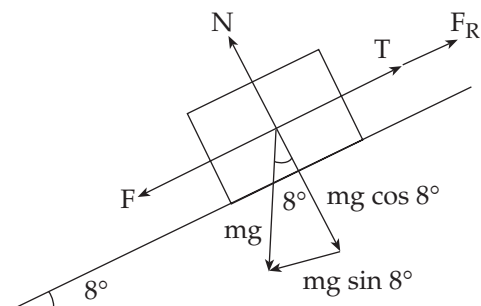


2003 Specialist Mathematics Written Examination 2 (analysis task) Suggested Answers and Solutions

Question 1

a



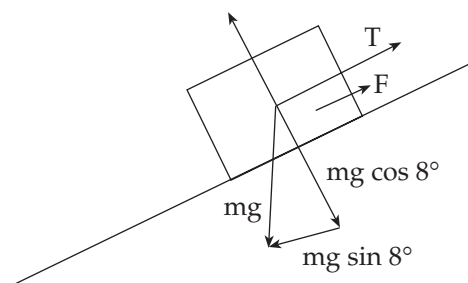
$$\begin{aligned}\text{Resultant force: } F_R &= ma \\ &= 1200 \times 0.25 \\ &= 300\end{aligned}$$

$$F_R = T - F - mg \sin 8^\circ$$

$$\begin{aligned}F &= \mu N \\ &= 0.09 \times 1200 \times 9.8 \cos 8^\circ\end{aligned}$$

$$\begin{aligned}T &= 300 + 1200 \times 9.8 (0.09 \cos 8^\circ + \sin 8^\circ) \\ &= 2985 \text{ N}\end{aligned}$$

b



$$F + T - mg \sin 8^\circ = 0$$

$$\begin{aligned}T &= mg \sin 8^\circ - F \\ &= 1200 \times 9.8 \sin 8^\circ - F \\ &= 1636.68 - F\end{aligned}$$

c i $T = 1636.68 - F$

$$\begin{aligned}T &= 1636.68 - \mu N \\ &= 1636.68 - 0.09 \times 1200 \times 9.8 \cos 8^\circ \\ &= 589\end{aligned}$$

c ii $T = 1636.68 - 0.15 \times 1200 \times 9.8 \cos 8^\circ$
 $= -110$

In this instance $T = 0$ because friction can only act to oppose motion.

Question 2

$$\begin{aligned}\text{a } \cos\left(\frac{\pi}{12}\right) &= \cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right) \\ &= \cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right) \\ &= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2} \\ &= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} \\ &= \frac{\sqrt{2}}{\sqrt{2}} \times \frac{\sqrt{3}}{2\sqrt{2}} + \frac{\sqrt{2}}{\sqrt{2}} \times \frac{1}{2\sqrt{2}} \\ &= \frac{\sqrt{6} + \sqrt{2}}{4}\end{aligned}$$

$$\begin{aligned}\sin\left(\frac{\pi}{12}\right) &= \sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right) \\ &= \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) - \cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right) \\ &= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2} \\ &= \frac{\sqrt{3} - 1}{2\sqrt{2}} \\ &= \frac{\sqrt{6} - \sqrt{2}}{4}\end{aligned}$$

$$\mathbf{b} \quad u = \frac{\sqrt{6} + \sqrt{2}}{4} + \frac{\sqrt{6} - \sqrt{2}}{4}i$$

$$v = \bar{u} = \frac{\sqrt{6} + \sqrt{2}}{4} - \frac{\sqrt{6} - \sqrt{2}}{4}i$$

$$v - u = 0 - 2\left(\frac{\sqrt{6} - \sqrt{2}}{4}\right)i$$

$$\approx 0 - 0.518i$$

$$\text{Arg}(v - u) = -\frac{\pi}{2}$$

$$\mathbf{c} \quad u = \text{cis}\left(\frac{\pi}{12}\right)$$

$$v = \text{cis}\left(-\frac{\pi}{12}\right)$$

$$\frac{v}{u} = \frac{1}{1} \text{cis}\left(-\frac{\pi}{12} - \frac{\pi}{12}\right)$$

