

2003 Specialist Mathematics Written Examination 1 (facts, skills and applications) Suggested Answers and Solutions

Part I (Multiple-choice) Answers

1. D	2. B	3. E	4. A	5. E
6. A	7. D	8. A	9. A	10. C
11. D	12. E	13. B	14. B	15. B
16. C	17. C	18. E	19. A	20. D
21. D	22. E	23. C	24. C	25. B
26. C	27. E	28. B	29. D	30. E

Question 1

[D]

Graph translated 4 units to the right so has the equation:

$$\frac{(x-4)^2}{a^2} - \frac{y^2}{b^2} = 1$$

at $y = 0$ and $x = 2$

$$\frac{-2^2}{a^2} - 0 = 1$$

$$a^2 = 4$$

$$\text{Asymptote: } y = \pm \frac{bx}{a} = \pm \frac{3}{2}x$$

$$\Rightarrow b = 3$$

$$\frac{(x-4)^2}{4} - \frac{y^2}{9} = 1$$

Question 2

[B]

$$\sin^2(x) + \cos^2(x) = 1$$

$$\frac{1}{25} + \cos^2(x) = 1$$

$$\cos^2(x) = \frac{24}{25}$$

$$= \pm \frac{2\sqrt{6}}{5}$$

$$\cot(x) = \frac{\cos(x)}{\sin(x)}$$

$$= \frac{\pm 2\sqrt{6}}{\frac{1}{5}} = \pm 2\sqrt{6}$$

$$\cot(x) \text{ is positive when } \pi \leq x \leq \frac{3\pi}{2}$$

$$\therefore \cot(x) = \pm 2\sqrt{6}$$

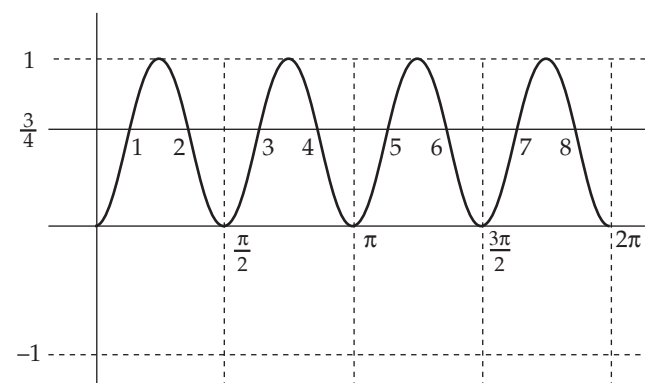
Question 3

[E]

Using a graphics calculator:

$$y_1 = \sin^2(2x)$$

$$y_2 = \frac{3}{4}$$



There are 8 points of intersection.
Hence there are 8 solutions.

OR

$$\sin^2 2x = \frac{3}{4}$$

$$\therefore \sin 2x = \pm \frac{\sqrt{3}}{2} \quad 0 \leq x \leq 2\pi$$

so 8 solutions.

Question 4**[A]**

$$y = \sin^{-1}\left(\frac{4}{x}\right)$$

$$\text{Let } u = 4x^{-1}$$

$$\frac{du}{dx} = -4x^{-2}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \frac{-4}{x^2 \sqrt{1 - \frac{16}{x^2}}}$$

$$= \frac{-4}{x^2 \sqrt{\frac{x^2}{x^2} - \frac{16}{x^2}}}$$

$$= \frac{-4}{\frac{x^2}{x} \sqrt{x^2 - 16}}$$

$$= \frac{-4}{x\sqrt{x^2 - 16}}$$

Question 5**[E]**

$$r \operatorname{cis} \theta$$

$$r = \sqrt{x^2 + y^2}$$

$$\tan \theta = \frac{y}{x}$$

$$\text{For } r: r = \sqrt{3+1} \\ = 2$$

$$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{\pi}{6}$$

$-\sqrt{3} - i$ is in the 3rd quadrant

$$\therefore \theta = \frac{7\pi}{6}$$

$$-\sqrt{3} - i = 2\operatorname{cis}\left(\frac{7\pi}{6}\right)$$

Question 6**[A]**

$$z^2 = 4\operatorname{cis}\left(\frac{4\pi}{3}\right)$$

$$z = 2\operatorname{cis}\left(\frac{2\pi}{3} + n\pi\right)$$

$$z_1 = 2\operatorname{cis}\left(\frac{2\pi}{3}\right)$$

$$z_2 = 2\operatorname{cis}\left(\frac{5\pi}{3}\right)$$

$$z_1 = 2\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right) = 2\left(-\frac{1}{2} + \frac{\sqrt{3}i}{2}\right)$$

