

MAV Specialist Mathematics Examination 2 Solutions

Question 1 a. i.

$$\begin{aligned} z^2 &= (1 - 2i)^2 \\ &= 1^2 - 4i - 4 \\ &= -3 - 4i \end{aligned} \quad [\text{A}]$$

Question 1 a. ii.

$$\begin{aligned} P(z) &= 0 \\ z^2 - 5z + 7 + i &= 0 \\ z &= \frac{5 \pm \sqrt{25 - 4(7+i)}}{2} \\ &= \frac{5 \pm \sqrt{-3 - 4i}}{2} \end{aligned}$$

Using the result from a. i. [M]

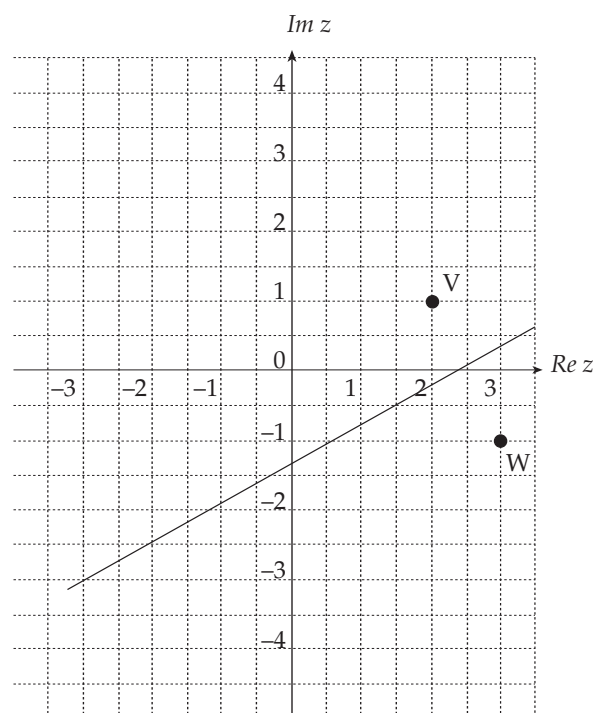
$$z = \frac{5 \pm (1 - 2i)}{2}$$

$$z = 2 + i \quad [\text{A}]$$

$$\text{or } z = 3 - i \quad [\text{A}]$$

Question 1 b.

1 mark for both of their points correctly positioned. [A]



Question 1 c. i.

$$S = \{z: |z - v| = |z - w|, z \in \mathbb{C}\}$$

$$|x + iy - 2 - i| = |x + iy - 3 + i| \quad [\text{M}]$$

$$|x - 2 + iy - i| = |x - 3 + iy + i| \quad [\text{M}]$$

$$\begin{aligned} (x-2)^2 + (y-1)^2 &= (x-3)^2 + (y+1)^2 \\ x^2 - 4x + 4 + y^2 - 2y + 1 &= x^2 - 6x + 9 + y^2 + 2y + 1 \\ -4x - 2y + 5 &= -6x + 2y + 10 \\ -4y &= -2x + 5 \\ y &= \frac{x}{2} - \frac{5}{4} \end{aligned} \quad [\text{A}]$$

Question 1 c. ii.

On graph, to gain mark should be a straight line with y -intercept $(0, -1.25)$ and x -intercept $(2.5, 0)$ [A]

Question 2 a. i.

$$\frac{d}{dx}(x \tan^{-1} x) = \tan^{-1} x + \frac{x}{1+x^2} \quad [\text{A}]$$

Question 2 a. ii.

$$\begin{aligned} \tan^{-1} x &= \frac{d}{dx}(x \tan^{-1} x) - \frac{x}{1+x^2} \\ \int \tan^{-1} x dx &= \int \left(\frac{d}{dx}(x \tan^{-1} x) - \frac{x}{1+x^2} \right) dx \quad [\text{M}] \\ &= x \tan^{-1}(x) - \frac{1}{2} \log_e(1+x^2) \quad [\text{A}] \end{aligned}$$

Question 2 b. i.

$$\begin{aligned} v(t) &= \frac{9 \tan^{-1} \sqrt{t}}{\sqrt{t}} \\ v(3) &= \frac{9 \tan^{-1} \sqrt{3}}{\sqrt{3}} \\ &= \frac{9 \times \frac{\pi}{3}}{\sqrt{3}} = \sqrt{3} \pi \end{aligned} \quad [\text{A}]$$

Question 2 b. ii.