$$= \text{cis}\left(-\frac{2\pi}{12}\right)$$

$$= \text{cis}\left(-\frac{\pi}{6}\right)$$

$$\text{Arg}\left(\frac{v}{u}\right) = -\frac{\pi}{6}$$

$$\mathbf{d} \quad (z - u)(z - v) = 0$$

$$z^2 - uz - vz + uv = 0$$

$$z^2 - (u + v)z + uv = 0$$

$$\text{But } z^2 + az + b = 0$$

$$\text{so } b = uv = 1^2 \text{cis}\left(\frac{\pi}{12} - \frac{\pi}{12}\right)$$

$$= 1 \text{cis}0$$

$$= 1(\cos 0 + i \sin 0)$$

$$= 1(1 + 0i)$$

$$= 1$$

$$u = \frac{\sqrt{6} + \sqrt{2}}{4} + \frac{\sqrt{6} - \sqrt{2}}{4}i$$

$$v = \frac{\sqrt{6} + \sqrt{2}}{4} - \frac{\sqrt{6} - \sqrt{2}}{4}i$$

$$-a = u + v = \frac{\sqrt{6} + \sqrt{2}}{4} + \frac{\sqrt{6} + \sqrt{2}}{4} + 0$$

$$= 2 \times \frac{\sqrt{6} + \sqrt{2}}{4}$$

$$= \frac{\sqrt{6} + \sqrt{2}}{2}$$

$$a = \frac{-\sqrt{6} - \sqrt{2}}{2}$$

Question 3

$$\mathbf{a} \quad \overrightarrow{OC} = \underline{c} = -2\underline{i} + 5\underline{j} - 2\underline{k}$$

$$\overrightarrow{OB} = \underline{b} = 2\underline{i} + 6\underline{j} + 2\underline{k}$$

$$\overrightarrow{OA} = \underline{a} = 4\underline{i} + \underline{j} + 4\underline{k}$$

$$\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC}$$

$$= -\underline{b} + \underline{c} = -4\underline{i} - \underline{j} - 4\underline{k}$$

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$$

$$= -\underline{a} + \underline{b} = -2\underline{i} + 5\underline{j} - 2\underline{k}$$

$$\overrightarrow{OC} = \overrightarrow{AB} \text{ and } \overrightarrow{OA} = \overrightarrow{CB}$$

Therefore opposite sides are parallel and equal in length.

$$|\overrightarrow{AB}| = \sqrt{4 + 25 + 4} = \sqrt{33}$$

$$|\overrightarrow{BC}| = \sqrt{16 + 1 + 16} = \sqrt{33}$$

$\therefore$ All sides are of equal length and opposite sides are parallel.

$\therefore$ Base OABC is a rhombus.

$$\mathbf{b} \quad \overrightarrow{OC} = \underline{c} = -2\underline{i} + 5\underline{j} - 2\underline{k}$$

$$\overrightarrow{OA} = \underline{a} = 4\underline{i} + \underline{j} + 4\underline{k}$$

Let $\theta = \angle AOC$

$$\cos \theta = \frac{\underline{a} \cdot \underline{c}}{|\underline{a}| |\underline{c}|}$$