$$z_2 = 2\left(\cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right)\right) = 2\left(\frac{1}{2} - \frac{\sqrt{3}i}{2}\right)$$

$$z_1 = -1 + \sqrt{3}i$$

$$z_2 = 1 - \sqrt{3}i$$

Question 7**[D]**

Alternatives A, B and E can be eliminated because they are not linear factors.

$$\text{For C} \quad z + 2 = 0$$

$$\text{Then} \quad z = (-2)$$

$$P(-2) = -8 - 8 - 8 - 8 \neq 0$$

$\therefore z + 2$ is not a factor.

This suggests that $z + 2i$ is the correct alternative.

Just to check:

$$z + 2i = 0$$

$$\text{when } z = -2i$$

$$P(-2i) = 8i + 8 - 8i - 8 = 0$$

$\therefore z + 2i$ is a linear factor

OR

$$P(z) = z^3 - 2z^2 + 4z - 8$$

$$= z^2(z - 2) + 4(z - 2)$$

$$= (z - 2)(z^2 + 4)$$

$$= (z - 2)(z - 2i)(z + 2i)$$

$\therefore z + 2i$ is a linear factor.

Question 8**[A]**

Equation of a circle is:

$$(x+3)^2 + y^2 = 9$$

$$x^2 + 6x + 9 + y^2 = 9$$

$$x^2 + y^2 + 6x + 9 = 9$$

Expanding $(z+a)(\bar{z}+b)$

$$z\bar{z} + a\bar{z} + bz + ab$$

$$a(x-iy) + b(x+iy)$$

$$ax - aiy + bx + biy = 6x$$

$$\Rightarrow a + b = 6$$

$$ab = 9$$

$$\therefore a = b = 3$$

Equation of a circle: $(z+3)(\bar{z}+3) = 9$

Note: this solution is given to allow an understanding of the mathematics.

Question 9**[A]**

Alternative 1:

$$|z-1| = |z+1|$$

$$|x+iy-1| = |x+iy+i|$$

$$|(x-1)+iy| = |x+i(y+1)|$$

$$(x-1)^2 + y^2 = x^2 + (y+1)^2$$

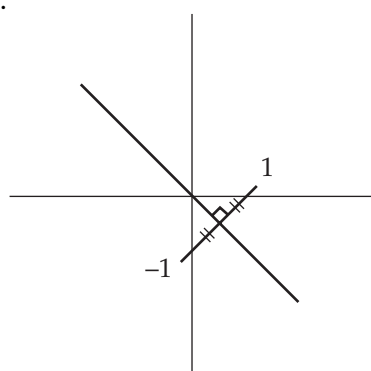
$$x^2 - 2x + 1 + y^2 = x^2 + y^2 + 2y + 1$$

$$-2x = 2y$$

$$y = -x$$

Alternative 2:

$$|z-1| = |z+1|$$

S is the region equal distance from $z=1$ and $z=-i$.Plot $z=1$ and $z=-i$ and draw a line the perpendicular bisector of the line joining $z=1$ and $z=-i$.**Question 10****[C]**

