The Mathematical Association of Victoria

Specialist Mathematics

2001 Written Examinations

Solutions

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"Cliveden", 61 Blyth Street, Brunswick, 3056
Phone: (03) 9380 2399 Fax: (03) 9389 0399
E-mail: office@mav.vic.edu.au website: http://www.mav.vic.edu.au

2001 Specialist Maths
Written Examination 1(facts, skills and applications)
Suggested answers and solutions

Part 1 Multiple-choice Answers

1. E	2. E	3. D	4. A	5. A
6. E	7. C	8. A	9. E	10. C
11. C	12. D	13. B	14. D	15. C
16. B	17. C	18. D	19. E	20. B
21. B	22. E	23. A	24. A	25. C
26. D	27. B	28. B	29. D	30. A

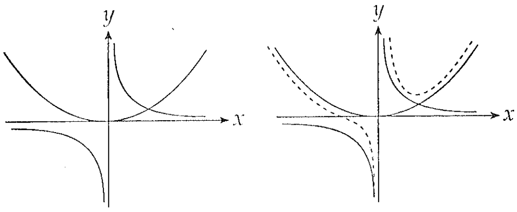
1. The ellipse shown is a translation of an ellipse $\frac{x^2}{4}+\frac{y^2}{1}=1$ from centre (0, 0) to a new centre at (2, –1)

The equation of the translated ellipse is:

$\frac{(x-2)^2}{4}+\frac{(y+1)^2}{1}=1$ **[E]**

2. Think of $f(x)=\frac{x^3+16}{4x}$ as composed of two fractions $\frac{x^3}{4x}$ and $\frac{16}{4x}$ that is $\frac{x^2}{4}$ and $\frac{4}{x}$.

Essentially, the values of $f(x)$ are obtained by the addition of ordinates for the two graphs:



For x small positive, $\frac{4}{x}$ will dominate, giving the function a vertical asymptote.

For x large positive, $\frac{x^2}{4}$ will dominate, giving the graph a shape close to that of the parabola.

For x small negative, $\frac{4}{x}$ will dominate, giving the function a vertical asymptote.

For x large negative, $\frac{x^2}{4}$ will dominate, giving the graph a shape close to that of the parabola. **[E]**

3. $\frac{d}{dx}\left(Tan^{-1}x\right)=\frac{1}{1+x^2}$

If $u=\frac{1}{2x}\quad\frac{du}{dx}=-\frac{1}{2x^2}$

$\frac{d}{dx}\left(Tan^{-1}u\right)=\frac{d}{du}\left(Tan^{-1}u\right).\frac{du}{dx}$

$=\frac{1}{1+u^2}.-\frac{1}{2x^2}$

$=\frac{1}{1+\frac{1}{4x^2}}.-\frac{1}{2x^2}$

$=\frac{4x^2}{4x^2+1}.-\frac{1}{2x^2}$

$=-\frac{2}{4x^2+1}$ **[D]**

4. $\cos(x)=-\frac{1}{10}$ for $\frac{\pi}{2}<x<\pi$

Note that x is in the second quadrant, making $\sin(x)$ positive and so $\operatorname{cosec}(x)$ positive.

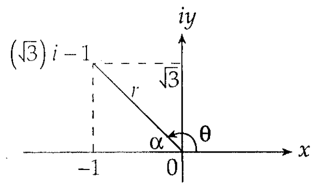
$\sin^2 x=1-\cos^2 x$

$\sin(x)=\frac{\sqrt{100-1}}{10}=\frac{\sqrt{99}}{10}=\frac{3\sqrt{11}}{10}$

$\operatorname{cosec}(x)=\frac{1}{\sin(x)}$

$\operatorname{cosec}(x)=\frac{10}{3\sqrt{11}}\times\frac{\sqrt{11}}{\sqrt{11}}=\frac{10\sqrt{11}}{33}$ **[A]**

5.



$r=\sqrt{3+1}=2\qquad \text{Tan}\,\alpha=\frac{\sqrt{3}}{1}$

$\alpha=\frac{\pi}{3}$

$\theta=\pi-\alpha$

$\theta=\pi-\frac{\pi}{3}=\frac{2\pi}{3}$ OR $-\pi-\frac{\pi}{3}=-\frac{4\pi}{3}$

Required polar form is $2cis\left(-\frac{4\pi}{3}\right)$, since there is no answer corresponding to $2cis\left(\frac{2\pi}{3}\right)$ **[A]**

6. $z^3 - 8 = 0$

$z^3 = 8$

One root of the equation will be $z = 2$, let $u = 2$

The other two roots will be equally spaced on a circle in the Argand diagram with $z = 2$ as an 'anchor' value.