$$= \frac{(-8 + 5 - 8)}{\sqrt{33} \times \sqrt{33}}$$

$$= \frac{-11}{33}$$

$$\theta = \cos^{-1}\left(-\frac{1}{3}\right) = 109.5^\circ$$

c i Let $\underline{e} = p\underline{i} + q\underline{j} + r\underline{k}$

If $\underline{e}$ is a unit vector

$$p^2 + q^2 + r^2 = 1 \quad 1$$

Given $\underline{e}$ is perpendicular to $\overrightarrow{OA}$

We know $\underline{e} \cdot \overrightarrow{OA} = 0$

$$\Rightarrow 4p + q + 4r = 0 \quad 2$$

Given $\underline{e}$ is perpendicular to $\overrightarrow{OC}$

We know $\underline{e} \cdot \overrightarrow{OC} = 0$

$$\Rightarrow -2p + 5q - 2r = 0 \quad 3$$

Solving 2 and 3 simultaneously

$$11q = 0$$

$$\Rightarrow q = 0 \text{ (As required)}$$

$$\therefore p^2 + r^2 = 1 \quad 4$$

$$\text{and } 4p + 4r = 0 \quad 5$$

From 5 $p = -r$

Substituting $p = -r$ into 4

$$p^2 + p^2 = 1$$

$$\Rightarrow p^2 = \frac{1}{2}$$

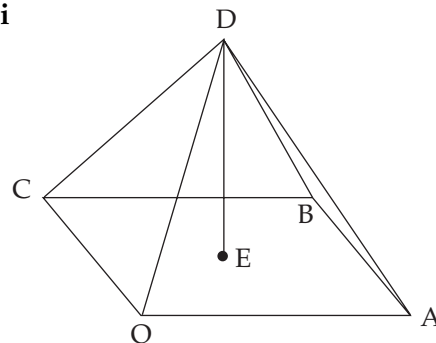
$$p = \pm \frac{1}{\sqrt{2}}$$

Given $p > 0$

$$\therefore p = \frac{1}{\sqrt{2}}$$

$$\text{and } r = -\frac{1}{\sqrt{2}}$$

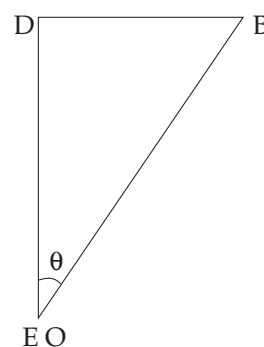
c ii



We know that $\overrightarrow{ED}$ is perpendicular to the base and thus is perpendicular to $\overrightarrow{OC}$ and $\overrightarrow{OA}$.

This suggests that $\overrightarrow{ED}$ is parallel to $\underline{e}$ from previous question.

$$\text{Where } \underline{e} = \frac{1}{\sqrt{2}}(\underline{i} - \underline{k})$$



$$|\overrightarrow{DE}| = |\overrightarrow{OD}| \cos \theta$$

we know that $\underline{e}$ is a unit vector

$$\therefore |\overrightarrow{DE}| = |\overrightarrow{OD}| |\underline{e}| \cos \theta$$

$$\Rightarrow |\overrightarrow{DE}| = |\overrightarrow{OD}| \cdot \underline{e}$$

$$\overrightarrow{OD} \cdot \underline{e} = \left(3\underline{i} + 4\underline{j} - \frac{\underline{k}}{3} \right) \cdot \frac{1}{\sqrt{2}}(\underline{i} - \underline{k})$$

