

Question 1

- a. i. From the diagram, $\vec{OQ} = 7\vec{i} + 24\vec{j}$. (1 mark)

- ii. Similarly, $\vec{OP} = 7\vec{i} + 24\vec{j} + 2\vec{k}$. (1 mark)

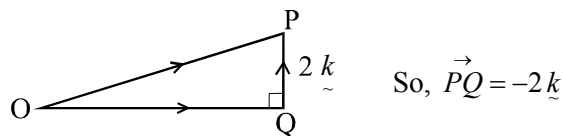
- b. i. The vector component of $\vec{OP}$ perpendicular to $\vec{OQ}$ is given by

$$\begin{aligned} & \vec{OP} - \left(\vec{OP} \cdot \frac{\vec{OQ}}{|\vec{OQ}|} \right) \frac{\vec{OQ}}{|\vec{OQ}|} \quad (1 \text{ mark}) \\ &= 7\vec{i} + 24\vec{j} + 2\vec{k} - \left\{ (7\vec{i} + 24\vec{j} + 2\vec{k}) \cdot \frac{1}{25}(7\vec{i} + 24\vec{j}) \right\} \frac{1}{25}(7\vec{i} + 24\vec{j}) \\ &= 7\vec{i} + 24\vec{j} + 2\vec{k} - \frac{625}{25} \times \frac{1}{25}(7\vec{i} + 24\vec{j}) \\ &= 7\vec{i} + 24\vec{j} + 2\vec{k} - 7\vec{i} - 24\vec{j} \\ &= 2\vec{k} \end{aligned}$$

as required (1 mark)

- ii. Method 1

From the diagram, $\vec{PQ}$ is the component of $\vec{OP}$ perpendicular to $\vec{OQ}$.



(1 mark)

(1 mark)

Method 2

$$\begin{aligned} \vec{PQ} &= \vec{PO} + \vec{OQ} \quad (1 \text{ mark}) \\ &= -7\vec{i} - 24\vec{j} - 2\vec{k} + 7\vec{i} + 24\vec{j} \\ &= -2\vec{k} \end{aligned}$$

(1 mark)

c. Now, $\vec{RT} = \vec{RO} + \vec{OT}$

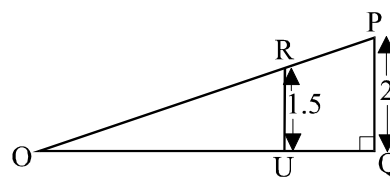
Now, from the diagram, $\vec{OT} = 3\hat{i} + 25\hat{j}$.

(1 mark)

We need to find $\vec{RO}$.

Now, $\triangle ORU$ is similar to $\triangle OPQ$.

So, $\vec{OR} = \frac{3}{4}(\vec{OP})$ (1 mark)



Now, $\vec{RT} = \vec{RO} + \vec{OT}$

$$= -\vec{OR} + \vec{OT}$$

$$= -\frac{3}{4}(\vec{OP}) + \vec{OT} \quad (1 \text{ mark})$$

$$= -\frac{3}{4}(7\hat{i} + 24\hat{j} + 2\hat{k}) + 3\hat{i} + 25\hat{j}$$

$$= -5.25\hat{i} - 18\hat{j} - 1.5\hat{k} + 3\hat{i} + 25\hat{j}$$

$$= -2.25\hat{i} + 7\hat{j} - 1.5\hat{k} \text{ as required} \quad (1 \text{ mark})$$

d. We are looking for the angle of deflection $\angle PRT$. We want to find the angle between the vectors $\vec{RT}$ and $\vec{RP}$. Since $\vec{RP}$ runs parallel to $\vec{OP}$, we seek to find the angle between the vectors $\vec{RT}$ and $\vec{OP}$. (1mark)

$$\vec{RT} = -2.25\hat{i} + 7\hat{j} - 1.5\hat{k}$$

and $\vec{OP} = 7\hat{i} + 24\hat{j} + 2\hat{k}$

So, $\cos \theta = \frac{-15.75 + 168 - 3}{\sqrt{56.3125}\sqrt{629}}$ (1 mark)

$$= 0.7930$$

$$\theta = 37^\circ 32'$$

So the angle of deflection is $37^\circ 32'$ (to the nearest minute)

(1mark)

Total 13 marks

Question 2

- a. 10.00am corresponds to 3 hours after work started so $t = 3$.