$$x = \int_0^3 \frac{9 \tan^{-1} \sqrt{t}}{\sqrt{t}} dt$$

$$\text{Let } u = \sqrt{t}$$

[M]

$$\text{When } t = 3, u = \sqrt{3}$$

$$\text{When } t = 0, u = 0$$

$$\frac{du}{dt} = \frac{1}{2\sqrt{t}}$$

[M]

$$x = \int_0^3 \frac{9 \tan^{-1} u}{\sqrt{t}} \times \frac{2\sqrt{t} du}{dt} dt$$

$$= 18 \int_0^{\sqrt{3}} \tan^{-1} u du$$

[A]

$$= 18 \left[u \tan^{-1} u - \frac{1}{2} \log_e (1 + u^2) \right]_0^{\sqrt{3}}$$

$$= 18 \left[\left(\sqrt{3} \tan^{-1} \sqrt{3} - \frac{1}{2} \log_e (1 + \sqrt{3}^2) \right) - 0 \right]$$

[M]

$$= 18 \left[\sqrt{3} \times \frac{\pi}{3} - \log_e (2) \right]$$

$$= 6\pi\sqrt{3} - 18 \log_e (2)$$

[A]

Question 3 a.

QR is tangential, therefore $\angle OQR$ is a right angle

[M]

$$\cos \theta = \frac{OQ}{OR}$$

$$= \frac{2}{x}$$

Answer given, so
need to see working [M]

$$\theta = \cos^{-1} \left(\frac{2}{x} \right)$$

Question 3 b.

Arc Length

$$C = r\theta^c$$

$$\theta^c = \pi - \alpha$$

$$= \pi - \cos^{-1} \left(\frac{2}{x} \right)$$

$$r = 2$$

$$C = 2 \left(\pi - \cos^{-1} \left(\frac{2}{x} \right) \right) \quad \text{Answer given, so need to see working [M][A]}$$

Question 3 c.

$$C = 2 \left(\pi - \cos^{-1} \left(\frac{2}{x} \right) \right)$$

$$\text{Let } u = \frac{2}{x}, \quad \frac{du}{dx} = \frac{-2}{x^2}$$

[M]

$$C = 2 \left(\pi - \cos^{-1} u \right)$$

$$\frac{dC}{du} = \frac{2}{\sqrt{1-u^2}}$$

[M]

$$\frac{dC}{dx} = \frac{dC}{du} \times \frac{du}{dx}$$

$$= \frac{2}{\sqrt{1-\frac{4}{x^2}}} \times \frac{-2}{x^2}$$

$$= \frac{2}{\frac{1}{x} \sqrt{x^2-4}} \times \frac{-2}{x^2}$$

$$= \frac{2}{\sqrt{x^2-4}} \times \frac{-2}{x}$$

[A]

$$= \frac{-4}{x\sqrt{x^2-4}}$$

Question 3 d.

Since $\angle OQR$ is a right angle

$$QR^2 = x^2 - 4$$

$$QR = \sqrt{x^2 - 4}$$

$$V = 2 \left(\pi - \cos^{-1} \left(\frac{2}{x} \right) \right) + \sqrt{x^2 - 4}$$

[A]

Question 3 e.

$$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt} \quad [\text{M}]$$

$$\text{Given } \frac{dx}{dt} = -3$$

$$V = 2\left(\pi - \cos^{-1}\left(\frac{2}{x}\right)\right) + \sqrt{x^2 - 4}$$

$$\frac{dV}{dx} = \frac{-4}{x\sqrt{x^2-4}} + \frac{x}{\sqrt{x^2-4}} \quad [\text{M}]$$

$$\begin{aligned} \frac{dV}{dx} &= \frac{-4+x^2}{x\sqrt{x^2-4}} \\ &= \frac{\sqrt{x^2-4}}{x} \end{aligned}$$

$$\frac{dV}{dt} = \frac{\sqrt{x^2-4}}{x} \times -3 \quad [\text{M}]$$

$$\text{At } x = 6$$

$$\begin{aligned} \frac{dV}{dt} &= \frac{\sqrt{36-4}}{6} \times -3 \\ &= -\frac{\sqrt{32}}{2} \\ &= -2\sqrt{2} \end{aligned} \quad [\text{A}]$$

Question 4 a. i.

$$\frac{dT}{dt} = k(T - S) \quad [\text{A}]$$

Question 4 a. ii.

$$\begin{aligned} \frac{dT}{dt} &= k(T - S) \\ \frac{dt}{dT} &= \frac{1}{k(T - S)} \end{aligned} \quad [\text{M}]$$

$$t + c = \frac{1}{k} \log_e(T - S)$$

$$kt + kc = \log_e(T - S)$$

$$e^{kt+kc} = T - S$$

$$\text{Let } A = e^{kc}$$

$$T = Ae^{kt} + S \quad [\text{A}]$$

Question 4 b. i.

$$\text{At } t = 0 \quad A + S = 20 \quad [\text{A}]$$

$$\text{At } t = 10 \quad Ae^{10k} + S = 15 \quad [\text{A}]$$

$$\text{At } t = 20 \quad Ae^{20k} + S = 11 \quad [\text{A}]$$

Question 4 b. ii.