$$\int_0^{\frac{\pi}{6}} \cos^3(2x) dx$$

$$= \int_0^{\frac{\pi}{6}} \cos 2x (1 - \sin^2(2x)) dx$$

Let $u = \sin(2x)$

$$\frac{du}{dx} = 2 \cos(2x)$$

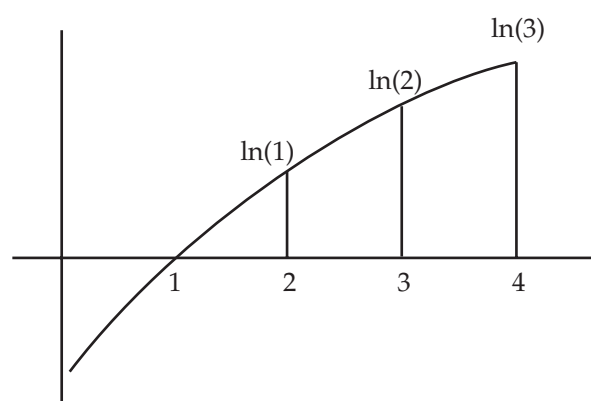
$$dx = \frac{du}{2 \cos(2x)}$$

$$u = \sin\left(2 \times \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

$$u = \sin(0) = 0$$

$$\int_0^{\frac{\sqrt{3}}{2}} \cos(2x) (1 - u^2) \frac{du}{2 \cos(2x)}$$

$$= \frac{1}{2} \int_0^{\frac{\sqrt{3}}{2}} 1 - u^2 du$$

Question 11**[D]**

$$\text{Area 1} = \frac{ba}{2} = \frac{1 \times \ln(2)}{2}$$

$$\begin{aligned} \text{Area 2} &= \frac{1}{2}(a+b)h \\ &= \frac{1}{2}((\ln(2) + \ln(3)) \times 1) \end{aligned}$$

$$\text{Area 3} = \frac{1}{2}((\ln(3) + \ln(4)) \times 1)$$

$$\begin{aligned} \text{Total area} &= \frac{\ln(2)}{2} + \frac{\ln(2) + \ln(3)}{2} + \frac{\ln(3) + \ln(4)}{2} \\ &= \frac{1}{2}(\ln(2) + \ln(2) + \ln(3) + \ln(3) + \ln(4)) \\ &= \frac{1}{2} \ln(2^2 \times 3^2 \times 4) \\ &= \ln(2 \times 3 \times 2) \\ &= \ln(12) \quad \therefore a = 12 \end{aligned}$$

Question 12**[E]**

$$\begin{aligned}
 \text{Area} &= \int_0^1 2 \cos\left(\frac{\pi x}{2}\right) - (x^2 - 1) dx \\
 &= \int_0^1 2 \cos\left(\frac{\pi x}{2}\right) - x^2 + 1 dx \\
 &= \left[\frac{4}{\pi} \sin\left(\frac{\pi x}{2}\right) - \frac{x^3}{3} + x \right]_0^1 \\
 &= \left(\frac{4}{\pi} \sin\left(\frac{\pi}{2}\right) - \frac{1}{3} + 1 \right) - (0) \\
 &= \frac{4}{\pi} + \frac{2}{3}
 \end{aligned}$$

Question 13**[B]**

$$\begin{aligned}
 \frac{3}{x(3-x)} &= \frac{A}{x} + \frac{B}{(3-x)} \\
 &= \frac{A(3-x) + Bx}{x(3-x)}
 \end{aligned}$$