Now $\frac{dl}{dt} = 2te^{\frac{-t^2}{10}}$

When $t = 3$, $\frac{dl}{dt} = 6e^{\frac{-9}{10}}$
 $= 2.44$ metres/ hour

So, at 10.00am, the rate at which the path is being laid is 2.44 metres/hour (correct to 2 decimal places).

(1 mark)

- b. i. We are looking for $\frac{dl}{dt}$ to be a maximum. This will occur when $\frac{d^2l}{dt^2} = 0$.

Now $\frac{dl}{dt} = 2te^{\frac{-t^2}{10}}$

So, $\frac{d^2l}{dt^2} = 2t \times \frac{-2t}{10} e^{\frac{-t^2}{10}} + 2e^{\frac{-t^2}{10}}$
 $= \frac{-2t^2}{5} e^{\frac{-t^2}{10}} + 2e^{\frac{-t^2}{10}}$

When $\frac{d^2l}{dt^2} = 0$

we have $\frac{-2t^2}{5} e^{\frac{-t^2}{10}} + 2e^{\frac{-t^2}{10}} = 0$

(1mark)

$$e^{\frac{-t^2}{10}} \left(-\frac{2t^2}{5} + 2 \right) = 0$$

Now, $e^{\frac{-t^2}{10}} \neq 0$ for $t \in \mathbb{R}$

So, $\frac{-2t^2}{5} + 2 = 0$

$$-2t^2 + 10 = 0$$

$$2(5 - t^2) = 0$$

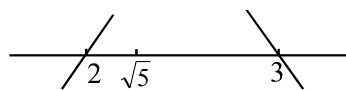
$$t = \pm\sqrt{5}$$

Since $t \geq 0$, $t = \sqrt{5}$ ($\cong 2.23$)

Check for a maximum.

When $t = 2$, $\frac{d^2l}{dt^2} = 0.2681$

When $t = 3$, $\frac{d^2l}{dt^2} = -0.6505$



So we have a maximum at $t = \sqrt{5}$.

(1 mark)

So the fastest rate occurs at $t = \sqrt{5}$ hours.

$$\begin{aligned}
 \text{Now } \sqrt{5} \text{ hours} &= \sqrt{5} \times 60 \text{ minutes} \\
 &= 134 \text{ minutes} \quad (\text{to the nearest minute}) \\
 &= 2 \text{ hour } 14 \text{ minutes}
 \end{aligned}$$

So the time at which the path is being laid fastest is 9.14am.

(1 mark)

- ii. From **part i.**, the fastest rate at which the landscape gardener lays the path occurs at $t = \sqrt{5}$.

$$\begin{aligned}
 \text{So } \frac{dl}{dt} &= 2\sqrt{5}e^{\frac{-5}{10}} \\
 &= 2.7
 \end{aligned}$$

So, the fastest rate at which the landscape gardener lays the path is 2.7 metres/hour (correct to one decimal place).

(1 mark)

- c. We need to solve, simultaneously, the equations

$$\frac{dl}{dt} = 2te^{\frac{-t^2}{10}} \quad \text{and} \quad \frac{dl}{dt} = 2 \quad \text{for } t > 2.$$

Using a graphics calculator we find that the two respective graphs intersect at (3.5655584, 2).

So, after 3 hours and 34 minutes (to the nearest minute) or at 10.34am, the rate at which the path is being laid drops below 2 metres per hour.