$$S = 20 - A \quad \dots (1)$$

$$Ae^{10k} + 20 - A = 15$$

$$A(e^{10k} - 1) = -5 \quad \dots (2) \quad [\text{M}]$$

$$Ae^{20k} + 20 - A = 11$$

$$A(e^{20k} - 1) = -9 \quad \dots (3) \quad [\text{M}]$$

$$(2) + (1) \quad \frac{A(e^{20k} - 1)}{A(e^{10k} - 1)} = \frac{-9}{-5} \quad k \neq 0 \quad [\text{M}]$$

$$5(e^{10k})^2 - 5 = 9e^{10k} - 9$$

$$5(e^{10k})^2 - 9e^{10k} + 4 = 0$$

$$(5e^{10k} - 4)(e^{10k} - 1) = 0$$

OR

$$\frac{A(e^{10k} - 1)(e^{10k} + 1)}{A(e^{10k} - 1)} = \frac{-9}{-5} \quad [\text{M}]$$

$$e^{10k} + 1 = \frac{9}{5}$$

$$e^{10k} = \frac{4}{5}$$

$$\begin{aligned} (5e^{10k} - 4) \quad \text{OR} \quad e^{10k} &= \frac{4}{5} \\ k &= \frac{\log_e\left(\frac{4}{5}\right)}{10} \quad \text{OR} \quad k = \frac{\log_e\left(\frac{4}{5}\right)}{10} \end{aligned} \quad [\text{M}]$$

$$\text{Substituting } k = \frac{\log_e\left(\frac{4}{5}\right)}{10} \text{ into (2)}$$

$$A \left(e^{10 \frac{\log_e\left(\frac{4}{5}\right)}{10}} - 1 \right) = -5$$

$$\frac{-A}{5} = -5$$

$$A = 25 \quad [\text{M}]$$

Substituting $A = 25$ into (1)

$$S = -5$$

$$T = 25e^{\frac{t \log_e \left(\frac{4}{5} \right)}{10}} - 5 \quad [\text{M}]$$

Question 4 c. i.

$$T = 25e^{\frac{t \log_e \left(\frac{4}{5} \right)}{10}} - 5$$

As $t \rightarrow \infty$, $T \rightarrow -5^\circ\text{C}$ Outside Temperature is -5°C [C]**Question 4 c. ii.**

$$0 = 25e^{\frac{t \log_e \left(\frac{4}{5} \right)}{10}} - 5$$

$$25e^{\frac{t \log_e \left(\frac{4}{5} \right)}{10}} = 5 \quad [\text{M}]$$

$$e^{\frac{t \log_e \left(\frac{4}{5} \right)}{10}} = 0.2$$

$$\frac{t \log_e \left(\frac{4}{5} \right)}{10} = \log_e(0.2)$$

 $t = 72 \text{ minutes } 8 \text{ seconds (to the nearest second)}$ [C]
Question 5 a. i.