$$3 \equiv A(3-x) + Bx$$

$$\text{at } x = 3$$

$$3 = 3B$$

$$\Rightarrow B = 1$$

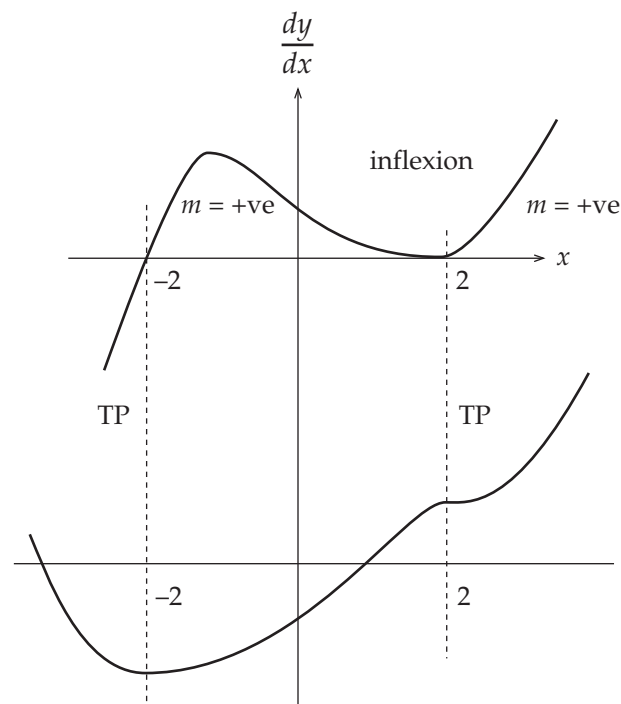
$$\text{at } x = 0$$

$$3 = 3A$$

$$\Rightarrow A = 1$$

$$\int \frac{3}{x(3-x)} dx = \int \frac{1}{x} + \frac{1}{(3-x)} dx$$

$$= \ln x - \ln(3-x)$$

Question 14**[B]****Question 15****[B]**

$$f'(x) = 2 \sin^2\left(\frac{x}{2}\right) - 1$$

$$= -\left(1 - \sin^2\left(\frac{x}{2}\right)\right)$$

$$\text{but } \cos(2a) = 1 - \sin^2(a)$$

$$\text{let } a = \frac{x}{2}$$

$$\therefore \cos x = 1 - \sin^2\left(\frac{x}{2}\right)$$

$$\text{so } f'(x) = -\cos(x)$$

$$f(x) = -\sin(x) + c$$

$$f\left(\frac{\pi}{2}\right) = 0 = -1 + c$$

$$c = 1$$

$$\therefore f(x) = 1 - \sin(x)$$

Question 16

$$r = \frac{h}{5}$$

$$V = \frac{1}{3} \pi r^2 h$$

$$= \frac{1}{3} \pi \left(\frac{h}{5} \right)^2 h$$

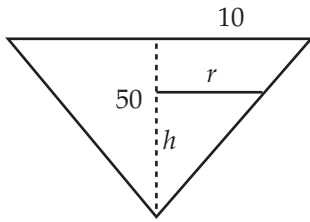
$$= \frac{\pi h^3}{3 \times 25}$$

$$\frac{dv}{dh} = \frac{\pi h^2}{25}$$

$$\frac{dh}{dt} = \frac{dh}{dv} \times \frac{dv}{dt}$$

$$= \frac{25}{\pi h^2} \times 600$$

$$= \frac{15\,000}{\pi h^2}$$

**Question 17**

$$\frac{dy}{dx} = f(x) = \cos\left(\frac{x}{2}\right)$$

$$x_{n+1} = x_n + h$$

$$y_{n+1} = y_n + hf(x_n)$$

$$x_0 = 0$$

$$y_0 = 2$$

$$h = 0.2$$

$$x_1 = 0.2 \quad y_1 = 2 + 0.2f(0) = 2.2$$

$$x_2 = 0.4 \quad y_2 = 2.2 + 0.2 \cos\left(\frac{0.2}{2}\right) \\ = 2.2 + 0.2 \cos(0.1)$$

Question 18

$$\overrightarrow{AC} = \underline{p} + \underline{q}$$

$$\overrightarrow{CA} = \underline{r} + \underline{s}$$

$$\Rightarrow \overrightarrow{AC} = -\underline{r} - \underline{s}$$

$$\therefore \underline{p} + \underline{q} = -\underline{r} - \underline{s}$$

OR

$$\underline{p} + \underline{q} + \underline{r} + \underline{s} = 0$$

$$\therefore \underline{p} + \underline{q} = -\underline{r} - \underline{s}$$

[C]

Question 19

$$\underline{a} = \underline{i} - 2\underline{j} + 3\underline{k}$$

$$|\underline{a}| = \sqrt{1 + 4 + 9} \\ = \sqrt{14}$$

$$\hat{\underline{a}} = \frac{\underline{a}}{|\underline{a}|} = \frac{1}{\sqrt{14}} (\underline{i} - 2\underline{j} + 3\underline{k})$$

$$-\hat{\underline{a}} = -\frac{1}{\sqrt{14}} (\underline{i} - 2\underline{j} + 3\underline{k})$$

$$= \frac{1}{\sqrt{14}} (-\underline{i} + 2\underline{j} - 3\underline{k})$$

[A]

