

Year 2003

VCE

Specialist Mathematics Trial Examination 2

Detailed Suggested Solutions

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Question 1

i. $u = \frac{1}{4}(\sqrt{3} - i) = \frac{1}{2} \operatorname{cis}\left(\frac{-\pi}{6}\right)$ $v = \sqrt{2} \operatorname{cis}\left(\frac{\pi}{4}\right) = 1 + i$ $\Rightarrow uv = \frac{1}{4}(\sqrt{3} - i)(1 + i)$ $\Rightarrow uv = \frac{1}{4}(\sqrt{3} - i + \sqrt{3}i - i^2)$ $\Rightarrow uv = \frac{1}{4}((\sqrt{3} + 1) + (\sqrt{3} - 1)i)$	ii. $uv = \frac{1}{2} \operatorname{cis}\left(\frac{-\pi}{6}\right) \times \sqrt{2} \operatorname{cis}\left(\frac{\pi}{4}\right)$ $\Rightarrow uv = \frac{\sqrt{2}}{2} \operatorname{cis}\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$ $\Rightarrow uv = \frac{\sqrt{2}}{2} \operatorname{cis}\left(\frac{\pi}{12}\right)$
iii. $\frac{\sqrt{2}}{2} \operatorname{cis}\left(\frac{\pi}{12}\right) = \frac{\sqrt{2}}{2} \cos\left(\frac{\pi}{12}\right) + i \frac{\sqrt{2}}{2} \sin\left(\frac{\pi}{12}\right)$ $= \frac{1}{4}(\sqrt{3} + 1) + i \frac{1}{4}(\sqrt{3} - 1)$ Equating imaginary parts $\frac{\sqrt{2}}{2} \sin\left(\frac{\pi}{12}\right) = \frac{1}{4}(\sqrt{3} - 1)$ $\Rightarrow \sin\left(\frac{\pi}{12}\right) = \frac{1}{4} \times \frac{2}{\sqrt{2}}(\sqrt{3} - 1) \times \frac{\sqrt{2}}{\sqrt{2}}$ $\Rightarrow \sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{2}}{4}(\sqrt{3} - 1)$	iv. $\sin\left(\frac{\pi}{12}\right) = \sin 15^\circ$ $= \sin(45^\circ - 30^\circ)$ $= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$ $= \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \times \frac{1}{2}$ $= \frac{\sqrt{2}}{4}(\sqrt{3} - 1)$

Question 2

i. A (0,11) B $\left(\frac{7}{2}, 19\right)$	ii. When $x = 0, y = 11 \Rightarrow c = 11$ When $x = \frac{7}{2}, y = 19$ $y = ax^2 + c$ $\Rightarrow 19 = \frac{49}{4}a + 11$ $\Rightarrow \frac{49}{4}a = 8$ $\Rightarrow a = \frac{32}{49}$
iii. $y = \frac{32}{49}x^2 + 11$ $\Rightarrow y - 11 = \frac{32}{49}x^2$ $\Rightarrow x^2 = \frac{49}{32}(y - 11)$ $V = \pi \int_a^b x^2 dy$ $\Rightarrow V = \pi \int_{11}^{19} \frac{49}{32}(y - 11) dy$ $\Rightarrow V = \frac{49\pi}{32} \left[\frac{1}{2}y^2 - 11y \right]_{11}^{19}$ $\Rightarrow V = \frac{49\pi}{32} \left[\left(\frac{1}{2} \times 19^2 - 11 \times 19\right) - \left(\frac{1}{2} \times 11^2 - 11 \times 11\right) \right]$ $\Rightarrow V = 49\pi$	iv. At A(0,11) $y = 11 = A + 1$ $\Rightarrow A = 10$ At B $\left(\frac{7}{2}, 19\right)$ $\Rightarrow y = 10 + e^{\frac{7k}{2}} = 19$ $\Rightarrow e^{\frac{7k}{2}} = 9$ $\Rightarrow \frac{7k}{2} = \log_e 9$ $\Rightarrow k = \frac{2}{7} \log_e 9$
v. $y = 10 + e^{0.6278x}$ $\Rightarrow x = \frac{1}{0.6278} \log_e (y - 10)$ $V = \pi \int x^2 dy = \pi \int_{11}^{19} \frac{1}{(0.6278)^2} (\log_e (y - 10))^2 dy$ $Y_1 = \frac{1}{0.6278} \log_e (y - 10)$ $\pi * \text{fnInt}(Y_1^2, X, 11, 19) = 158.63$ (TI – 83)	vi. $V = \pi \int_{11}^h \frac{49}{32}(y - 11) dy$ $\Rightarrow V = \frac{49\pi}{32} \left[\frac{1}{2}(y - 11)^2 \right]_{11}^h$ $\Rightarrow V = \frac{49\pi}{64} [(h - 11)^2 - 0]$ $\Rightarrow V = \frac{49\pi}{64} (h - 11)^2$ shown