Diagram E is the only diagram that shows this properly. **[E]**

7. All the equations $P(z) = 0$ for options A to E have real coefficients, so if $z = 3i$ is a solution, another solution **must** be $z = -3i$.

$$P(z) = (z - 3i)(z + 3i)(z - a)$$
$$= (z^2 + 9)(z - a)$$

Among the options available, only $a = 0$ will provide an answer.

$$P(z) = (z^2 + 9)z$$
$$= z^3 + 9z$$

[C]

8. $\{z: (z - 2)(\bar{z} - 2) = 4, z \in C\}$

let $z = x + yi$

$\bar{z} = x - yi$

$$(z - 2)(\bar{z} - 2) = 4$$
$$(x + iy - 2)(x - iy - 2) = 4$$
$$(x - 2 + iy)(x - 2 - iy) = 4$$
$$(x - 2)^2 + y^2 = 4$$

This is the equation of a circle with original equation $x^2 + y^2 = 4$, translated two units to the right, that is with a new centre at (2, 0) and radius still 2 units. **[A]**

9. The line S is the set of points which lie on the perpendicular bisector of the line joining (0, 2i) and (2, 0). In other words, the points that lie on S are equidistant from (0, -2i) and (2, 0). For any point z on the line, therefore, $|z - 2| = |z + 2i|$ **[E]**

10. Using the function of a function rule:

$$\frac{d}{dx}\left[\log_e(\log_e x)\right]$$
$$= \frac{1}{\log_e x} \cdot \frac{1}{x}$$

AND

$$\frac{d}{dx}\left[\log_e(\log_e ax)\right]$$
$$= \frac{1}{\log_e(ax)} \cdot \frac{a}{ax}$$

So $\frac{d}{dx}\left[\log_e(\log_e 2x)\right]$

$$= \frac{1}{x \log_e(2x)}$$

[C]

11. $\int_0^{\frac{\pi}{6}} \cos^2(2x).\cos(2x)dx$

$\cos^2(2x).\cos(2x)$ can be rewritten as

$\left(1 - \sin^2(2x)\right).\cos(2x)$

So by letting $u = \sin(2x)$ with $\frac{du}{dx} = 2\cos 2x$,

$\cos^2(2x).\cos(2x)$ can be re-written as

$(1 - u^2).\frac{1}{2}.\frac{du}{dx}$

The integral can therefore be expressed as:

$$\int_a^b (1 - u^2).\frac{1}{2}.\frac{du}{dx}.dx$$

$$\frac{1}{2}\int_a^b (1 - u^2)\,du \text{ where}$$

$a = \sin 0$ therefore $a = 0$

$b = \sin 2\left(\frac{\pi}{6}\right)$ therefore $b = \frac{\sqrt{3}}{2}$ **[C]**

12. $\int_{\sqrt{3}}^{3} \frac{3}{x^2+3}\,dx$

$\sqrt{3}\int_{\sqrt{3}}^{3} \frac{\sqrt{3}}{x^2+\left(\sqrt{3}\right)^2}\,dx$

$\sqrt{3}\left[Tan^{-1}\left(\frac{x}{\sqrt{3}}\right)\right]_{\sqrt{3}}^{3}$

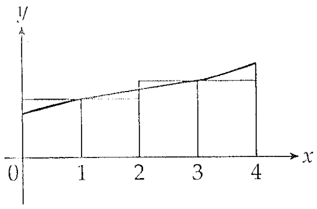
$\sqrt{3}\left[Tan^{-1}\left(\frac{3}{\sqrt{3}}\right)-Tan^{-1}\left(\frac{\sqrt{3}}{\sqrt{3}}\right)\right]$

$\sqrt{3}\left[Tan^{-1}\left(\sqrt{3}\right)-Tan^{-1}(1)\right]$

$\sqrt{3}\left[\frac{\pi}{3}-\frac{\pi}{4}\right]$

$\frac{\pi\sqrt{3}}{12}$ **[D]**

13.



Using two equal intervals to approximate the area, the first rectangle has base 2 units and height 1 + cos (1); and the second rectangle has a base of 2 units and height 3 + cos (3).

