

$$5. \underline{u} = \underline{i} - \underline{j} - \underline{k}$$

$$\underline{v} = \underline{i} + a \underline{j} + \underline{k}$$

The angle between $\underline{u}$ and $\underline{v}$ is known to be 60° .

$$\underline{u} \cdot \underline{v} = |\underline{u}| |\underline{v}| \cos 60^\circ$$

$$1 - a - 1 = \sqrt{3} \sqrt{2 + a^2} \cdot \frac{1}{2}$$

$$-2a = \sqrt{3(2 + a^2)}$$

$$4a^2 = 6 + 3a^2$$

$$a^2 = 6$$

$$a = \pm\sqrt{6}$$

Consideration of the earlier equation,

$$-2a = \sqrt{3(2 + a^2)}$$

shows that the value of $a = -\sqrt{6}$ must be chosen since the right hand side is positive,

$$a = -\sqrt{6}$$

$$6. a = -2 + \sqrt{v^2 + 5}$$

$$\frac{dv}{dt} = -2 + \sqrt{v^2 + 5}$$

$$\frac{dv}{dt} = \frac{1}{-2 + \sqrt{v^2 + 5}}$$

The time taken for the velocity to go from

$$v = 3 \text{ to } v = 10 \text{ is } t = \int_3^{10} \frac{dv}{-2 + \sqrt{v^2 + 5}}$$

Using graphics calculator, $t = 1.7$ seconds.

2002 Specialist Mathematics Written Examination 2 (Analysis task) Suggested answers and solutions

$$1. a. \overrightarrow{OC} = 2\underline{i} - 6\underline{j}$$

$$\overrightarrow{OS} = 8\underline{i} - 4\underline{j}$$

$$\overrightarrow{CS} = \overrightarrow{CO} + \overrightarrow{OS}$$

$$= -\overrightarrow{OC} + \overrightarrow{OS}$$

$$= -2\underline{i} + 6\underline{j} + 8\underline{i} - 4\underline{j}$$

$$= 6\underline{i} + 2\underline{j}$$

$$|\overrightarrow{CS}| = \sqrt{36 + 4}$$

$$= \sqrt{40}$$

$$= 2\sqrt{10}$$

$$= 6.32\dots$$

Distance between the cargo ship and the sailing ship at 12:00 midday is 6.3 kilometres to the nearest tenth of a kilometre.

$$b. i. \overrightarrow{OP} = 6m\underline{i} - 2m\underline{j}$$

$$\overrightarrow{PS} = \overrightarrow{PO} + \overrightarrow{OS}$$

$$= -\overrightarrow{OP} + \overrightarrow{OS}$$

$$= -6m\underline{i} + 2m\underline{j} + 8\underline{i} - 4\underline{j}$$

$$= (8 - 6m)\underline{i} + (2m - 4)\underline{j}$$

$$\overrightarrow{OP} \cdot \overrightarrow{PS} = 6m(8 - 6m)\underline{i} \cdot \underline{i} - 2m(2m - 4)\underline{j} \cdot \underline{j}$$

$$= 48m - 36m^2 - 4m^2 + 8m$$

$$= 56m - 40m^2$$

$$b. ii. \overrightarrow{OP} \cdot \overrightarrow{PS} = 56m - 40m^2$$

When $\overrightarrow{OP} \cdot \overrightarrow{PS} = 0$

$\overrightarrow{OP}$ is perpendicular to $\overrightarrow{PS}$ if $OP \neq 0$ and $PS \neq 0$

If $\overrightarrow{OP}$ is perpendicular to $\overrightarrow{PS}$ then this will be the sailing ship's closest point to the shore line.

