$

$$-100 = 0 - 4.9t^2$$

$$t = \sqrt{\frac{100}{4.9}} = 4.52 \text{ sec}$$

ii. $v^2 = u^2 + 2as$

$$v^2 = 0 + 2 \times -9.8 \times -100$$

$$= 1960$$

$$v = \pm\sqrt{1960}$$

downward speed is 44.27 m/s

b. with air resistance

i. by Newton's Second Law of Motion

$$5a = 5g - 0.01v^2 \quad \text{using} \quad a = v \frac{dv}{dx}$$

$$v \frac{dv}{dx} = \frac{49 - 0.01v^2}{5} \quad \text{shown}$$

ii. $v \frac{dv}{dx} = \frac{49 - 0.01v^2}{5}$

$$\frac{dv}{dx} = \frac{49 - 0.01v^2}{5v} \quad \text{inverting both sides}$$

$$\frac{dx}{dv} = \frac{5v}{49 - 0.01v^2} \quad \text{integrating with respect to } v$$

$$x = \int \frac{5v dv}{49 - 0.01v^2} = -\frac{5}{0.02} \log_e (49 - 0.01v^2) + C$$

$$\text{but when } x = 0 \quad v = 0 \quad \Rightarrow \quad C = \frac{5}{0.02} \log_e 49$$

$$x = \frac{5}{0.02} \log_e 49 - \frac{5}{0.02} \log_e (49 - 0.01v^2) = 250 \log_e \left(\frac{49}{49 - 0.01v^2} \right)$$

$$0.01v^2$$



$$5g$$

iii. Now the hammer hits the ground when $x = 100$

$$100 = 250 \log_e \left(\frac{49}{49 - 0.01v^2} \right)$$

$$e^{\frac{10}{25}} = \frac{49}{49 - 0.01v^2}$$

$$49 - 0.01v^2 = 49e^{-\frac{10}{25}}$$

$$0.01v^2 = 49 - 49e^{-\frac{10}{25}}$$

$$v = \sqrt{4900 \left(1 - e^{-\frac{10}{25}} \right)} = 40.1924$$

$$v = 40.19 \text{ m/s}$$

iv. $a = \frac{dv}{dt} = \frac{49 - 0.01v^2}{5} \quad \frac{dt}{dv} = \frac{5}{49 - 0.01v^2}$

$$t = \int \frac{5 \, dv}{49 - 0.01v^2}$$

by partial fractions $\frac{5}{49 - 0.01v^2} = \frac{A}{7 + 0.1v} + \frac{B}{7 - 0.1v}$

$$\frac{5}{49 - 0.01v^2} = \frac{A(7 - 0.1v) + B(7 + 0.1v)}{49 - 0.01v^2} = \frac{7(A + B) + 0.1v(B - A)}{49 - 0.01v^2}$$

so that $7(A + B) = 5$ and $A - B = 0$ so $A = B = \frac{5}{14}$

$$t = \frac{5}{14} \int \left(\frac{1}{7 - 0.1v} + \frac{1}{7 + 0.1v} \right) dv$$

$$t = \frac{5}{14} \left[\frac{1}{0.1} \left[-\log_e(7 - 0.1v) + \log_e(7 + 0.1v) \right] \right] + C$$

but when $t = 0$ $v = 0 \Rightarrow C = 0$

$$t = \frac{50}{14} \log_e \left(\frac{7 + 0.1v}{7 - 0.1v} \right)$$

$$t = \frac{25}{7} \log_e \left(\frac{70 + v}{70 - v} \right)$$

v. when $T = ?$ when $v = 40.1924$

$$t = \frac{25}{7} \log_e \left(\frac{70+v}{70-v} \right)$$

$$t = \frac{25}{7} \log_e \left(\frac{70+40.1924}{70-40.1924} \right)$$

$t = 4.67$ sec (correct to two decimal places)

vi. now transposing $t = \frac{25}{7} \log_e \left(\frac{70+v}{70-v} \right)$ to make v the subject