(1 mark)

$$\begin{aligned}
 \text{d. } \int 2te^{\frac{-t^2}{10}} dt & \qquad \text{let } u = \frac{-t^2}{10} \\
 & \qquad \frac{du}{dt} = \frac{-2t}{10} \\
 & \qquad = \frac{-t}{5} \\
 & = \int -10 \frac{du}{dt} e^u dt \quad \text{(1 mark)} \\
 & = -10 \int e^u du \\
 & = -10e^u + c \\
 & = -10e^{\frac{-t^2}{10}} + c \\
 & \qquad \qquad \qquad \text{(1 mark)}
 \end{aligned}$$

- e. We need to evaluate

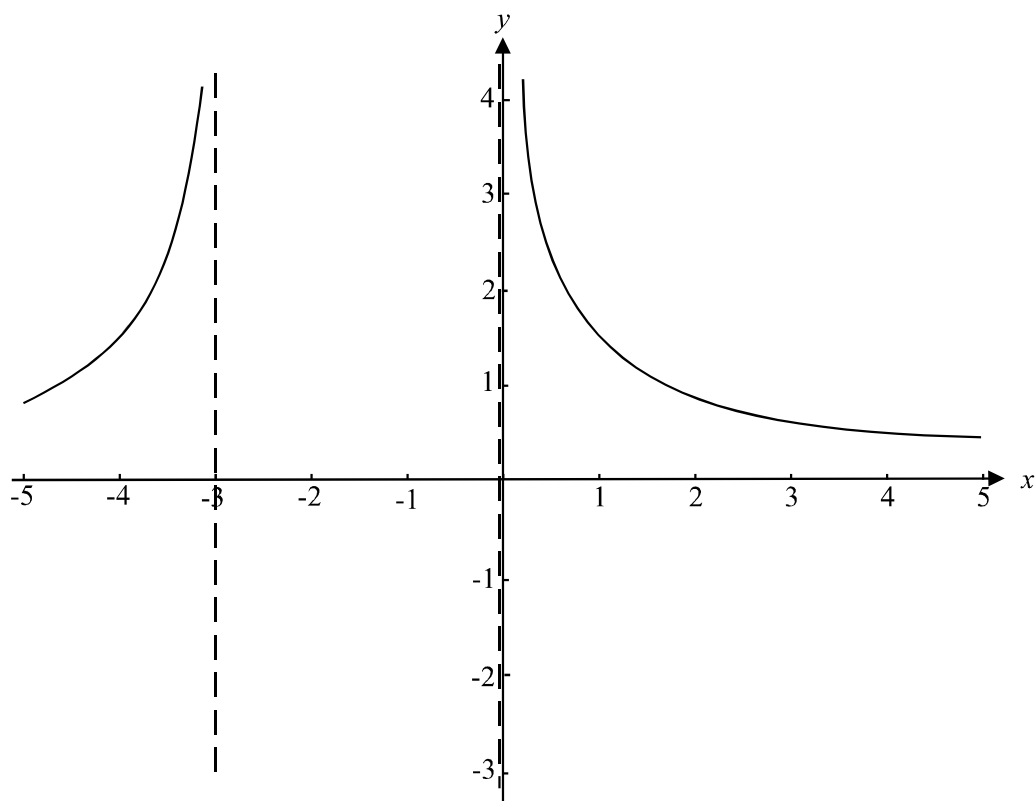
$$\begin{aligned}
 & \int_0^5 2te^{\frac{-t^2}{10}} dt \quad \text{(1 mark)} \\
 & = -10 \left[e^{\frac{-t^2}{10}} \right]_0^5 \\
 & = -10 \{ e^{-2.5} - e^0 \} \\
 & = 9.2 \text{ metres correct to 1 decimal place}
 \end{aligned}$$

(1 mark)

Total 10 marks

Question 3

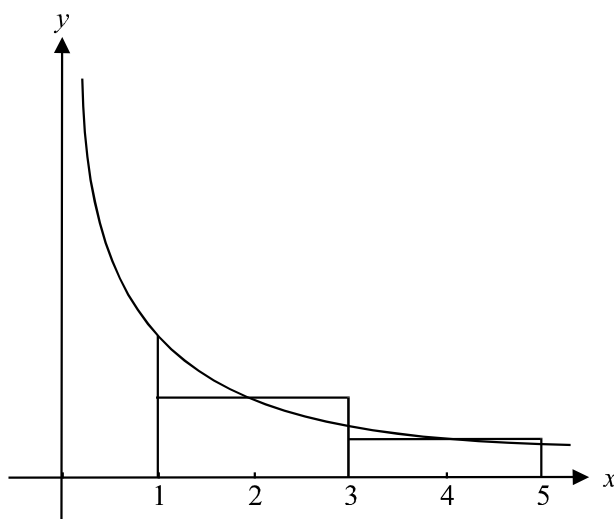
a.



