

Question 1

- a. The highest point vertically above the base of the crane when the arm is inclined at its maximum angle is 5 metres. **(1 mark)**

The $\hat{k}$ coordinate or the vertical component for each of the given points is 5 and we

are told that the points A , B , C and D are on the outer ring of the cone. Hence the highest point when the arm is inclined at its maximum angle lies anywhere on this circle. **(1 mark)**

- b. Since the edge of the solid shape is a circle, the length of the crane when fully extended is given by $|\vec{OA}|$ or $|\vec{OB}|$ etc.

$$\vec{OA} = 3\hat{i} + 5\hat{k}$$

$$\begin{aligned}\text{So } |\vec{OA}| &= \sqrt{9 + 25} \\ &= \sqrt{34} \text{ metres}\end{aligned}$$

(1 mark)

- c. i.

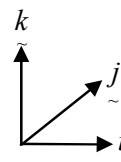
$$\begin{aligned}\vec{AB} &= \vec{AO} + \vec{OB} \\ &= -3\hat{i} - 5\hat{k} - \hat{i} - 2\sqrt{2}\hat{j} + 5\hat{k} \\ &= -4\hat{i} - 2\sqrt{2}\hat{j}\end{aligned}$$

$$\begin{aligned}\vec{AD} &= \vec{AO} + \vec{OD} \\ &= -3\hat{i} - 5\hat{k} + \hat{i} + 2\sqrt{2}\hat{j} + 5\hat{k} \\ &= -2\hat{i} + 2\sqrt{2}\hat{j}\end{aligned}$$

$$\begin{aligned}\text{So } \vec{AB} \cdot \vec{AD} &= 8 - 8 \\ &= 0\end{aligned}$$

(1 mark)

(1 mark)



- ii. From part i. $\angle BAD = 90^\circ$.
So points BAD lie on a semicircle with BD the diameter. (1 mark)

$$\begin{aligned}\text{Also, } \vec{BC} &= \vec{BO} + \vec{OC} \\ &= \underline{i} + 2\sqrt{2}\underline{j} - 5\underline{k} - 2\underline{i} - \sqrt{5}\underline{j} + 5\underline{k} \\ &= -\underline{i} + (2\sqrt{2} - \sqrt{5})\underline{j}\end{aligned}$$

$$\begin{aligned}\text{and } \vec{CD} &= \vec{CO} + \vec{OD} \\ &= 2\underline{i} + \sqrt{5}\underline{j} - 5\underline{k} + \underline{i} + 2\sqrt{2}\underline{j} + 5\underline{k} \\ &= 3\underline{i} + (\sqrt{5} + 2\sqrt{2})\underline{j}\end{aligned}$$

$$\begin{aligned}\text{So, } \vec{BC} \cdot \vec{CD} &= -3 + (2\sqrt{2} - \sqrt{5}) \times (\sqrt{5} + 2\sqrt{2}) \\ &= -3 + 2\sqrt{10} + 8 - 5 - 2\sqrt{10} \\ &= 0\end{aligned}$$

- So $\angle BCD = 90^\circ$
So points BCD lie on a semicircle with BD the diameter. (1 mark)
Since BCD and BAD lie on the same plane, A, B, C, D lie on a circle. (1 mark)

d. $\vec{OD} = \underline{i} + 2\sqrt{2}\underline{j} + 5\underline{k}$

and $\vec{OE} = -\underline{i} - \underline{j} + 4\underline{k}$

$$\begin{aligned}\text{So } \vec{OD} \cdot \vec{OE} &= -1 - 2\sqrt{2} + 20 \\ &= 19 - 2\sqrt{2}\end{aligned} \quad (1 \text{ mark})$$

Also $\vec{OD} \cdot \vec{OE} = |\vec{OD}| |\vec{OE}| \cos \theta$

$$\begin{aligned}\text{So } 19 - 2\sqrt{2} &= \sqrt{1 + 8 + 25} \sqrt{1 + 1 + 16} \cos \theta \\ \cos \theta &= 0.6536 \dots \\ \theta &= 49^\circ 11' \text{ (to the nearest minute)}\end{aligned}$$