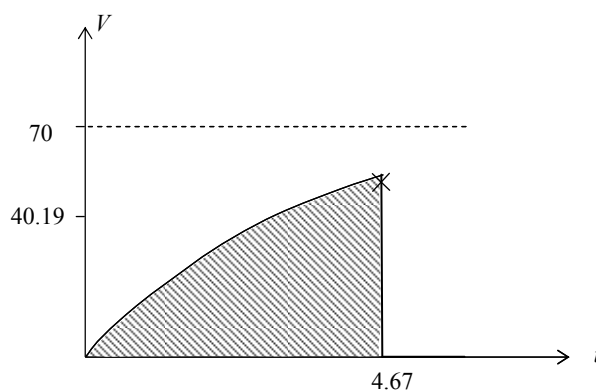
$$e^{-0.28t} = \frac{70-v}{70+v} \quad (70+v)e^{-0.28t} = 70-v$$

$$70e^{-0.28t} + ve^{-0.28t} = 70-v \quad v + ve^{-0.28t} = 70 - 70e^{-0.28t}$$

$$v(1 + e^{-0.28t}) = 70(1 - e^{-0.28t})$$

$$v = v(t) = \frac{70(1 - e^{-0.28t})}{1 + e^{-0.28t}} \quad 0 \leq t \leq T$$

vii.



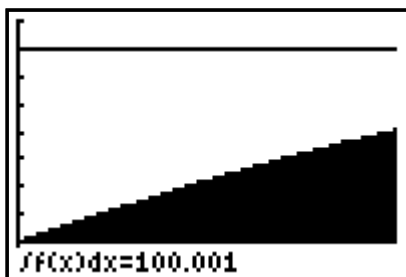
as check only

```

4.67
Plot1 Plot2 Plot3
\Y1=0*(1-e^(-0.28X))/(1+e^(-0.28X))
\Y2=70
\Y3=
\Y4=
\Y5=
    
```

```

WINDOW
Xmin=
Xmax=4.67
Xscl=1
Ymin=-10
Ymax=80
Yscl=10
Xres=1
    
```



Question 4

i. $\underline{r} \cdot \underline{k} = 0 \Rightarrow 4 + 4 \sin(\pi t) = 0 \text{ or } 4 \sin(\pi t) = -4$

$$\sin \pi t = -1$$

$$\pi t = \frac{3\pi}{2}$$

$$t = \frac{3}{2}$$

ii. $\dot{\underline{r}}(t) = 10\underline{i} + 90\underline{j} + 4\pi \cos(\pi t)\underline{k}$

$$\dot{\underline{r}}(0) = 10\underline{i} + 90\underline{j} + 4\pi\underline{k}$$

$$|\dot{\underline{r}}(0)| = \sqrt{10^2 + 90^2 + 16\pi^2}$$

$$|\dot{\underline{r}}(0)| = 91.42 \text{ m/s}$$

now the angle α at which it is hit is given by $\tan \alpha = \frac{4\pi}{\sqrt{10^2 + 90^2}}$

$$\alpha = \text{Tan}^{-1}\left(\frac{4\pi}{\sqrt{10^2 + 90^2}}\right) = 7.9^\circ$$

so $\alpha = 8^\circ$ to the nearest degree.

iii. $\underline{r}\left(\frac{3}{2}\right) = 15\underline{i} + 135\underline{j}$

$$\left|\underline{r}\left(\frac{3}{2}\right)\right| = \sqrt{15^2 + 135^2}$$

$$\left|\underline{r}\left(\frac{3}{2}\right)\right| = 136 \text{ m}$$

iv. at max height $4 \sin \pi t + 4 = 8$ when

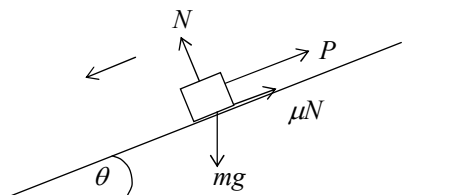
$$\sin(\pi t) = 1$$

$$\pi t = \frac{\pi}{2}$$

$$t = \frac{1}{2}$$

Now $\underline{r}\left(\frac{1}{2}\right) = 5\underline{i} + 45\underline{j} + 8\underline{k}$

i.