Area of the two rectangles added together

2 (1 + 0.5403) + 2 (3 – 0.98999)

= 3.0806 + 4.0200

= 7.1006 which gives 7.101 correct to three decimal places. **[B]**

14. Let $y_1 = 2\sin^2(x)$

$y_2 = \sin(2x)$

The area of a small sector of the shaded region will be given by:

$SA \approx (y_2 - y_1)\delta x$ *for* $0 \le x \le \frac{\pi}{4}$

and $SA \approx (y_1 - y_2)\delta x$ *for* $\frac{\pi}{4} \le x \le \pi$

So the required definite integral is in two parts:

$$\int_0^{\frac{\pi}{4}} (\sin(2x) - 2\sin^2(x))dx + \int_{\frac{\pi}{4}}^{\pi} (2\sin^2(x) - \sin(2x))dx$$

The second integral needs to be re-expressed as:

$$-\int_{\frac{\pi}{4}}^{\pi} \sin(2x) - 2\sin^2(x)dx$$

to be in a form that matches. **[D]**

15. The graph shown is that of $y = f'(x)$. From the graph, it can be deduced that the original function $f(x)$ has a stationary point of inflexion at $x = -3$; that is, the gradient is negative either side of $x = -3$ and is zero at $x = -3$. It can also be deduced that $f(x)$ has a local minimum at $x = 0$; that is, the gradient goes from negative for x just less than zero, and becomes positive for x just a bit more than zero. **[C]**

16. $\frac{dv}{dt} = -0.05(v^2 - 5)$

$\frac{dt}{dv} = \frac{1}{-0.05(v^2-5)}$

$= \frac{-20}{v^2-5}$

$t = \int_{v_1}^{v_2} \frac{-20}{v^2-5}\,dv$

So the time for the velocity to fall from 50m/s to 3m/s will be given by:

$t = \int_{50}^{3} \frac{-20}{v^2-5}dv$

$= 20\int_{50}^{3} \frac{1}{v^2-5}\,dv$ **[B]**

17. $f'(x) = \frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}$

$$f(0) = 1$$
$$f(0+0.2) \approx 0.2f'(0) + f(0)$$
$$\approx 0.2 \times 1 + 1$$
$$\approx 1.2$$
$$f(0.2+0.2) \approx 0.2f'(0.2) + f(0.2)$$
$$\approx \frac{0.2}{\sqrt{1.04}} + 1.2$$
$$\approx 0.2 \times 0.98058 + 1.2$$
$$\approx 1.3961$$

[C]

18. $V = 0.03\pi h^3$

$$\frac{dV}{dh} = 0.09\pi h^2$$

We are also given that:

$\frac{dV}{dt} = -0.1\sqrt{h}\ (\text{m}^3/\text{hr})$ Note: $\frac{dV}{dt} < 0$

$$\frac{dh}{dt} = \frac{dh}{dV} \cdot \frac{dV}{dt}$$
$$= \frac{1}{0.09\pi h^2} \cdot -0.1\sqrt{h}$$
$$= -\frac{1}{0.9\pi h^{\frac{3}{2}}}$$

[D]

19. $\frac{dv}{dt} = 6\sin(2t)\ \text{m/s}^2$ at $t = 0, \frac{dv}{dt} = 0, x = 0$

$$v = -3\cos(2t) + c$$
$$t = 0, \quad 0 = -3 + c$$
$$v = 3 - 3\cos(2t), \qquad \text{since } v = \frac{dx}{dt}$$
$$x = 3t - \frac{3}{2}\sin(2t) + c$$
$$t = 0, x = 0, c = 0$$
$$x = 3t - 1.5\sin(2t)$$

[E]

20. $\frac{dv}{dx} = \frac{1}{v}$

Note that $\frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = v.\frac{dv}{dx}$, hence

from $\frac{dv}{dx} = \frac{1}{v}$, it follows that

$v.\frac{dv}{dx} = 1$, i.e $\frac{dv}{dt} = 1$

If acceleration is constant (= 1)
then velocity will be increasing.