(1 mark)

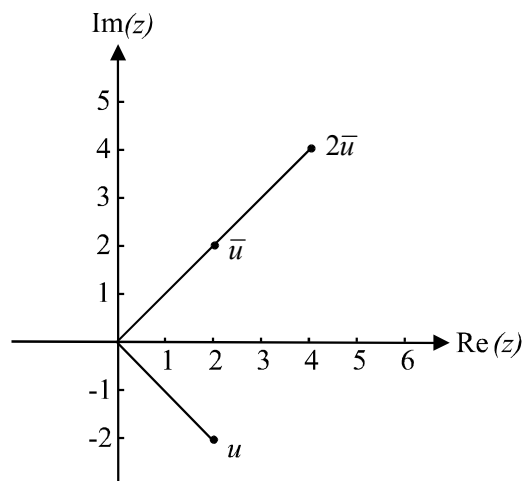
The arm of the crane moves through an angle of $49^\circ 11'$.

(1 mark)

Total 11 marks**Question 2**

a. i. $u = 2 - 2i$ so $\bar{u} = 2 + 2i$
 $u\bar{u} = (2 - 2i)(2 + 2i)$
 $= 4 + 4$
 $= 8$ (1 mark)

ii. $\text{Arg} u + \text{Arg}(2\bar{u})$
 $= \frac{-\pi}{4} + \frac{\pi}{4}$
 $= 0$ (1 mark)



b. i. If u is a solution then,

$$(2 - 2i)^4 - 6(2 - 2i)^3 + 19(2 - 2i)^2 - 28(2 - 2i) + 24 = 0$$

$$LS = -8i \times -8i - 6 \times -8i(2 - 2i) + 19 \times -8i - 56 + 56i + 24$$

$$= -64 + 96i + 96 - 152i - 56 + 56i + 24$$

$$= 0$$

$$= RS$$

Have shown

(1 mark)

ii. Since the coefficients of the *LHS* of the equation are real, $2 + 2i$ is also a solution (conjugate root theorem).

Now, $(z - 2 - 2i)(z - 2 + 2i)$

$$= z^2 - 2z + 2iz - 2z + 4 - 4i - 2iz + 4i + 4$$

$$= z^2 - 4z + 8 \text{ which is a quadratic factor.}$$

(1 mark)

$$\begin{array}{r} z^2 - 2z + 3 \\ z^2 - 4z + 8 \overline{) z^4 - 6z^3 + 19z^2 - 28z + 24} \\ \underline{z^4 - 4z^3 + 8z^2} \\ -2z^3 + 11z^2 - 28z \\ \underline{-2z^3 + 8z^2 - 16z} \\ 3z^2 - 12z + 24 \\ \underline{3z^2 - 12z + 24} \\ 0 \end{array}$$

Let $p(z) = z^4 - 6z^3 + 19z^2 - 28z + 24$

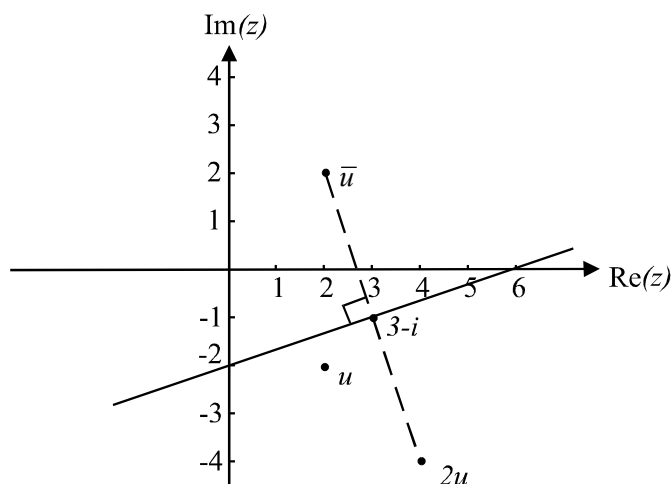
(1 mark)

$$\begin{aligned} \text{So } p(z) &= (z^2 - 4z + 8)(z^2 - 2z + 3) \\ &= (z - 2 - 2i)(z - 2 + 2i)(z^2 - 2z + 1 - 1 + 3) \\ &= (z - 2 - 2i)(z - 2 + 2i)(z - 1 - \sqrt{2}i)(z - 1 + \sqrt{2}i) \end{aligned}$$