Alternative 1

$$x = 3 \cos 2t \quad y = \sin 4t$$

$$\frac{x}{3} = \cos 2t \quad y = 2 \sin 2t \cos 2t \quad [\text{M}]$$

$$x^2 = 9 \cos^2 2t$$

$$\frac{x^2}{9} = \cos^2 2t$$

$$= 1 - \sin^2 2t$$

$$\sin^2 2t = 1 - \frac{x^2}{9} \quad [\text{M}]$$

$$y = 2 \times \sqrt{1 - \frac{x^2}{9}} \times \frac{x}{3}$$

$$= \frac{2x\sqrt{9-x^2}}{9}$$

$$y^2 = \frac{4x^2(9-x^2)}{81} \quad [\text{M}]$$

Alternative 2

$$x = 3 \cos 2t$$

$$y^2 = \frac{4x^2(9-x^2)}{81}$$

$$RHS = \frac{4x^2(9-x^2)^2}{81}$$

$$= \frac{4 \times 9 \cos^2 2t \times (9 - 9 \cos^2 2t)}{81} \quad [\text{M}]$$

$$= \frac{4 \times 9 \cos^2 2t \times 9(1 - \cos^2 2t)}{81}$$

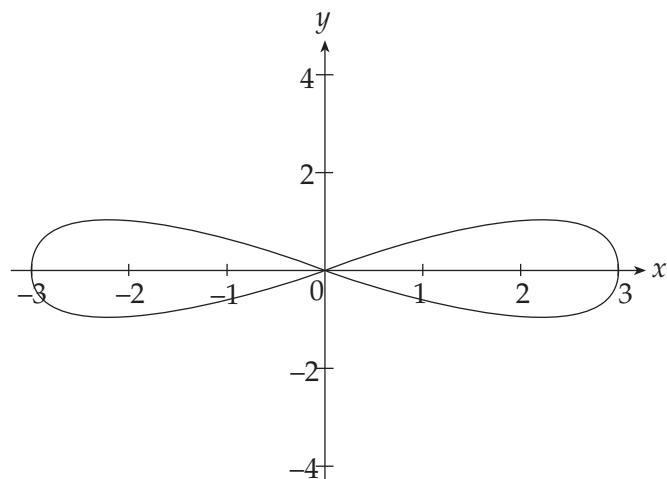
$$= 4 \cos^2 2t \sin^2 2t$$

$$= (2 \cos 2t \sin 2t)^2$$

$$= \sin^2 4t \quad [\text{M}]$$

$$LHS = RHS$$

$$\therefore y^2 = \frac{4x^2(9-x^2)}{81} \quad [\text{M}]$$

Question 5 a. ii.

2 marks for complete graph showing both positive and negative components of the graph, correct axial intercepts and range.

1 mark for half the graph, with correct axial intercepts and range.

Question 5 b.

$$\tilde{r}(t) = 3 \cos(2t) \tilde{i} + \sin(4t) \tilde{j}$$

$$\tilde{\dot{r}}(t) = -6 \sin(2t) \tilde{i} + 4 \cos(4t) \tilde{j} \quad [\text{M}]$$

$$\tilde{\ddot{r}}(t) = -12 \cos(2t) \tilde{i} - 16 \sin(4t) \tilde{j} \quad [\text{A}]$$

Question 5 c. i.

$$\begin{aligned} \tilde{r}(t) \cdot \tilde{\ddot{r}}(t) &= -36 \cos^2 2t - 16 \sin^2 4t \\ &= -18(2 \cos^2 2t) - 16 \sin^2 4t \end{aligned} \quad [\text{M}]$$

$$= -18(\cos 4t + 1) - 16(1 - \cos^2 4t) \quad [\text{M}]$$

$$= -18(U + 1) - 16(1 - U^2) \quad [\text{M}]$$

$$= 2(8U^2 - 9U - 17) \quad [\text{A}]$$

Question 5 c. ii.

$$\begin{aligned} \tilde{r}(t) \cdot \tilde{\ddot{r}}(t) &= 2(8U^2 - 9U - 17) \\ &= 2(8U - 17)(U + 1) \end{aligned} \quad [\text{M}]$$

$$\tilde{r}(t) \cdot \tilde{\ddot{r}}(t) = 0 \text{ when}$$

$$U = \frac{17}{8}$$

$$U = -1$$

$$U = \cos 4t \quad [\text{M}]$$

Disregard $U = \frac{17}{8}$ because it is greater 1

$$\cos 4t = -1$$

$$4t = \pi + 2n\pi$$

$$t = \frac{\pi + 2n\pi}{4} \quad [\text{M}]$$

n is a positive integer or zero

Question 5 c. iii.

$$\tilde{\ddot{r}}\left(\frac{\pi}{4}\right) = -12 \cos\left(\frac{2\pi}{4}\right) \tilde{i} - 16 \sin\left(\frac{4\pi}{4}\right) \tilde{j}$$

$$= -12 \cos\left(\frac{\pi}{2}\right) \tilde{i} - 16 \sin(\pi) \tilde{j}$$

$$= 0 \tilde{i} - 0 \tilde{j} = \mathbf{0} \quad [\text{A}]$$

Question 5 d.

The position and acceleration vector will be perpendicular if $\vec{r}(t) \cdot \ddot{\vec{r}}(t) = 0$ given that neither

$$\vec{r}(t) = \vec{0} \quad \text{or} \quad \ddot{\vec{r}}(t) = \vec{0}. \quad [\text{M}]$$

$$\vec{r}(t) \cdot \ddot{\vec{r}}(t) = 0$$

$$\text{When } t = \frac{\pi + 2n\pi}{4}$$

$$\begin{aligned} & \ddot{\vec{r}}\left(\frac{\pi + 2n\pi}{4}\right) \\ &= -12\cos\left(2\frac{(\pi + 2n\pi)}{4}\right)\vec{i} - 16\sin\left(4\frac{(\pi + 2n\pi)}{4}\right)\vec{j} \\ &= -12\cos\left(\frac{(\pi + 2n\pi)}{2}\right)\vec{i} - 16\sin(\pi + 2n\pi)\vec{j} \quad [\text{M}] \\ &= 0\vec{i} - 0\vec{j} = \vec{0}, \text{ for all } n \text{ when } n \text{ is a positive integer.} \end{aligned}$$

This implies that $\ddot{\vec{r}}(t) = \vec{0}$ every time that

$\vec{r}(t) \cdot \ddot{\vec{r}}(t) = 0$, so the position and acceleration vector are never perpendicular, and so the train will not derail. [A]