Question 2

vii.

$$\text{Given } \frac{dv}{dt} = -k\sqrt{h}$$

$$V = \frac{49\pi}{64}(h-11)^2$$

Differentiate with respect to h

$$\Rightarrow \frac{dV}{dh} = \frac{49\pi}{32}(h-11)$$

Using a chainrule

$$\Rightarrow \frac{dh}{dt} = \frac{dh}{dV} \cdot \frac{dV}{dt} = \frac{-k\sqrt{h}}{\frac{49\pi}{32}(h-11)} = \frac{A\sqrt{h}}{h-11} \text{ where } A \text{ is a constant}$$

Inverting

$$\frac{dt}{dh} = \frac{h-11}{A\sqrt{h}}$$

$$\Rightarrow \int A dt = \int (h-11)h^{-\frac{1}{2}} dh$$

Integrating

$$\Rightarrow At = \int (h-11)h^{-\frac{1}{2}} dh$$

$$\Rightarrow At + c = \int (h^{\frac{1}{2}} - 11h^{-\frac{1}{2}}) dh = \frac{2}{3}h^{\frac{3}{2}} - 22h^{\frac{1}{2}}$$

$$\text{Now when } t=0, h=19 \Rightarrow c = \frac{-28\sqrt{19}}{3} \approx 40.683$$

$$\text{When } t=3, h=16 \Rightarrow 3A + c = -45\frac{1}{3} \Rightarrow A = 1.55$$

When the glass is empty $h=11$

$$At + c = \frac{-44\sqrt{11}}{3} \approx -48.64$$

$$\Rightarrow -1.55t - 40.683 = -48.64$$

$$\Rightarrow t = \frac{48.64 - 40.683}{1.55}$$

$$\Rightarrow t = 5.133 \text{ minutes}$$

$$\Rightarrow t = 5.1 \text{ minutes}$$

Question 3

a.

$$57.6 \text{ km/hr} = \frac{57.6 \times 1000}{60 \times 60} = 16 \text{ m/s}$$

$$14.4 \text{ km/hr} = \frac{14.4 \times 1000}{60 \times 60} = 4 \text{ m/s}$$

Hence, $u = 16, v = 4, a = -3, t = ?, s = ?$

$$v = u + at$$

$$4 = 16 - 3t$$

$$3t = 12$$

$$t = 4 \text{ seconds}$$

$$s = \left(\frac{u + v}{2} \right) t$$

$$s = \left(\frac{16 + 4}{2} \right) \times 4$$

$$s = 40 \text{ metres}$$

It takes 4 seconds to reach 14.4 km/hr

and travels 40 metres

b.(i)

$$ma = -R$$

$$\Rightarrow 4096a = -24v^3$$

$$a = v \frac{dv}{dx}$$

$$\Rightarrow 4096v \frac{dv}{dx} = -24v^3$$

$$\Rightarrow 512 \frac{dv}{dx} = -3v^3 \text{ shown}$$

b.(ii)

$$\frac{dv}{dx} = -\frac{3}{512}v^2$$

$$\int \frac{dv}{v^2} = -\frac{3}{512} \int dx$$

$$-\frac{3x}{512} = -\frac{1}{v} + c_1$$

Now when $v = 16, x = 0$

$$c_1 = \frac{1}{16}$$

$$-\frac{3x}{512} = -\frac{1}{v} + \frac{1}{16}$$

$$\frac{1}{v} = \frac{3x}{512} + \frac{1}{16} = \frac{3x + 32}{512}$$

$$v = \frac{512}{3x + 32}$$

Now when $v = 4$

$$3x + 32 = \frac{512}{4} = 128$$

$$3x = 96$$

$$x = 32 \text{ m}$$

Question 3

b.(iii)