The other three solutions are $z = 2 + 2i, 1 \pm \sqrt{2}i$

(1 mark)

c. Method 1
Graphing.



(1 mark)

z is any complex number such that the distance from z to $\bar{u}$ equals the distance from z to $2u$.

From the graph, we see that all such numbers z lie on a straight line shown on the Argand diagram.

This line crosses the Real axis at 6. So when $y = 0, x = 6$.

(1 mark)

Method 2

Algebraically.

$$\begin{aligned} |z - \bar{u}| &= |z - 2u| \\ |x + yi - 2 - 2i| &= |x + yi - 4 + 4i| \\ \sqrt{(x-2)^2 + (y-2)^2} &= \sqrt{(x-4)^2 + (y+4)^2} \\ x^2 - 4x + 4 + y^2 - 4y + 4 &= x^2 - 8x + 16 + y^2 + 8y + 16 \\ 4x - 12y &= 24 \\ x - 3y &= 6 \end{aligned} \quad \text{(1 mark)}$$

When $y = 0, x = 6$

(1 mark)

d.

$$\begin{aligned} u^n + (\bar{u})^n &= 0 \\ (2 - 2i)^n + (2 + 2i)^n &= 0 \\ \left(2\sqrt{2}\operatorname{cis}\left(\frac{-\pi}{4}\right)\right)^n + \left(2\sqrt{2}\operatorname{cis}\left(\frac{\pi}{4}\right)\right)^n &= 0 \\ (2\sqrt{2})^n \operatorname{cis}\left(\frac{-\pi n}{4}\right) + (2\sqrt{2})^n \operatorname{cis}\left(\frac{\pi n}{4}\right) &= 0 \quad \text{De Moivre's Theorem} \end{aligned} \quad \text{(1 mark)}$$

(1 mark)

since $2\sqrt{2}^n \neq 0$

$$\begin{aligned} \cos\left(\frac{-\pi n}{4}\right) + i \sin\left(\frac{-\pi n}{4}\right) + \cos\left(\frac{\pi n}{4}\right) + i \sin\left(\frac{\pi n}{4}\right) &= 0 \\ \cos\left(\frac{\pi n}{4}\right) - i \sin\left(\frac{\pi n}{4}\right) + \cos\left(\frac{\pi n}{4}\right) + i \sin\left(\frac{\pi n}{4}\right) &= 0 \\ 2 \cos\left(\frac{\pi n}{4}\right) &= 0 \end{aligned} \quad \text{(1 mark)}$$

$$\begin{aligned} \frac{\pi n}{4} &= \dots, \frac{-\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \dots \\ &= \frac{(2k+1)\pi}{2}, \quad k \in J \end{aligned}$$