Question 20

$$\overrightarrow{OS} = -c\underline{i} + 2c\underline{j}$$

$$\overrightarrow{OR} = -2\underline{i} + \underline{j}$$

$$\overrightarrow{RS} = -\overrightarrow{OR} + \overrightarrow{OS}$$

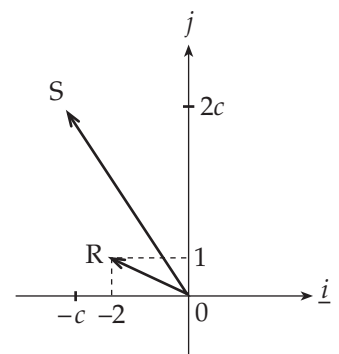
$$= 2\underline{i} - \underline{j} - c\underline{i} + 2c\underline{j}$$

$$= (2 - c)\underline{i} + (2c - 1)\underline{j}$$

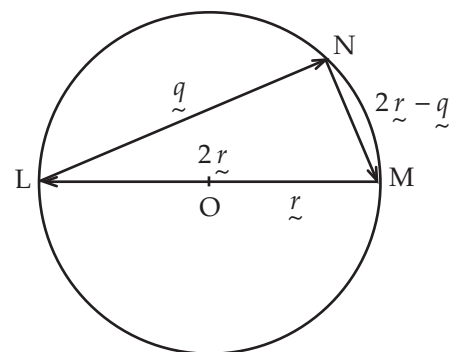
$$\overrightarrow{OS} \cdot \overrightarrow{RS} = -c(2 - c) + 2c(2c - 1)$$

$$= -2c + c^2 + 4c^2 - 2c$$

$$= 5c^2 - 4c$$



[D]

Question 21 $\angle LMN$ is a right angle.

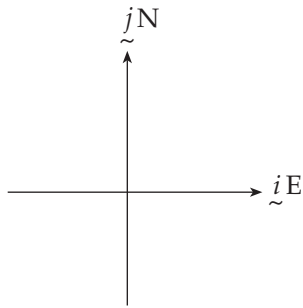
$$\therefore \underline{q} \cdot (2\underline{r} - \underline{q}) = 0$$

$$\Rightarrow 2\underline{r} \cdot \underline{q} - \underline{q} \cdot \underline{q} = 0$$

$$\Rightarrow 2\underline{r} \cdot \underline{q} = \underline{q} \cdot \underline{q}$$

[E]

[D]

Question 22**[E]**

$$\underline{r} = (5t - 8)\underline{i} + (t^2 - 5t + 6)\underline{j}$$

$$\underline{s} = (t^2 - t)\underline{i} + (3 - t)\underline{j}$$

r is north or south of s when:

$$5t - 8 = t^2 - t$$

$$\Rightarrow t^2 - 6t + 8 = 0$$

$$(t - 2)(t - 4) = 0$$

$$t = 2, 4$$

at $t = 2$

$$L \rightarrow \underline{r} = 2\underline{i} + 0\underline{j}$$

$$M \rightarrow \underline{s} = 2\underline{i} + \underline{j}$$

so L is south of M

at $t = 4$

$$L \rightarrow \underline{r} = 12\underline{i} + 2\underline{j}$$

$$M \rightarrow \underline{s} = 12\underline{i} - \underline{j}$$

$\therefore L$ is north of m when $t = 4$.

Question 23**[C]**

$$\underline{r}(t) = 4t\underline{i} - e^{2t}\underline{j} + 5\underline{k}$$

$$\dot{\underline{r}}(t) = 4\underline{i} - 2e^{2t}\underline{j}$$

$$\dot{\underline{r}}(0) = 4\underline{i} - 2\underline{j}$$

$$|\dot{\underline{r}}(0)| = \sqrt{16 + 4}$$

$$= \sqrt{20}$$

Question 24**[C]**

$$\ddot{\underline{r}}(t) = \cos(t)\underline{i} - \sin(t)\underline{j}$$

$$\dot{\underline{r}}(t) = \sin(t)\underline{i} + \cos(t)\underline{j} + \underline{c}$$

$$\dot{\underline{r}}(0) = \underline{i} + \underline{j}$$

$$\underline{i} + \underline{j} = 0\underline{i} + \underline{j} + \underline{c}$$