(1 mark) asymptotes
(1 mark) shape
(1 mark)

b. $d = (-\infty, -3) \cup (0, \infty)$

c.



If $f(x) = \frac{3}{\sqrt{x(x+3)}}$

Area approximation $= (3-1)f(2) + (5-3)f(4)$

$= 2f(2) + 2f(4)$

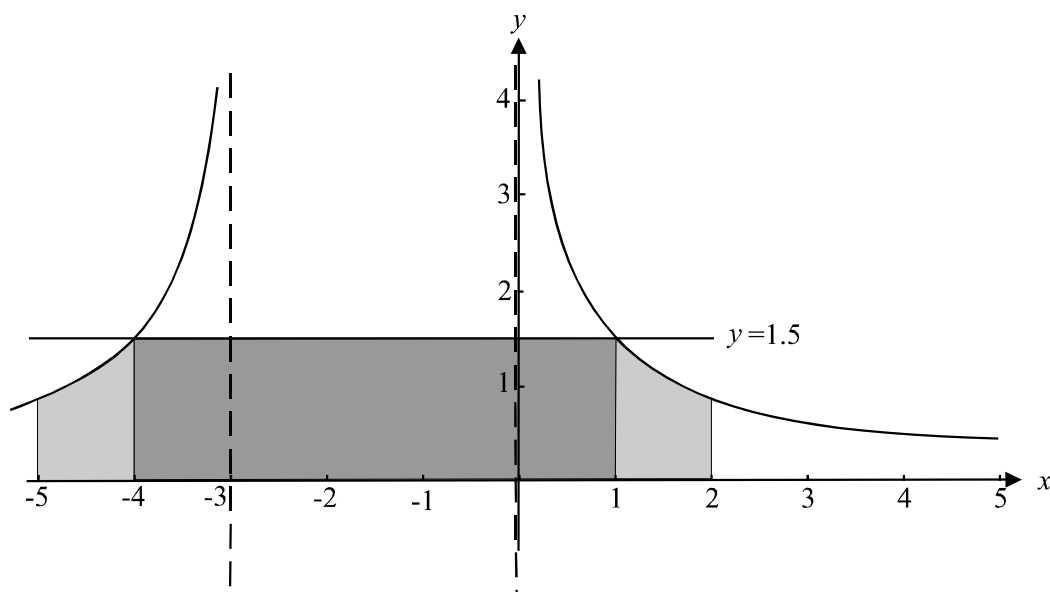
$= 1.897367 + 1.133893$ (correct to 6 decimal places)

$= 3.031260$

$= 3.03$ (correct to 3 significant figures)

(1 mark)

- d. The area required is shown in the diagram below broken into 3 sections.



To find the middle, darker area, we need to find the points of intersection between

$$y = \frac{3}{\sqrt{x(x+3)}} \text{ and } y = 1.5$$

$$\text{So, } 1.5 = \frac{3}{\sqrt{x(x+3)}}$$

$$1.5\sqrt{x(x+3)} = 3$$

$$\sqrt{x(x+3)} = 2$$

$$x(x+3) = 4$$

$$x^2 + 3x - 4 = 0$$

$$(x+4)(x-1) = 0$$

$$x = -4 \text{ or } x = 1$$

So the middle section has an area of $1.5 \times 5 = 7.5$ square units.

(1 mark)

$$\text{Total area required} = \int_{-5}^{-4} \frac{3}{\sqrt{x(x+3)}} dx + \int_1^2 \frac{3}{\sqrt{x(x+3)}} dx + 7.5$$

(1 mark)

Using a calculator, we obtain

$$= 1.177152 + 1.177152 + 7.500000 \text{ (each to 6 decimal places)}$$

$$= 9.85 \text{ square units (correct to 3 significant figures)}$$

(1 mark)

e.

$$\begin{aligned}\text{Volume required} &= \pi \int_1^5 y^2 dx \\ &= \pi \int_1^5 \frac{9}{x(x+3)} dx\end{aligned}$$