$$\frac{\pi n}{4} = \frac{(2k+1)\pi}{2}$$

$$\text{So,} \quad n = 2(2k+1), \quad k \in J$$

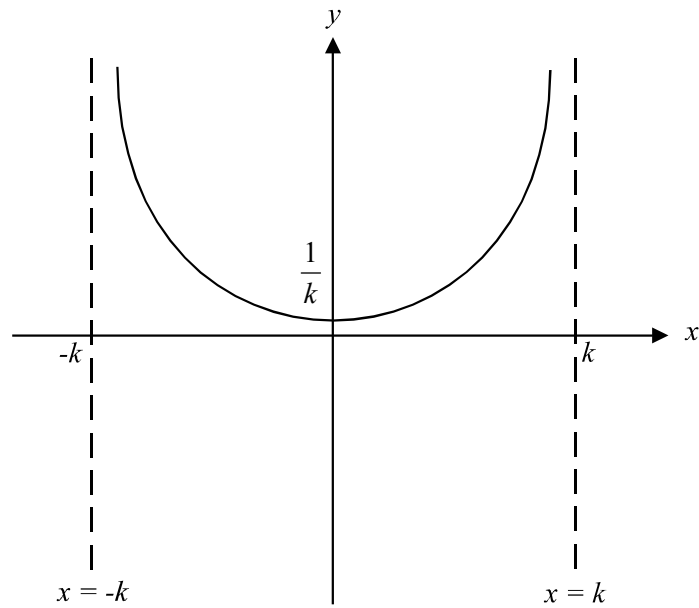
$$\text{or} \quad n = \dots, -2, 2, 6, \dots$$

(1 mark)

Total 12 marks

Question 3

a. i.



$$y = \frac{1}{\sqrt{k^2 - x^2}}$$

when $x = 0$,

$$\begin{aligned} y &= \frac{1}{\sqrt{k^2 - 0}} \\ &= \frac{1}{k} \end{aligned}$$

(1 mark) correct shape
(1 mark) labelling clearly
 asymptotes and y-intercept

ii. From the graph, $r_f = \left[\frac{1}{k}, \infty \right)$.

(1 mark)

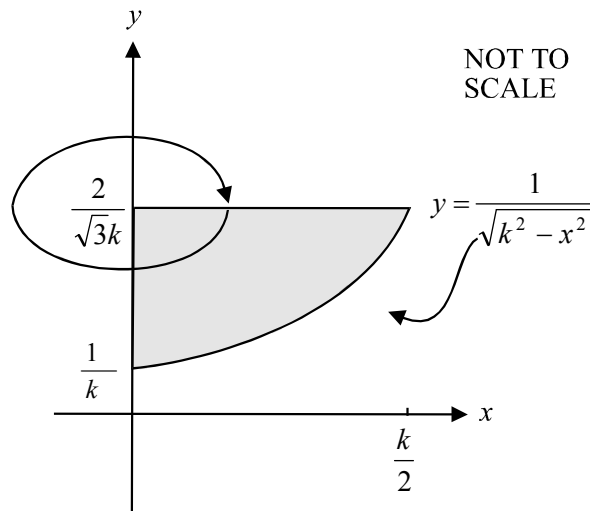
b. Area required = $\int_0^{\frac{k}{2}} \frac{1}{\sqrt{k^2 - x^2}} dx$

(1 mark)

$$\begin{aligned} &= \left[\sin^{-1} \left(\frac{x}{k} \right) \right]_0^{\frac{k}{2}} \\ &= \sin^{-1} \left(\frac{k}{2k} \right) - \sin^{-1}(0) \\ &= \sin^{-1} \left(\frac{1}{2} \right) \\ &= \frac{\pi}{6} \text{ square units} \end{aligned}$$

(1 mark)

- c. Since we are rotating about the y -axis, the terminals of integration will lie on the y -axis and the volume of rotation, V , is given by $V = \pi \int x^2 dy$.



One of the terminals is $\frac{2}{\sqrt{3}k}$ (given in the question).

To find the other let $x = 0$.

So,

$$y = \frac{1}{\sqrt{k^2 - 0}}$$

$$= \frac{1}{k}$$

(1 mark) – correct terminals

For the integrand,

$$y = \frac{1}{\sqrt{k^2 - x^2}}$$

$$\sqrt{k^2 - x^2} = \frac{1}{y}$$

$$k^2 - x^2 = \frac{1}{y^2}$$

$$x^2 = k^2 - \frac{1}{y^2}$$

$$x^2 = k^2 - y^{-2}$$

(1 mark) – correct integrand

So required volume

$$= \pi \int_{\frac{1}{k}}^{\frac{2}{\sqrt{3}k}} (k^2 - y^{-2}) dy$$

(1 mark) – correct form
i.e. $\pi \int x^2 dy$

$$= \pi \left[k^2 y + y^{-1} \right]_{\frac{1}{k}}^{\frac{2}{\sqrt{3}k}}$$

$$= \pi \left\{ \left(k^2 \times \frac{2}{\sqrt{3}k} + \frac{\sqrt{3}k}{2} \right) - \left(k^2 \times \frac{1}{k} + k \right) \right\} \quad \textbf{(1 mark)}$$