Let $\overrightarrow{OP} \cdot \overrightarrow{PS} = 0$

$$\Rightarrow 56m - 40m^2 = 0$$

$$m(56 - 40m) = 0$$

$$\therefore m = 0 \text{ or } m = \frac{56}{40} = 1.4$$

Disregard $m = 0$ because for $m = 0$,

$$\overrightarrow{OP} = 0$$

$$P(6 \times 1.4, -2 \times 1.4)$$

$$= (8.4, -2.8)$$

$$b. iii. P(8.4, -2.8)$$

$$S(8, -4)$$

$$d_{PS} = \sqrt{(8.4 - 8)^2 + (-2.8 + 4)^2}$$

$$= \sqrt{(0.4)^2 + (1.2)^2}$$

$$= \sqrt{0.16 + 1.44}$$

$$= \sqrt{1.60}$$

$$= 1.26$$

$= 1.3$ (to the nearest tenth of a kilometre)

$$c. i. \underline{v}_c = 15\underline{i} - 5\underline{j}$$

$$\underline{s}_c = 15t\underline{i} - 5t\underline{j} + c$$

$$\text{at } t = 0, \underline{s}_c = 2\underline{i} - 6\underline{j}$$

$$\therefore c = 2\underline{i} - 6\underline{j}$$

$$\Rightarrow \underline{s}_c = 15t\underline{i} - 5t\underline{j} + 2\underline{i} - 6\underline{j}$$

$$= (15t + 2)\underline{i} - (5t + 6)\underline{j}$$

c. ii. $\underline{v}_s = 12\underline{i} + (3\sin(t) - 8)\underline{j}$
 $\underline{s}_s = 12t\underline{i} + (-3\cos(t) - 8t)\underline{j} + \underline{c}$
 $= 12t\underline{i} - (3\cos(t) + 8t)\underline{j} + \underline{c}$
 at $t = 0$ $\underline{s}_s = 8\underline{i} - 4\underline{j}$
 $\Rightarrow 8\underline{i} - 4\underline{j} = -3\underline{j} + \underline{c}$
 $\underline{c} = 8\underline{i} - \underline{j}$
 $\therefore \underline{s}_s = 12t\underline{i} - (3\cos(t) + 8t)\underline{j} + 8\underline{i} - \underline{j}$
 $= (12t + 8)\underline{i} - (3\cos(t) + 8t + 1)\underline{j}$

d. When $t = 2$

$$\underline{s}_c(2) = (15 \times 2 + 2)\underline{i} - (5 \times 2 + 6)\underline{j}$$

$$= 32\underline{i} - 16\underline{j}$$

$$\underline{s}_s(2) = (12 \times 2 + 8)\underline{i} - (3\cos(2) + 8 \times 2 + 1)\underline{j}$$

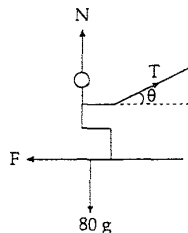
$$= 32\underline{i} - (3\cos(2) + 16 + 1)\underline{j}$$

$$= 32\underline{i} - 15.75\underline{j}$$

$\underline{s}_c$ and $\underline{s}_s$ have the same position in east west line because both have the same $\underline{i}$ value.

On the north south line $\underline{s}_c < \underline{s}_s$ therefore the cargo ship is directly south of the sailing ship.

2. a.



b. $a = 2\text{m/s}^2$ $\theta = 60^\circ$

$$\underline{i}: T \cos \theta - \mu N = F$$

$$T \cos 60^\circ - 0.3N = 80 \times 2 \quad \textcircled{1}$$

$$\underline{j}: N + T \sin \theta - 80g = 0$$

$$N = 80g - T \sin 60^\circ \quad \textcircled{2}$$

Substitute $\textcircled{2}$ into $\textcircled{1}$

$$T \cos 60^\circ - 0.3(80g - T \sin 60^\circ) = 160$$

$$T \cos 60^\circ - 24g + 0.3T \sin 60^\circ = 160$$

$$T(\cos 60^\circ + 0.3 \sin 60^\circ) = 160 + 24g$$

$$T = \frac{160 + 24g}{\cos 60^\circ + 0.3 \sin 60^\circ}$$

$$= 520\text{N (to the nearest integer)}$$

c. i. $T = \frac{160 + 24g}{\cos \theta + 0.3 \sin \theta}$

$$T = (160 + 24g)(\cos \theta + 0.3 \sin \theta)^{-1}$$

$$\frac{dT}{d\theta} = (160 + 24g) \times -1(\cos \theta + 0.3 \sin \theta)^{-2}$$