$$= \frac{3}{\sqrt{2}} + \frac{1}{3\sqrt{2}}$$

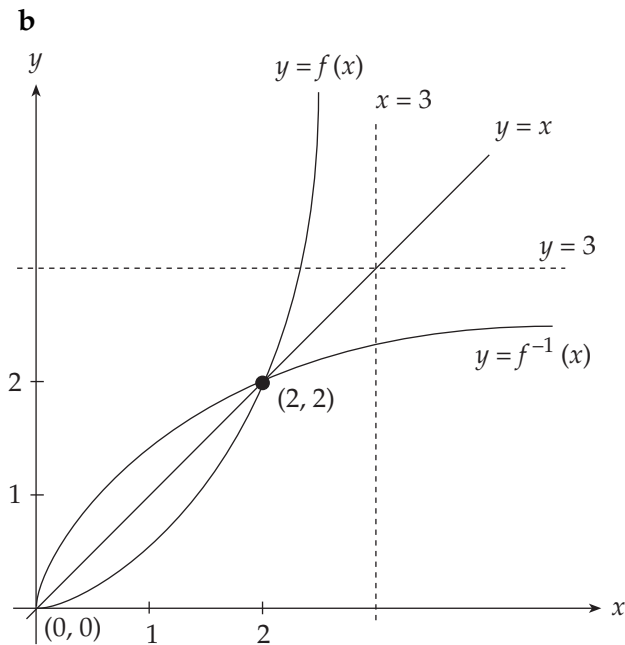
$$= \frac{9+1}{3\sqrt{2}}$$

$$= \frac{10}{3\sqrt{2}} = \frac{10\sqrt{2}}{6}$$

$$= \frac{5\sqrt{2}}{3}$$

Question 4

a $f(z) = -2 + 2 \sec\left(\frac{\pi}{3}\right)$
 $= -2 + 4 = 2$



c $y = -2 + 2 \sec\left(\frac{\pi x}{6}\right)$
 To find rule of inverse function, interchange x for y , and solve for y .

$$x = -2 + 2 \sec\left(\frac{\pi y}{6}\right)$$

$$\Rightarrow x + 2 = \frac{2}{\cos\left(\frac{\pi y}{6}\right)}$$

$$\Rightarrow \cos\left(\frac{\pi y}{6}\right) = \frac{2}{x + 2}$$

$$\Rightarrow \frac{\pi y}{6} = \cos^{-1}\left(\frac{2}{x + 2}\right)$$

$$\Rightarrow y = \frac{6}{\pi} \cos^{-1}\left(\frac{2}{x + 2}\right)$$

$$\therefore a = \frac{6}{\pi}$$

d $A = \int_0^2 \frac{\pi}{6} \cos^{-1}\left(\frac{2}{x + 2}\right) + 2 - 2 \sec\left(\frac{\pi x}{6}\right) dx$
 $= 1.939$ (using Graphics calculator)

e i Let $y = \log_e(u)$

where $u = \frac{1 + \sin kx}{\cos kx}$

$$\frac{du}{dx} = \frac{k \cos^2 kx + (1 + \sin kx)k \sin kx}{\cos^2 kx}$$

$$= \frac{k(\cos^2 kx + \sin kx + \sin^2 kx)}{\cos^2 kx}$$

$$= \frac{k(1 + \sin kx)}{\cos^2 kx}$$

$$\frac{dy}{du} = \frac{1}{u}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \frac{\cos kx}{1 + \sin kx} \times \frac{k(1 + \sin kx)}{\cos^2 kx}$$

$$= k \sec(kx) \text{ (as required)}$$

e ii $A = \int_0^2 \frac{\pi}{6} \cos^{-1}\left(\frac{2}{x + 2}\right) + 2 - 2 \sec\left(\frac{\pi x}{6}\right) dx$

Due to symmetry about $y = x$

This can be re-written as:

$$A = 2 \int_0^2 x - \left(-2 + 2 \sec\left(\frac{\pi x}{6}\right)\right) dx$$

$$= 2 \int_0^2 x + 2 - 2 \sec\left(\frac{\pi x}{6}\right) dx$$

$$= 2 \int_0^2 x + 2 - 2 \times \frac{6}{\pi} \times \frac{\pi}{6} \sec\left(\frac{\pi x}{6}\right) dx$$

$$= 2 \left[\frac{x^2}{2} + 2x - \frac{12}{\pi} \log_e \left(\frac{1 + \sin\left(\frac{\pi x}{6}\right)}{\cos\left(\frac{\pi x}{6}\right)} \right) \right]_0^2$$

$$= 2 \left[\left(2 + 4 - \frac{12}{\pi} \log_e \left(\frac{2 + \sqrt{3}}{\frac{1}{2}} \right) \right) - \right.$$