$$= \pi \left(\frac{2k}{\sqrt{3}} + \frac{\sqrt{3}k}{2} - 2k \right)$$

$$= \pi \left(\frac{4k + 3k - 4\sqrt{3}k}{2\sqrt{3}} \right)$$

$$= \frac{\pi k}{2\sqrt{3}} (7 - 4\sqrt{3})$$

$$= \frac{\sqrt{3}\pi k}{6} (7 - 4\sqrt{3}) \quad \text{cubic units}$$

(1 mark)

Total 10 marks

Question 4

- a. The pump was repaired when $\frac{dv}{dt} = 50$.

$$\begin{aligned}\text{So } 50 &= \frac{500}{(t+1)(10-t)} \\ (t+1)(10-t) &= 10 \\ -t^2 + 9t + 10 &= 10 \\ -t(t-9) &= 0 \\ t &= 0 \text{ or } t = 9 \\ \text{So, } a &= 9.\end{aligned}$$

(1 mark)

b.
$$\begin{aligned}\frac{dv}{dt} &= \frac{500}{(t+1)(10-t)} \\ &= \frac{500}{(-t^2 + 9t + 10)} \\ \frac{d^2v}{dt^2} &= \frac{(-t^2 + 9t + 10) \times 0 - 500(-2t + 9)}{(-t^2 + 9t + 10)^2} \\ &= \frac{-500(-2t + 9)}{(-t^2 + 9t + 10)^2}\end{aligned}$$

(1 mark)

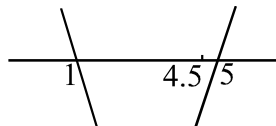
$$\begin{aligned}\text{max/min when } \frac{d^2v}{dt^2} &= 0 \\ -500(-2t + 9) &= 0 \\ t &= 4.5\end{aligned}$$

(1 mark)

Check max/min

$$\text{When } t=1, \quad \frac{d^2v}{dt^2} = -10 \cdot 8$$

$$\text{When } t=5, \quad \frac{d^2v}{dt^2} = 0 \cdot 5$$



We have a min at 4.5.

(1 mark)

So the minimum rate at which the pump was pumping during its period of

$$\text{malfunction was } \frac{500}{5 \cdot 5 \times 5 \cdot 5} = 16 \cdot 53 \text{ litres per hour (correct to 2 decimal places).}$$

(1 mark)

c. For $\int \frac{500}{(t+1)(10-t)} dt$ we need to break up the fraction using partial fractions.

$$\text{Let } \frac{500}{(t+1)(10-t)} \equiv \frac{A}{t+1} + \frac{B}{10-t}$$

$$\equiv \frac{A(10-t) + B(t+1)}{(t+1)(10-t)} \quad (1 \text{ mark})$$

$$\text{True iff } 500 \equiv A(10-t) + B(t+1)$$

$$\text{Put } t = 10,$$

$$500 \equiv 11B, \quad B = \frac{500}{11}$$

$$\text{Put } t = -1,$$

$$500 \equiv 11A, \quad A = \frac{500}{11}$$

(1 mark)

$$\text{So } \frac{500}{(t+1)(10-t)} \equiv \frac{500}{11(t+1)} + \frac{500}{11(10-t)}$$

$$\text{So, } \int \frac{500}{(t+1)(10-t)} dt = \frac{500}{11} \int \left(\frac{1}{t+1} + \frac{1}{10-t} \right) dt$$

$$= \frac{500}{11} (\log_e(t+1) - \log_e(10-t)) + c$$

$$= \frac{500}{11} \log_e \left(\frac{t+1}{10-t} \right)$$

(because we are asked for an antiderivative.)