$$(-\sin \theta + 0.3 \cos \theta)$$

$$= \frac{-(160 + 24g)(-\sin \theta + 0.3 \cos \theta)}{(\cos \theta + 0.3 \sin \theta)^2}$$

$$\cos \theta = -0.3 \sin \theta$$

$$\text{because } 0 < \theta < 90^\circ$$

$$\text{Let } \frac{dT}{d\theta} = 0$$

$$\sin \theta = 0.3 \cos \theta$$

$$\tan \theta = 0.3$$

$$\theta = 16.6999\dots$$

$$\theta = 16.7^\circ \text{ (to nearest tenth of a degree)}$$

c. ii. $T = \frac{160 + 24g}{\cos \theta + 0.3 \sin \theta}$ (from previous work)

$$\theta = 16.6999\dots$$

$$T_{\min} = \frac{160 + 24g}{\cos(16.999) + 0.3 \sin(16.999)}$$

$$T_{\min} = 378.53\dots$$

$$T_{\min} = 379 \text{ N (to the nearest Newton)}$$

3. a. i. $u = 0$ $t = 8$ $s = 400$
 constant acceleration, so we can use

$$s = ut + \frac{1}{2}at^2$$

$$400 = 0 + \frac{1}{2}(8)^2$$

$$a = \frac{2(400)}{64}$$

$$a = 12.5 \text{ m/s}^2$$

a. ii. $u = 0$ $t = 8$ $s = 400$ $a = 12.5$
 constant acceleration, so we can use

$$v = u + at$$

$$v = 0 + 12.5 \times 8$$

$$v = 100 \text{ (value as required)}$$

b. i. $F = ma = -(5000 + 0.5v^2)$

$$m = 400$$

$$400a = -(5000 + 0.5v^2)$$

$$a = -\left(\frac{5000}{400} + \frac{0.5v^2}{400}\right)$$

$$= -\left(\frac{10\,000 + v^2}{800}\right)$$

b. ii. $a = -\left(\frac{10\,000 + v^2}{800}\right)$

$$a = v \frac{dv}{dx} = -\frac{(10^4 + v^2)}{800}$$

$$\frac{dv}{dx} = -\frac{(10^4 + v^2)}{800v}$$

b. iii. $\frac{dv}{dx} = -\frac{(10^4 + v^2)}{800v}$

$$\Rightarrow \frac{dx}{dv} = -\frac{800v}{10^4 + v^2}$$

$$dx = -\frac{800v}{10^4 + v^2} \cdot dv$$

$$x = \int -\frac{800v}{10^4 + v^2} \cdot dv$$

$$\text{Let } u = 10^4 + v^2$$

$$\frac{du}{dv} = 2v$$

$$dv = \frac{du}{2v}$$

$$x = -\int \frac{800v}{u} \times \frac{du}{2v}$$

$$= -\int \frac{400}{u} du$$

$$x = -400 \log_e u + c$$

$$= -400 \log_e (10^4 + v^2) + c$$

$$\text{at } x = 0 \quad v = 100 \text{ (from a. ii.)}$$

$$c = 400 \log_e (10^4 + 10^4)$$

$$= 400 \log_e (2 \times 10^4)$$

$$\therefore x = -400 \log_e (10^4 + v^2) + 400(2 \times 10^4)$$

$$= 400 \log_e \left(\frac{2 \times 10^4}{10^4 + v^2} \right)$$

$$\text{at } v = 0$$

$$x = 400 \log_e \left(\frac{2 \times 10^4}{10^4} \right)$$

$$= 400 \log_e 2$$

$$= 277 \text{ metres}$$

c. From 3 b. i. we know that:

$$a = -\left(\frac{10^4 + v^2}{800}\right)$$

$$\Rightarrow \frac{dv}{dt} = -\left(\frac{10^4 + v^2}{800}\right)$$

$$\Rightarrow \frac{dt}{dv} = \frac{-800}{10^4 + v^2}$$

$$\frac{dt}{dv} = \frac{-800}{10^2} \times \frac{10^2}{(10^2)^2 + v^2}$$

$$t = -8 \tan^{-1} \frac{v}{100} + c$$

$$\text{at } t = 0 \quad v = 100$$

$$c = 8 \tan^{-1}(1)$$

$$= 8 \times \frac{\pi}{4}$$

$$= 2\pi$$

$$t = -8 \tan^{-1} \frac{v}{100} + 2\pi$$

$$\text{at } v = 0$$

$$t = 0 + 2\pi$$

$$= 6.283$$

$$= 6 \text{ seconds (to the nearest second)}$$

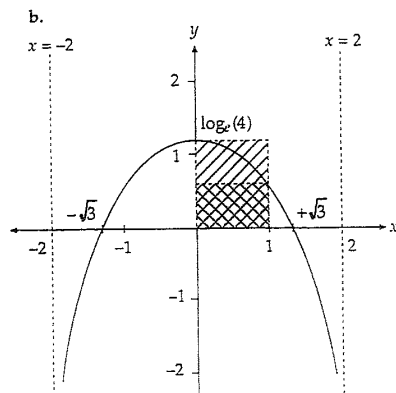
4. a. $(2-x)(2+x) > 0$

$$\Rightarrow 2-x > 0 \text{ and } 2+x > 0$$

$$\Rightarrow -x > -2 \text{ and } x > -2$$

$$x < 2$$

$$\therefore D_{DM}f = D = (-2, 2) \text{ or } \{x: -2 < x < 2\}$$



c. Rectangle $l \times w = \log_e 4 \times 1 = \log_e 4$

Rectangle $l \times w = \log_e(4-1) = \log_e 3$

$$w = 1$$

$$l \times w = \log_e 3 \times 1 = \log_e 3$$

$$\therefore \log_e 3 < A < \log_e 4$$

d. i. $y = x \log_e(4-x^2)$

$$\frac{dy}{dx} = \log_e(4-x^2) + x \times \frac{-2x}{4-x^2}$$

$$= \log_e(4-x^2) - \frac{2x^2}{4-x^2}$$

d. ii. $\frac{x^2}{4-x^2} = -1 + \frac{4}{4-x^2}$

$$\frac{4}{4-x^2} = \frac{4}{(2-x)(2+x)} = \frac{A}{2-x} + \frac{B}{2+x}$$

$$4 \equiv A(2+x) + B(2-x)$$

$$\text{at } x = -2$$

$$4 = 4B \quad B = 1$$

$$\text{at } x = 2$$

$$4 = 4A \quad A = 1$$

$$\therefore \frac{x^2}{4-x^2} = -1 + \frac{1}{2-x} + \frac{1}{2+x}$$

$$\int \frac{x^2}{4-x^2} dx = \int -1 + \frac{1}{2-x} + \frac{1}{2+x} dx$$

$$= -x - \log_e(2-x) + \log_e(2+x) + c$$

$$= \log_e \frac{(2+x)}{(2-x)} - x$$

d. iii. $\int_0^1 \log_e(4-x^2) dx$

From part d. i.

$$\frac{dx \log_e(4-x^2)}{dx} = \log_e(4-x^2) - \frac{2x^2}{4-x^2}$$

$$\Rightarrow \int \log_e(4-x^2) dx = x \log_e(4-x^2) + \int \frac{2x^2}{4-x^2} dx$$

using result of part d. ii.