$$\underline{c} = \underline{i}$$

$$= (\sin(t) + 1)\underline{i} + \cos(t)\underline{j}$$

Question 25**[B]**

$$F - \mu N = F_R$$

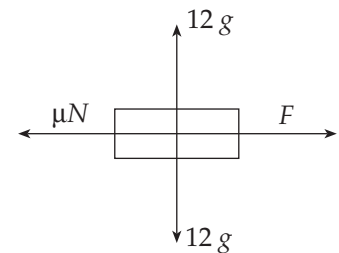
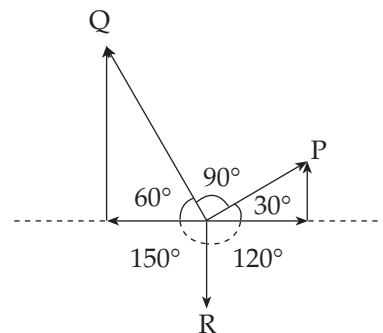
$$F_R = 0.5 \times 12 = 6\text{N}$$

$$66 - 12g\mu = 6$$

$$60 = 12g\mu$$

$$\mu = \frac{60}{12g}$$

$$= 0.51$$

**Question 26****[C]**

Using the components vertical and horizontal:

$$\overrightarrow{OQ} + \overrightarrow{OP} = \overrightarrow{RO}$$

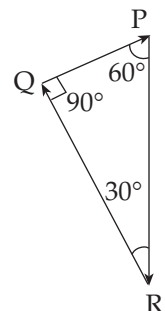
$$P = R \cos 60^\circ$$

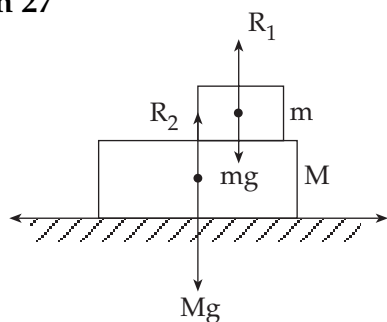
$$\text{and } Q = R \cos 30^\circ$$

By a process of elimination:

$$R = Q \cos 30^\circ$$

is not true.



Question 27**[E]**

Consider m : gravity **acts on** it with a force of mg .

Block M **acts on** m to produce an equal and opposite force R_1 .

Consider M : gravity **acts on** it with a force of Mg .

The ground **acts on** M to produce an equal and opposite force R_2 .

Question 28**[B]**

Consider each alternative:

A: $t = v - 1$

At $t = 0, v = 1$ possible

At $v = 0, t = -1$ doesn't need to be considered as $t \geq 0$

B: $t = x^2 - 1$

At $t = 0, x^2 - 1 = 0$

$(x - 1)(x + 1) = 0$

$x = +1$ and $x = -1$ impossible

This suggests that at $t = 0$ x is holding two positions simultaneously.

C: $x = t^2 - 1$

At $t = 0, x = -1$ possible

At $x = 0, t = \pm 1$, only $t = 1$ can be considered as possible.