(1 mark)

d. The volume of water in litres, that the malfunctioning pump had pumped t hours after the malfunction began is given by

$$v = \int \frac{dv}{dt} dt$$

$$= \int \frac{500}{(t+1)(10-t)} dt$$

$$= \frac{500}{11} \log_e \left(\frac{t+1}{10-t} \right) + c \quad \text{from part c.}$$

$$\text{So } v = \frac{500}{11} \log_e \left(\frac{t+1}{10-t} \right) + c$$

(1 mark)

In this case, when $t = 0$, $v = 0$, that is, the malfunctioning pump hadn't yet pumped any water when $t = 0$ (the functioning pump had though, but we are only interested in the period of malfunction – i.e. domains are crucial).

$$0 = \frac{500}{11} \log_e \left(\frac{1}{10} \right) + c$$

$$\text{So } v = \frac{500}{11} \log_e \left(\frac{t+1}{10-t} \right) - \frac{500}{11} \log_e \left(\frac{1}{10} \right)$$

$$= \frac{500}{11} \log_e \left(\frac{10(t+1)}{10-t} \right) \quad \text{as required} \quad (1 \text{ mark})$$

(Alternatively, you could use the definite integral $v = \int_0^t \frac{dv}{dt} dt$.)

- e. The volume of water pumped by the malfunctioning pump after 2 hours is given by

$$v = \frac{500}{11} \log_e \frac{10(2+1)}{10-2}$$

$$= 60.07981091\dots$$

(1 mark)

If the pump had been operating normally, it would have pumped
 2×50 litres = 100 litres . So there were an extra 39.92 litres (correct to 2 decimal places) in the mine.

(1 mark)

- f. The additional water is the difference between what is normally pumped out and what can be pumped out during the period of the malfunction.

Let A = volume of additional water.

$$\text{So } A = 50t - \frac{500}{11} \log_e \frac{10(t+1)}{(10-t)}$$

Using a graphics calculator.

Graph $Y_1 = 50t$

$$\text{Graph } Y_2 = \frac{500}{11} \log_e \frac{10(t+1)}{(10-t)}$$

Graph $Y_1 - Y_2$

(1 mark)

Look at the graph and the table of values for $Y_1 - Y_2$.

Note that the largest value of $Y_1 - Y_2$ for $t \in [0,9]$ is 240.67 and this occurs at $t = 9$.

(1 mark)

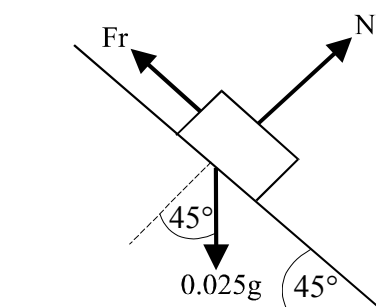
Hence during the period when the pump is malfunctioning the limit of additional water that the mine can cope with is not exceeded.

(1 mark)

Total 15 marks

Question 5

a.



(1 mark)

b. Resolve the forces

$$\underline{R} = m \underline{a}$$

$$(0.025g \sin 45^\circ - Fr)\underline{i} + (N - 0.025g \cos 45^\circ)\underline{j} = 0.025a \underline{i}$$

$$\text{Now } Fr = \mu N$$

$$\text{So, } \frac{0.025g}{\sqrt{2}} - \mu N = 0.025a \text{ and } N = \frac{0.025g}{\sqrt{2}}$$

(1 mark) (1 mark)

$$\frac{0.025g}{\sqrt{2}} - 0.5 \times \frac{0.025g}{\sqrt{2}} = 0.025a$$

$$\begin{aligned} a &= \frac{g}{\sqrt{2}} - \frac{g}{2\sqrt{2}} \\ &= \frac{g}{2\sqrt{2}} \\ &= \frac{\sqrt{2}g}{4} \text{ m/s}^2 \text{ as required} \end{aligned}$$