(1 mark)

$$\begin{aligned}\text{Let } \frac{9}{x(x+3)} &\equiv \frac{A}{x} + \frac{B}{x+3} \\ &\equiv \frac{A(x+3) + Bx}{x(x+3)}\end{aligned}$$

$$\text{True iff } 9 \equiv A(x+3) + Bx$$

$$\text{Put } x = -3, \text{ so } 9 = -3B \text{ so } B = -3$$

$$\text{Put } x = 0, \text{ so } 9 = 3A \text{ so } A = 3$$

$$\text{So, } \frac{9}{x(x+3)} \equiv \frac{3}{x} - \frac{3}{x+3}$$

$$\text{So, volume req'd} = \pi \int_1^5 \left(\frac{3}{x} - \frac{3}{x+3} \right) dx \quad (1 \text{ mark})$$

$$= \pi [3 \log_e x - 3 \log_e (x+3)]_1^5 \quad (1 \text{ mark})$$

$$= 3\pi \{(\log_e 5 - \log_e 8) - (\log_e 1 - \log_e 4)\}$$

$$= 3\pi (\log_e 5 - \log_e 8 + \log_e 4)$$

$$= 3\pi \log_e \frac{5 \times 4}{8}$$

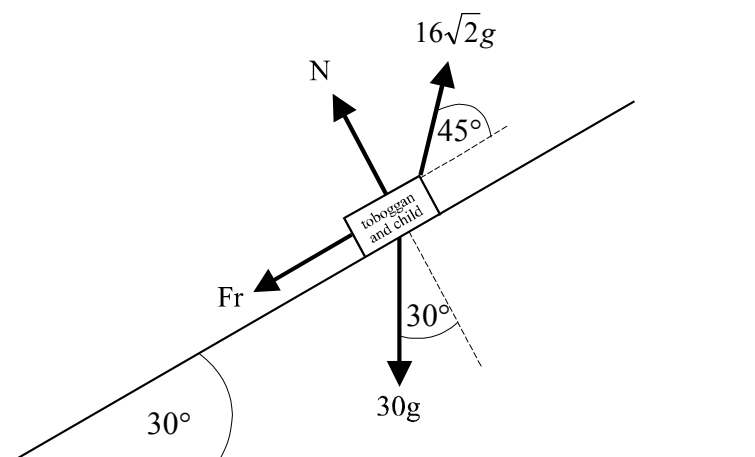
$$= 3\pi \log_e \frac{5}{2} \text{ cubic units}$$

(1 mark)

Total 12 marks

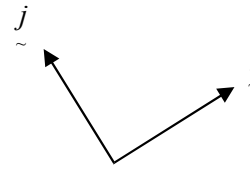
Question 4

a. i.



(2 marks)

- ii. Resolve forces, given that $\underline{\hat{i}}$ represents a unit vector parallel with the slope and $\underline{\hat{j}}$ represents a unit vector perpendicular to the slope.



Now, $\underline{\hat{R}} = m \underline{\hat{a}}$ becomes

$$(16\sqrt{2}g \cos 45^\circ - 30g \sin 30^\circ - Fr)\underline{\hat{i}} + (N + 16\sqrt{2}g \sin 45^\circ - 30g \cos 30^\circ)\underline{\hat{j}} = 30 \times 0 \underline{\hat{i}}$$

Note that since we have constant speed, $a = 0$.

(1 mark) for the left hand side of the equation

(1 mark) for the right hand side of the equation

$$\begin{aligned} \text{So } \frac{16\sqrt{2}g}{\sqrt{2}} - 15g - Fr &= 0 & \text{and} & & N + \frac{16\sqrt{2}g}{\sqrt{2}} - \frac{30\sqrt{3}g}{2} &= 0 \\ Fr &= g & \text{so} & & N &= -16g + 15\sqrt{3}g \\ & & & & &= g(15\sqrt{3} - 16) \end{aligned}$$

Now, $Fr = \mu N$

So, $\mu N = g$

(1 mark)

$$\begin{aligned} \mu &= \frac{g}{g(15\sqrt{3} - 16)} \\ &= \frac{1}{15\sqrt{3} - 16} \end{aligned}$$

as required

(1 mark)