$$\text{Use } a = \frac{dv}{dt} = -\frac{3}{512}v^3$$

$$\Rightarrow \int \frac{dv}{v^3} = -\frac{3}{512} \int dt$$

$$\Rightarrow -\frac{3t}{512} = \frac{1}{2}v^{-2} + c_2$$

Now when $v = 16, t = 0$

$$\Rightarrow c_2 = \frac{1}{512}$$

$$\Rightarrow \frac{1}{v^2} = \frac{3t}{256} + \frac{1}{256}$$

Now when $v = 4$

$$\frac{1}{16} = \frac{3t+1}{256}$$

$$\Rightarrow 3t+1 = \frac{256}{16} = 16$$

$$\Rightarrow 3t = 15$$

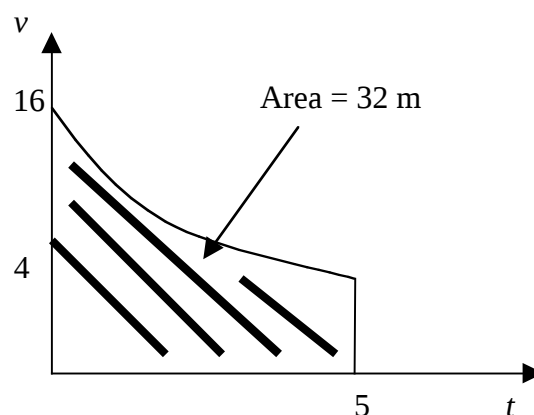
$$\Rightarrow t = 5 \text{ secs}$$

b.(iv)

$$\frac{1}{v^2} = \frac{3t+1}{256}$$

$$\Rightarrow v^2 = \frac{256}{3t+1}$$

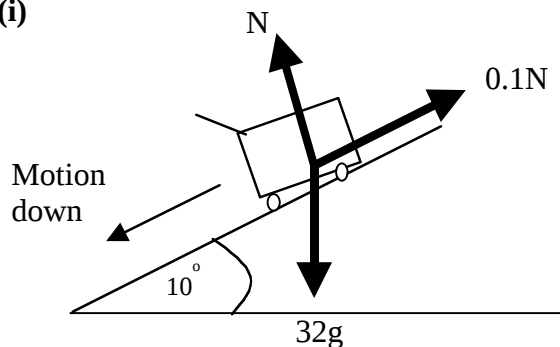
$$\Rightarrow v = \frac{16}{\sqrt{3t+1}} \text{ for } 0 \leq t \leq 5$$



Check on TI-83

Question 4

a.(i)



a.(ii)

Forces along the slope

$$32a = 32g \sin 10^\circ - 0.1N \quad (1)$$

Forces perpendicular to the slope

$$N - 32g \cos 10^\circ = 0 \quad (2)$$

$$\Rightarrow N = 32g \cos 10^\circ$$

Substitute into (1)

$$\Rightarrow 32a = 32g \sin 10^\circ - 0.1 \times 32g \cos 10^\circ$$

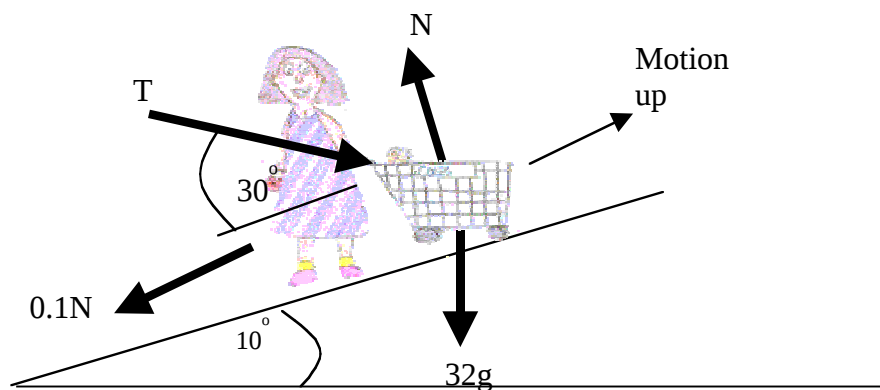
$$\Rightarrow a = g(\sin 10^\circ - 0.1 \cos 10^\circ)$$

$$\Rightarrow a = 9.8(\sin 10^\circ - 0.1 \cos 10^\circ)$$

$$\Rightarrow a = 0.74 \text{ m/s}^2$$

Question 4

b.(i)



b.(ii)