D: $x = v^2 - 1$

At $x = 0, v = \pm 1$ possible

At $v = 0, x = -1$ possible

E: $v = t - 1$

At $t = 0, v = -1$ possible

At $v = 0, t = 1$ possible

Question 29**[D]**

$$v = \cos(t) + \sqrt{3} \sin(t) - 1$$

$$\frac{dv}{dt} = -\sin(t) + \sqrt{3} \cos(t)$$

Turning point at $\frac{dv}{dt} = 0$

$$\sin(t) = \sqrt{3} \cos(t)$$

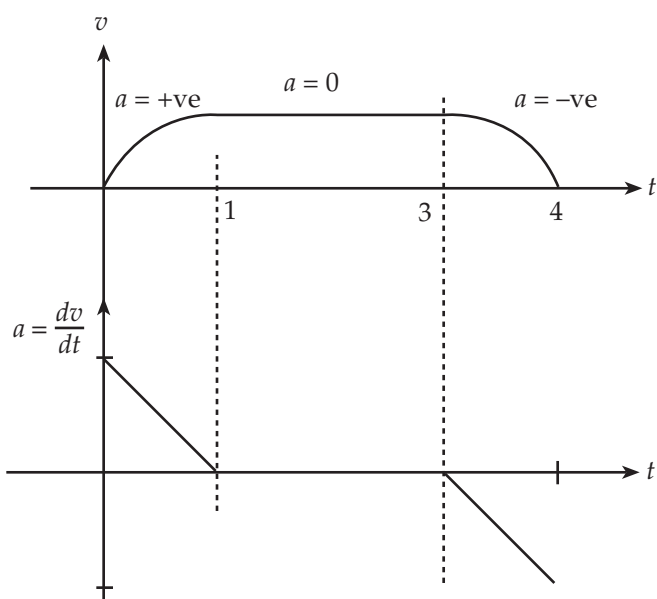
$$\tan(t) = \sqrt{3}$$

$$t = \frac{\pi}{3}, \frac{4\pi}{3}, \dots$$

$$v\left(\frac{\pi}{3}\right) = \frac{1}{2} + \frac{3}{2} - 1 = 1$$

$$v\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - \frac{3}{2} - 1 = -3$$

Maximum speed is 3 which occurs at $\frac{4\pi}{3}$.

Question 30**[E]**

Part II

Question 1

a $v = 4.5t + \cos(2t)$

$$\frac{dv}{dt} = 4.5 - 2\sin(2t)$$

b Max and min occurs when $\sin(2t) = \pm 1$
for $\sin(2t) = 1$

$$a = 4.5 - 2 = 2.5$$

for $\sin(2t) = -1$

$$a = 4.5 - (-2) = 6.5$$

using $a = 2.5$

$$F = ma$$

$$= 5 \times 2.5$$

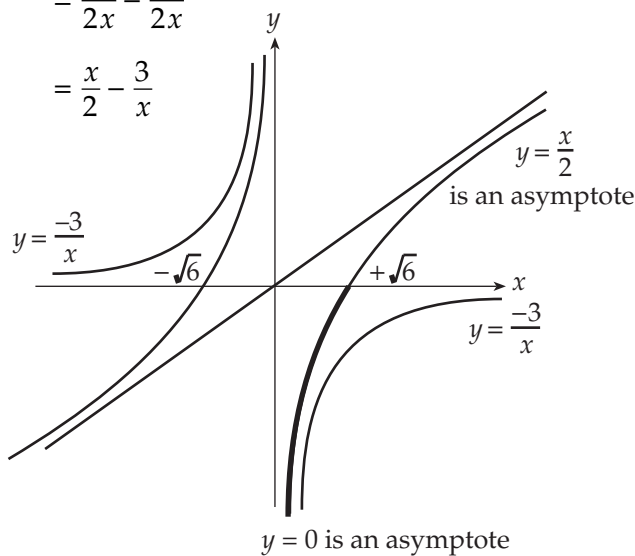
$$= 12.5 \text{ N}$$

Question 2

$$f(x) = \frac{x^2 - 6}{2x}$$

$$= \frac{x^2}{2x} - \frac{6}{2x}$$

$$= \frac{x}{2} - \frac{3}{x}$$



At $y = 0$, $0 = \frac{x^2 - 6}{2x} \quad x \neq 0$

$$0 = (x - \sqrt{6})(x + \sqrt{6})$$

x -intercepts at $x = \pm\sqrt{6}$

$$f'(x) = \frac{2x \times 2x - x(x^2 - 6)}{4x^2}$$

$$= \frac{2x^2 + 6}{4x^2}$$

$f'(x) \neq 0$ for all values of $x \therefore$ no turning point.