- b i.** Now, $\underline{\hat{R}} = m \underline{\hat{a}}$

This time,

$$(18\sqrt{2}g \cos 45^\circ - 30g \sin 30^\circ - Fr)\underline{\hat{i}} + (N + 18\sqrt{2}g \sin 45^\circ - 30g \cos 30^\circ)\underline{\hat{j}} = 30a \underline{\hat{i}}$$

(1 mark)

$$\begin{aligned} \text{So } \frac{18\sqrt{2}g}{\sqrt{2}} - 15g - Fr &= 30a & \text{and} & & N + 18g - 30g \times \frac{\sqrt{3}}{2} &= 0 \\ 30a &= 3g - \mu N & & & N &= 15\sqrt{3}g - 18g \\ &= 3g - \frac{1}{15\sqrt{3} - 16} \times g(15\sqrt{3} - 18) & & & &= g(15\sqrt{3} - 18) \\ &= 3g - g\left(\frac{15\sqrt{3} - 18}{15\sqrt{3} - 16}\right) \end{aligned}$$

$$a = 0.72 \text{ m/s}^2 \text{ correct to 2 decimal places}$$

(1 mark)

- ii. Since acceleration is constant, we have,

$$s = ut + \frac{1}{2}at^2$$

$$\begin{aligned}\text{So, } s &= 0 \times 20 + \frac{1}{2} \times 0.72 \times 20^2 \\ &= 144 \text{ metres}\end{aligned}$$

(1 mark)

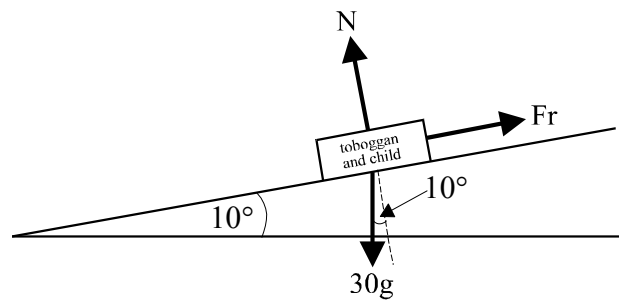
- iii. Again, since acceleration is constant we have

$$v = u + at$$

$$\begin{aligned}v &= 0 + 0.72 \times 20 \\ &= 14.4 \text{ m/s}\end{aligned}$$

(1 mark)

- c. i.



(2 marks)

- ii. Resolving the forces, we have

$$\underline{R} = m \underline{a}$$

$$(Fr - 30g \sin 10^\circ) \underline{i} + (N - 30g \cos 10^\circ) \underline{j} = 0 \underline{i}$$

$$\text{So, } Fr = 51.05 \text{ and } N = 289.53$$

(1 mark)

Now at the point of limiting equilibrium, when the toboggan is about to slide,

$$Fr = \mu N$$

$$\text{Now, } \mu N = 289.53 \times 0.2$$

$$= 57.906$$

and from above, $Fr = 51.05$.

$$\text{So } Fr < \mu N$$

So, the toboggan is not on the point of sliding down the slope.

(1 mark)

Total 14 marks

Question 5**a. i.**

$$z_1 = \sqrt{2} + \sqrt{2}i$$

$$r = \sqrt{(\sqrt{2})^2 + (\sqrt{2})^2}$$

$$= \sqrt{2+2}$$

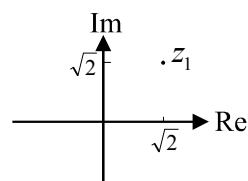
$$= 2$$

$$\theta = \tan^{-1} \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \tan^{-1} 1$$

$$= \frac{\pi}{4} \text{ since } z_1 \text{ is in the first quadrant.}$$

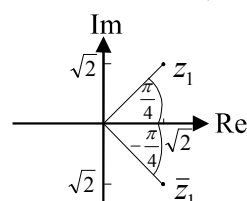
$$\text{So } z_1 = 2\text{cis} \frac{\pi}{4}$$

**(1 mark)****ii.** Method 1

$$z_1 = 2\text{cis} \frac{\pi}{4}$$

$$\text{so } \bar{z}_1 = 2\text{cis} \left(\frac{-\pi}{4} \right) \text{ since}$$

$\bar{z}_1$ is located by reflecting z_1 in the Real axis.