$$\left. \left(0 + 0 - \frac{12}{\pi} \log_e(1) \right) \right]$$

$$= 2 \left[6 - \frac{12}{\pi} \log_e(2 + \sqrt{3}) \right]$$

Question 5

a $\frac{dy}{dt} = a(100 - y)$

$$\frac{dt}{dy} = -1 \frac{-1}{a(100 - y)}$$

$$t + c = -\frac{1}{a} \log_e(100 - y)$$

let $-ca = d$

$$-at + d = \log_e(100 - y)$$

$$e^{-at+d} = 100 - y$$

$$e^d = A$$

$$Ae^{-at} = 100 - y$$

$$y = 100 - Ae^{-at}$$

We know that at $t = 0$, $y = 5$

$$5 = 100 - A$$

$$\Rightarrow A = 95$$

$$\therefore y = 100 - 95e^{-at} \text{ (as required)}$$

b $y = 10 + Ae^{-b(t-T)}$ 1

$$\frac{dy}{dt} = -bAe^{-b(t-T)}$$

From 1 we know that

$$Ae^{-b(t-T)} = y - 10$$

$$\therefore \frac{dy}{dt} = -b(y - 10)$$

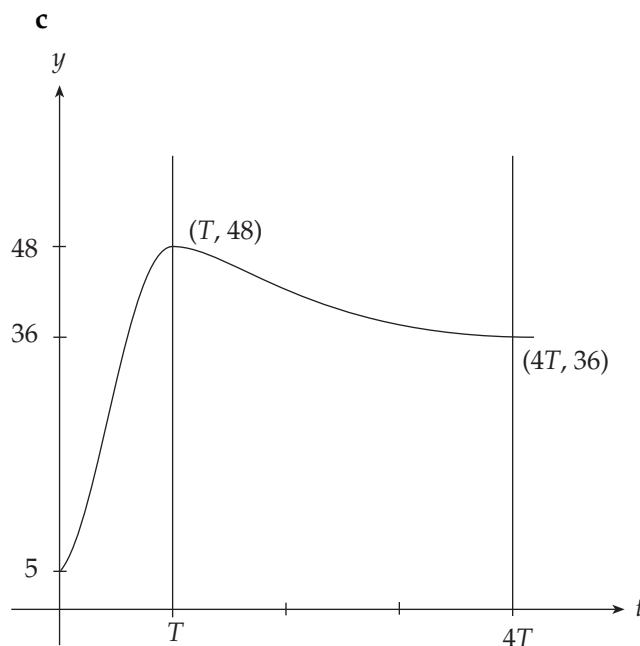
To evaluate A, use the initial conditions, that is: at $t = T$, $y = 48$

$$y = 10 + Ae^{-b(t-T)}$$

$$\therefore 48 = 10 + Ae^0$$

$$A = 38$$

$$\therefore y = 10 + 38Ae^{-b(t-T)}$$



d $y = 100 - 95e^{-at}$

at $t = T$, $y = 48$

$$48 = 100 - 95e^{-aT}$$

$$-52 = -95e^{-aT}$$

$$\frac{52}{95} = e^{-aT}$$

$$-aT = \log_e\left(\frac{52}{95}\right) \quad 1$$

$$y = 10 + 38e^{-b(t-T)}$$

at $t = 4T$, $y = 36$

$$36 = 10 + 38e^{-3bT}$$

$$26 = 38e^{-3bT}$$

$$-3bT = \log_e\left(\frac{26}{38}\right) \quad 2$$

$$1 \div 2$$

$$\frac{aT}{3bT} = \frac{\log_e\left(\frac{52}{95}\right)}{\log_e\left(\frac{26}{38}\right)}$$

$$\frac{a}{b} = 3 \frac{\log_e\left(\frac{52}{95}\right)}{\log_e\left(\frac{26}{38}\right)}$$

$$= 4.76 \text{ (correct to 3 significant figures).}$$