$$\int \log_e(4-x^2) dx = x \log_e(4-x^2) + 2 \int \frac{x^2}{4-x^2} dx$$

$$= x \log_e(4-x^2) - 2x - 2 \log_e(2-x) + 2 \log_e(2+x)$$

$$\int_0^1 \log_e(4-x^2) dx$$

$$= \log_e 3 - 2 - 2 \log_e 1 + 2 \log_e 3 - 0 + 0 + 2 \log_e 2 - 2 \log_e 2$$

$$= -2 + 3 \log_e 3$$

e. i. $y_{n+1} = y_n + hf(x_n)$

$$n+1 = 20 \quad n = 19 \quad h = 0.05$$

$$y_{20} = y_{19} + 0.05 f(x_{19})$$

$$y_{19} = 19 \times 0.5 \quad f(x) = \log_e(4-x^2)$$

$$y_{20} = y_{19} + 0.05 \log_e(4 - (19 \times 0.5)^2)$$

$$= y_{19} + 0.05 \log_e(3.0975)$$

e. ii. $y_{20} = 1.2464 + 0.05 \log_e(3.0975)$

$$= 1.3029 \text{ correct to 4 decimal places}$$

e. iii. y_{20} is an approximation to the solution of the differential equation

$$\frac{dy}{dx} = \log_e(4-x^2)$$

$$\text{We know } A = \int_0^1 \log_e(4-x^2) dx \text{ which is}$$

the solution of $\frac{dy}{dx}$ at $x = 1$ subtract the

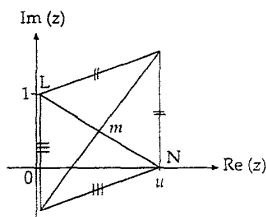
solution of $\frac{dy}{dx}$ at $x = 0$.

We know that $\frac{dy}{dx} = 0$ at $x = 0$

$$\text{Therefore } y_{20} \approx A = \int_0^1 \log_e(4-x^2) dx$$

5. a. i. When z lies on $\overline{LN}$ it is equal distance from L and N that means it is on the midpoint, m .

When z lies on either side of the $\overline{LN}$ it is equidistant from L and N , and forms an isosceles triangle. The line drawn from the vertex of an isosceles triangle bisecting the base of the triangle is perpendicular to the base.



a. ii. $|z - i| = |z - u|$
 $|x + iy - i| = |x + iy - u|$
 $|x + i(y - 1)| = |x - u + iy|$

$$\Rightarrow x^2 + (y - 1)^2 = (x - u)^2 + y^2$$

$$x^2 + y^2 - 2y + 1 = x^2 - 2xu + u^2 + y^2$$

$$-2y + 1 = -2xu + u^2$$

$$2y - 1 = 2ux - u^2$$

$$2y = 2ux - u^2 + 1$$

b. i. $y = ux - \frac{u^2}{2} + \frac{1}{2}$
at $x = u$

$$y = u^2 - \frac{u^2}{2} + \frac{1}{2}$$

$$= \frac{1}{2}(u^2 + 1)$$

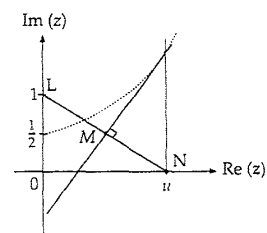
$$w = x + yi$$

$$w = u + \frac{1}{2}(u^2 + 1)i$$

b. ii. $w = u + \frac{1}{2}(u^2 + 1)i$

$$x = u \quad y = \frac{1}{2}(u^2 + 1)$$

$$y = \frac{1}{2}(x^2 + 1), \quad x > 0$$



c. $y = \frac{1}{2}(x^2 + 1)$

$$\frac{dy}{dx} = x$$

Tangent at $x = u \quad y = \frac{1}{2}(u^2 + 1)$

Given by $y - y_1 = \frac{dy}{dx}(x - x_1)$ at $x = u$

$$y = \frac{1}{2}(u^2 + 1) = u(x - u)$$

$$2y - u^2 - 1 = 2ux - 2u^2$$

$$2y = 2ux - u^2 + 1$$

From a. ii. we know that this is the equation of the perpendicular bisector of LN therefore the perpendicular bisector is tangent to the curve at w .