Method 2

$$z_1 = \sqrt{2} + \sqrt{2}i$$

$$\text{So, } \bar{z}_1 = \sqrt{2} - \sqrt{2}i$$

$$r = \sqrt{2+2}$$

$$= 2$$

$$\text{So, } \bar{z}_1 = 2\text{cis} \left(\frac{-\pi}{4} \right)$$

$$\theta = \tan^{-1} \left(\frac{-\sqrt{2}}{\sqrt{2}} \right)$$

$$= \tan^{-1}(-1)$$

$$= \frac{-\pi}{4} \text{ since } \bar{z}_1 \text{ is in the 4}^{\text{th}} \text{ quadrant.}$$

(1 mark)**iii.** To show : $\frac{z_1}{z_2} = -i$

$$\text{Left side} = \frac{z_1}{z_2}$$

$$= \frac{2\text{cis} \frac{\pi}{4}}{2\text{cis} \left(\frac{3\pi}{4} \right)}$$

$$= \text{cis} \left(\frac{-\pi}{2} \right)$$

(1mark)

$$= \cos \left(\frac{-\pi}{2} \right) + i \sin \left(\frac{-\pi}{2} \right)$$

$$= 0 + i \times -1$$

$$= -i$$

$$= \text{right side} \quad \text{Have shown.}$$

(1 mark)

- b.** Firstly, express z_2 in the form $x + yi$.

$$\begin{aligned} z_2 &= 2\text{cis}\left(\frac{3\pi}{4}\right) \\ &= 2\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right) \\ &= 2\left(-\frac{1}{\sqrt{2}} + i \times \frac{1}{\sqrt{2}}\right) \\ &= -\frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}}i \\ &= -\sqrt{2} + \sqrt{2}i \end{aligned}$$

(1 mark)

Now the equation is given by

$$\begin{aligned} &a(z - z_1)(z - z_2), \quad a \in \mathbb{R} \\ &= a(z - \sqrt{2} - \sqrt{2}i)(z + \sqrt{2} - \sqrt{2}i) \quad \text{(1 mark)} \\ &= a(z^2 + \sqrt{2}z - \sqrt{2}iz - \sqrt{2}z - 2 + 2i - \sqrt{2}iz - 2i - 2) \\ &= a(z^2 - 2\sqrt{2}iz - 4) \end{aligned}$$

So z_1 and z_2 could be the roots of any quadratic equation of the form

$$a(z^2 - 2\sqrt{2}iz - 4), \quad a \in \mathbb{R}.$$

(1 mark)

- c. i.** We know that $z_3 = \text{cis } \theta$

$$\begin{aligned} \text{So,} \quad (z_3)^n &= (\text{cis } \theta)^n \\ &= \text{cis}(n\theta) \quad \text{De Moivre's Theorem} \end{aligned}$$

(1 mark)

- ii.** To show: $\sin 3\theta = 3\cos^2 \theta \sin \theta - \sin^3 \theta$

From **part c. i.**, we have

$$(z_3)^n = \text{cis}(n\theta)$$

$$\text{So,} \quad (\text{cis } \theta)^n = \text{cis}(n\theta)$$

$$\text{or,} \quad \text{cis}(n\theta) = (\text{cis } \theta)^n$$

$$\text{So,} \quad \text{cis}(3\theta) = (\text{cis } \theta)^3 \quad \text{(1 mark)}$$

$$\begin{aligned} \cos(3\theta) + i\sin(3\theta) &= (\cos \theta + i\sin \theta)^3 \\ &= \cos^3 \theta + 3\cos^2 \theta i\sin \theta - 3\cos \theta \sin^2 \theta - i\sin^3 \theta \end{aligned}$$

(1 mark)

Equating imaginary parts, we have,

$$\sin 3\theta = 3\cos^2 \theta \sin \theta - \sin^3 \theta \quad \text{as required.} \quad \text{(1 mark)}$$

Have shown.

Total 11 marks