Forces along the slope. speed is constant so $a = 0$

$$T \cos 30^\circ - 0.1N - 32g \sin 10^\circ = 0 \quad (1)$$

Forces perpendicular to the slope

$$N - T \sin 30^\circ - 32g \cos 10^\circ = 0 \quad (2)$$

$$\Rightarrow N = T \sin 30^\circ + 32g \cos 10^\circ$$

Substitute into (1)

$$\Rightarrow T \cos 30^\circ - 0.1(T \sin 30^\circ + 32g \cos 10^\circ) - 32g \sin 10^\circ = 0$$

$$\Rightarrow T(\cos 30^\circ - 0.1 \sin 30^\circ) = 32g \sin 10^\circ + 32g \times 0.1 \cos 10^\circ$$

$$\Rightarrow T = \frac{32 \times 9.8(\sin 10^\circ + 0.1 \cos 10^\circ)}{\cos 30^\circ - 0.1 \sin 30^\circ} = 104.58 \text{ Newtons}$$

Question 5

i. $\bar{r} \cdot \bar{k} = 12 \sin\left(\frac{\pi t}{3}\right) = 0$ $\Rightarrow \frac{\pi t}{3} = 0, \pi, 2\pi$ $\Rightarrow t = 0, 3$ It take three seconds for the golf ball to hit the ground.	ii. $\bar{r}(t) = 8t\bar{i} + 50t\bar{j} + 12\sin\left(\frac{\pi t}{3}\right)\bar{k}$ $\Rightarrow \dot{\bar{r}}(t) = 8\bar{i} + 50\bar{j} + 12 \times \frac{\pi}{3} \cos\left(\frac{\pi t}{3}\right)\bar{k}$ $\Rightarrow \dot{\bar{r}}(0) = 8\bar{i} + 50\bar{j} + 4\pi\bar{k}$ $\Rightarrow \dot{\bar{r}}(0) = \sqrt{64 + 2500 + 16\pi^2}$ $\Rightarrow \dot{\bar{r}}(0) = 15.17 \text{ m/s}$
iii. $\bar{r}(3) = 24\bar{i} + 150\bar{j} + 0\bar{k}$ $\Rightarrow \bar{r}(3) = \sqrt{24^2 + 150^2} = 151.91$ $\Rightarrow \bar{r}(3) = 152 \text{ m}$	iv. $\bar{r} \cdot \bar{k} = \cos\frac{\pi t}{3} = 0$ $\Rightarrow \frac{\pi t}{3} = \frac{\pi}{2}$ $\Rightarrow t = \frac{3}{2}$ $\bar{r}(t) = 8t\bar{i} + 50t\bar{j} + 12\sin\left(\frac{\pi t}{3}\right)\bar{k}$ $\Rightarrow \bar{r}\left(\frac{3}{2}\right) = 12\bar{i} + 75\bar{j} + 12\bar{k}$

Question 6

i.

$$5x - 4 > 0$$

$$\Rightarrow 5x > 4$$

$$\Rightarrow \{x: x > \frac{4}{5}\} = \left(\frac{4}{5}, \infty\right)$$

ii.

Quotient Rule

$$\text{Let } u = x$$

$$\Rightarrow \frac{du}{dx} = 1$$

$$\text{Let } v = \sqrt{5x - 4} = (5x - 4)^{\frac{1}{2}}$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2} \times 5(5x - 4)^{-\frac{1}{2}} = \frac{5}{2\sqrt{5x - 4}}$$

$$\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{5x - 4} - \frac{5x}{2\sqrt{5x - 4}}}{5x - 4}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{5x - 4} \left[\frac{2(5x - 4) - 5x}{2\sqrt{5x - 4}} \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} (5x - 8)(5x - 4)^{-\frac{3}{2}}$$

iii.