ii. resolving perpendicular to the plane $N - mg \cos \theta = 0$

so that $N = mg \cos \theta$ (1)

resolving up and parallel to the plane $P + \mu N - mg \sin \theta = 0$

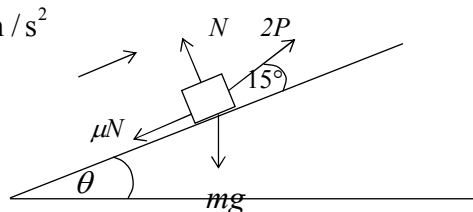
$$P = mg \sin \theta - \mu N$$

$$P = mg \sin \theta - \mu mg \cos \theta \quad \text{from (1) and } \mu = 0.5$$

$$P = mg(\sin \theta - 0.5 \cos \theta)$$

iii.

$$a = 1 \text{ m/s}^2$$



$$\mu = 0.5$$

iv. resolving perpendicular to the plane

$$N + 2P \sin 15^\circ - mg \cos \theta = 0 \quad (3)$$

resolving up and parallel to the plane, using Newton's Second Law of Motion

$$2P \cos 15^\circ - \mu N - mg \sin \theta = ma \quad (4)$$

from (3) $N = mg \cos \theta - 2P \sin 15^\circ$ into (4)

$$2P \cos 15^\circ - \mu(mg \cos \theta - 2P \sin 15^\circ) - mg \sin \theta = ma$$

$$2P(\cos 15^\circ + \mu \sin 15^\circ) - mg(\sin \theta + \mu \cos \theta) = ma$$

but from i.

$$P = mg(\sin\theta - 0.5 \cos\theta) \quad \text{and} \quad \mu = 0.5 \quad a = 1$$

$$2mg(\sin\theta - 0.5 \cos\theta)(\cos 15^\circ + 0.5 \sin 15^\circ) - mg(\sin\theta + 0.5 \cos\theta) = 1m$$

$$2g(\sin\theta - 0.5 \cos\theta)(\cos 15^\circ + 0.5 \sin 15^\circ) - g(\sin\theta + 0.5 \cos\theta) = 1$$

$$21.4686(\sin\theta - 0.5 \cos\theta) - 9.8(\sin\theta + 0.5 \cos\theta) = 1$$

$$(21.4686 - 9.8)\sin\theta - 0.5(21.4686 + 9.8)\cos\theta = 1$$

$$11.669\sin\theta - 15.634\cos\theta = 1$$

v. $11.669\sin\theta - 15.634\cos\theta = 1$

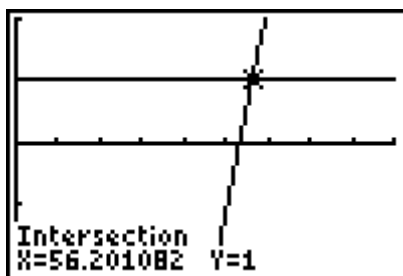
solving this equation on the TI-83 with the calculator in the DEGREES mode.

```

Plot1 Plot2 Plot3
\Y1=11.669sin(X)
-15.634cos(X)
\Y2=1
\Y3=
\Y4=
\Y5=
\Y6=
    
```

```

WINDOW
Xmin=
Xmax=90
Xscl=10
Ymin=-2
Ymax=2
Yscl=1
Xres=1
    
```



$$\theta = 56.2$$

$$\theta = 56^\circ$$

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