Question 3

$$y = xe^{3x}$$

$$\frac{dy}{dx} = e^{3x} + 3xe^{3x}$$

$$= e^{3x}(1 + 3x)$$

$$\frac{d^2y}{dx^2} = 3e^{3x}(1 + 3x) + 3e^{3x}$$

$$= 3e^{3x}(1 + 3x + 1)$$

$$= 3e^{3x}(2 + 3x)$$

Given $\frac{d^2y}{dx^2} + m\frac{dy}{dx} + ny = 0$

$$3e^{3x}(2 + 3x) + me^{3x}(1 + 3x) + nxe^{3x} = 0$$

$$\Rightarrow 3(2 + 3x) + m(1 + 3x) + nx = 0$$

$$\Rightarrow 6 + 9x + m + 3mx + nx = 0$$

$$\Rightarrow x(3m + n) + m = -9x - 6$$

$$\therefore m = -6$$

$$3m + n = -9$$

$$-18 + n = -9$$

$$n = 9$$

$$m = -6 \text{ and } n = 9$$

Question 4

a $\int_2^{-2} 1 - \frac{8}{x^2 + 4} dx$

$$= \int_2^{-2} 1 - 4 \times \frac{2}{x^2 + 2^2} dx$$

$$= \left[x - 4 \tan^{-1}\left(\frac{x}{2}\right) \right]_2^{-2}$$

$$= (-2 - 4 \tan^{-1}(-1)) - (2 - 4 \tan^{-1}(1))$$

$$= -2 + \pi - 2 + \pi$$

$$= -4 + 2\pi$$

$$b \quad v = \pi \int_{-1}^0 x^2 dy$$

$$y = 1 - \frac{8}{x^2 + 4}$$

$$\frac{8}{x^2 + 4} = 1 - y$$

$$x^2 = \frac{8}{1 - y} - 4$$

$$v = \pi \int_{-1}^0 \frac{8}{1 - y} - 4 dy$$

$$= -\pi \int_{-1}^0 \frac{-8}{1 - y} + 4 dy$$

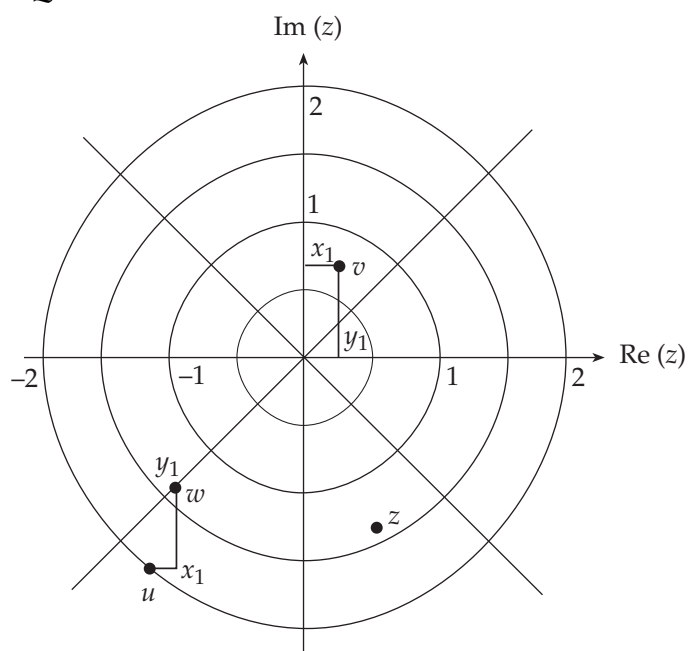
$$= -\pi [8 \ln(1 - y) + 4y]_{-1}^0$$

$$v = -\pi [(8 \ln(1) + 0) - (8 \ln(2) - 4)]$$

$$= \pi(8 \ln(2) - 4)$$

$$= 4.85 \text{ (3 significant figures)}$$

Question 5



$$a \quad z = \sqrt{2} \operatorname{cis} \theta$$

$$u = z^2 = (\sqrt{2})^2 \operatorname{cis} 2\theta$$

$$= 2 \operatorname{cis} 2\theta$$

From diagram:

$$\theta \approx -\frac{3\pi}{8}$$

$$\Rightarrow 2\theta = -\frac{3\pi}{4}$$

$$z^2 = 2 \operatorname{cis} \left(-\frac{3\pi}{4} \right)$$

$$b \quad z = \sqrt{2} \operatorname{cis} \theta$$

$$v = \frac{1}{z} = z^{-1} = \frac{1}{\sqrt{2}} \operatorname{cis}(-\theta)$$

$$= \frac{\sqrt{2}}{2} \operatorname{cis}(-\theta)$$

$$\text{using: } \theta = -\frac{3\pi}{8}$$

$$v = \frac{\sqrt{2}}{2} \operatorname{cis}\left(\frac{3\pi}{8}\right)$$

$$c \quad w = z^2 + \frac{1}{z}$$

$$w = u + v$$

This can be best achieved by adding x_1 to u followed by adding y_1 .

Refer to the diagram.