For stationary points

$$\frac{dy}{dx} = \frac{1}{2} (5x - 8)(5x - 4)^{-\frac{3}{2}} = 0$$

$$\Rightarrow x = \frac{8}{5}$$

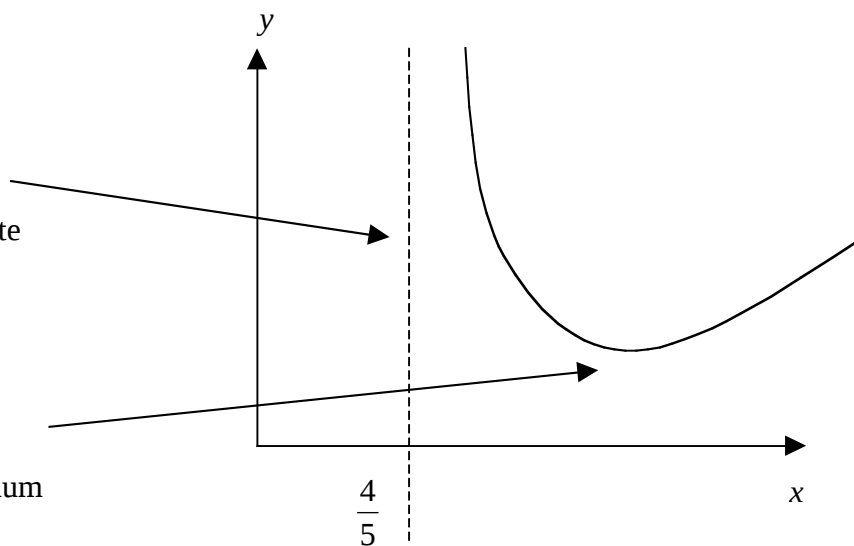
$$\Rightarrow x = \frac{4}{5} \text{ is a vertical asymptote}$$

$$y = \frac{x}{\sqrt{5x - 4}}$$

$$\Rightarrow f\left(\frac{8}{5}\right) = \frac{8}{10} = \frac{4}{5}$$

Turning point is $\left(\frac{8}{5}, \frac{4}{5}\right)$. Minimum

Range $\left(\frac{4}{5}, \infty\right)$



Question 6

iv.

At $x = 4, f(4) = 1$

$P(4,1)$

$$f'(x) = \frac{1}{2}(5x-8)(5x-4)^{-\frac{3}{2}}$$

$$f'(4) = \frac{12}{2 \times 16^{\frac{3}{2}}} = \frac{3}{32} = \text{gradient}$$

Tangent

$$y-1 = \frac{3}{32}(x-4)$$

$$\Rightarrow y = \frac{3x}{32} + \frac{5}{8}$$

v.

$$A_T = \int_1^8 \frac{x}{\sqrt{5x-4}} dx = 7.413 \quad (\text{PRGM TI-83})$$

$$a=1, b=8, n=7$$

Vi

$$A_T = \int_1^8 \frac{x}{\sqrt{5x-4}} dx = \int_1^8 x(5x-4)^{-\frac{1}{2}} dx$$

Let $u = 5x-4, \Rightarrow \frac{du}{dx} = 5$

$$\Rightarrow 5x = u+4$$

$$\Rightarrow x = \frac{1}{5}(u+4)$$

When $x=8, u=36$

When $x=1, u=1$

$$A = \frac{1}{5} \int_1^{36} (u+4) \times u^{-\frac{1}{2}} \times \frac{1}{5} du$$

$$\Rightarrow A = \frac{1}{25} \int_1^{36} (u^{\frac{1}{2}} + 4u^{-\frac{1}{2}}) du$$

$$\Rightarrow A = \frac{1}{25} \left[\frac{2}{3} u^{\frac{3}{2}} + 8u^{\frac{1}{2}} \right]_1^{36}$$

$$\Rightarrow A = \frac{1}{25} \left[\left(\frac{2}{3} \times 36^{\frac{3}{2}} + 8\sqrt{36} \right) - \left(\frac{2}{3} \times 1 + 8 \right) \right]$$

$$\Rightarrow A = 7\frac{1}{3}$$

Check $\text{fnInt}(Y_1, X, 1, 8) = 7.333$

End of suggested solutions 2003 Specialist Mathematics Trial Examination 2

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