(1 mark)

c. From part b., we know that a lolly accelerates at a constant rate down the metal chute hence we can use the formula

$$v^2 = u^2 + 2as$$

$$\begin{aligned} \text{So, } v^2 &= \left((2 - \sqrt{2}g)^{\frac{1}{2}} \right)^2 + 2 \times \frac{\sqrt{2}g}{4} \times 2 \\ &= 2 - \sqrt{2}g + \sqrt{2}g \\ &= 2 \\ v &= \sqrt{2} \text{ m/s} \end{aligned}$$

(1 mark) use of correct formula
(1 mark) correct answer

- d. When a lolly drops off the end of the metal chute, it goes into "freefall" where the only force acting on it is the gravitational force (since there are no resistance forces).

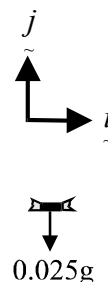
$$\text{So } \underline{\underline{R}} = m \underline{\underline{a}}$$

$$\text{becomes } -0.025g \underline{\underline{j}} = 0.025 \underline{\underline{a}}$$

$$\underline{\underline{a}} = -g \underline{\underline{j}}$$

$$\frac{d\underline{\underline{v}}}{dt} = -g \underline{\underline{j}}$$

$$\underline{\underline{v}} = -gt \underline{\underline{j}} + \underline{\underline{c}}$$



(1 mark)

When $t = 0$, i.e. when the lolly left the end of the metal chute, $v = \sqrt{2}\text{m/s}$

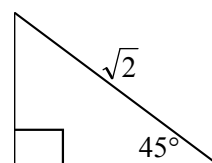
$$\text{and so } \underline{\underline{v}} = \sqrt{2} \cos 45^\circ \underline{\underline{i}} - \sqrt{2} \sin 45^\circ \underline{\underline{j}}$$

$$= \underline{\underline{i}} - \underline{\underline{j}}$$

$$\text{So, } \underline{\underline{v}} = -g \underline{\underline{t}} + \underline{\underline{c}}$$

$$\text{becomes } \underline{\underline{i}} - \underline{\underline{j}} = 0 \times \underline{\underline{j}} + \underline{\underline{c}}$$

$$\text{So, } \underline{\underline{c}} = \underline{\underline{i}} - \underline{\underline{j}}$$



(1 mark) for "initial" conditions

$$\text{So, } \underline{\underline{v}} = -gt \underline{\underline{j}} + \underline{\underline{i}} - \underline{\underline{j}}$$

$$= \underline{\underline{i}} - (1 + gt) \underline{\underline{j}}$$

as required.

(1 mark)

- e. From part d.,

$$\underline{\underline{v}} = \underline{\underline{i}} - (1 + gt) \underline{\underline{j}}$$

where $t = 0$ corresponds to the instant where it left the end of the metal chute and the end of the metal chute corresponds to the origin of motion.

So,

$$\underline{\underline{r}} = t \underline{\underline{i}} - \left(t + \frac{gt^2}{2} \right) \underline{\underline{j}} + \underline{\underline{c}}$$

At $t = 0$,

$$\underline{\underline{r}} = 0 \underline{\underline{i}} + 0 \underline{\underline{j}}$$

$$\text{So, } \underline{\underline{c}} = \underline{\underline{0}}$$

$$\text{So } \underline{\underline{r}} = t \underline{\underline{i}} - \left(t + \frac{gt^2}{2} \right) \underline{\underline{j}}$$

(1 mark)

So, when a lolly lands 0.4m vertically below the origin of motion, (i.e. the end of the metal chute),

$$-\left(t + \frac{gt^2}{2}\right) = -0.4$$

$$\frac{gt^2}{2} + t - 0.4 = 0$$

$$t = 0.2013...$$

(Take the positive value of t)

(1 mark)

Since the horizontal distance from the end of the metal chute is given by t , that horizontal distance is 0.2m (to 1 decimal place).

(1 mark)

Total 12 marks