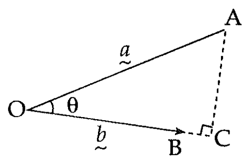
[B]

21. The given vector $\underset{\sim}{i} - 2\underset{\sim}{j} + 5\underset{\sim}{k}$ has magnitude $\sqrt{1^2 + 2^2 + 5^2} = \sqrt{30}$, so the unit vector parallel to the given vector is $\frac{1}{\sqrt{30}}(\underset{\sim}{i} - 2\underset{\sim}{j} + 5\underset{\sim}{k})$, and so a vector of magnitude 6 will be given by:

$\frac{6}{\sqrt{30}}(\underset{\sim}{i} - 2\underset{\sim}{j} + 5\underset{\sim}{k})$ or $\frac{\sqrt{30}}{5}(\underset{\sim}{i} - 2\underset{\sim}{j} + 5\underset{\sim}{k})$ **[B]**

22. $\underset{\sim}{a} = 3\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$

$\underset{\sim}{b} = -\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}$



The vector resolute of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$ is $\underset{\sim}{\text{OC}}$.

$$\underset{\sim}{\text{OC}} = \left(\left|\underset{\sim}{a}\right|\cos\theta\right)\frac{\underset{\sim}{b}}{\left|\underset{\sim}{b}\right|}$$
$$= \left(\left|\underset{\sim}{a}\right|\left|\underset{\sim}{b}\right|\cos\theta\right)\frac{\underset{\sim}{b}}{\left|\underset{\sim}{b}\right|\left|\underset{\sim}{b}\right|}$$
$$= \left(\underset{\sim}{a}.\underset{\sim}{b}\right).\frac{\underset{\sim}{b}}{14}$$
$$= (-3-6+2).\frac{\left(-\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}\right)}{14}$$
$$= -\frac{7}{14}(-\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k})$$
$$= \frac{1}{2}\left(\underset{\sim}{i} - 3\underset{\sim}{j} - 2\underset{\sim}{k}\right)$$

[E]

23. $\underset{\sim}{a} = -m\,\underset{\sim}{i} - 2\,\underset{\sim}{j} + \underset{\sim}{k}$

$\underset{\sim}{b} = \underset{\sim}{i} - n\,\underset{\sim}{j} - 6\,\underset{\sim}{k}$

If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular then $\underset{\sim}{a}\,.\,\underset{\sim}{b} = 0$

$\underset{\sim}{a}\,.\,\underset{\sim}{b} = -m + 2n - 6$

Hence $m - 2n + 6 = 0$

The only option which satisfies this condition is:

$m = -2, n = 2$ **[A]**

24. $\underset{\sim}{p} = 2\,\underset{\sim}{q} - 3\,\underset{\sim}{r}$

This implies a linear relationship between all three vectors $\underset{\sim}{p}$, $\underset{\sim}{q}$ and $\underset{\sim}{r}$.

In other words, if two of the vectors are known, the third can be deduced. **[A]**

25. $\underset{\sim}{r}(t) = (t-3)\,\underset{\sim}{i} - \left(\sqrt{t}\right)\underset{\sim}{j}$

$$\left|\underset{\sim}{r}(t)\right| = (t-3)^2 + \left(\sqrt{t}\right)^2$$

$$= t^2 - 6t + 9 + t$$

$$= t^2 - 5t + 9$$

$\left|\underset{\sim}{r}(t)\right|$ is smallest when $t^2 - 5t + 9$ is a minimum, i.e. when $\frac{d}{dt}\left(t^2 - 5t + 9\right) = 0$

$2t - 5 = 0$

$t = 2.5$ **[C]**

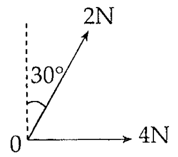
26. $\underset{\sim}{r}(t) = e^{-2t}\,\underset{\sim}{i} + \left(\sin(\pi t)\right)\underset{\sim}{j} + 2\,\underset{\sim}{k}$

Need to find $\underset{\sim}{v}(t)$ and substitute for $t = 0$ to get the initial direction of motion of the particle.

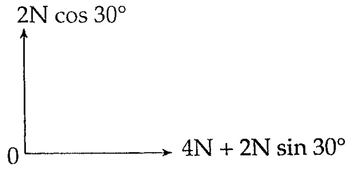
$\underset{\sim}{v}(t) = -2e^{-2t}\,\underset{\sim}{i} + \left(\pi\cos(\pi t)\right)\underset{\sim}{j}$

$\underset{\sim}{v}(0) = -2\,\underset{\sim}{i} + \pi\,\underset{\sim}{j}$ **[D]**

27.



Resolving horizontally and vertically



Magnitude of vertical component is

$2N\frac{\sqrt{3}}{2} = \sqrt{3}N$

Horizontal component is $4N + N = 5N$

Magnitude of resultant force is

$\sqrt{3 + 25} = \sqrt{28} = 2\sqrt{7}$ **[B]**

28. Constant force means constant acceleration.

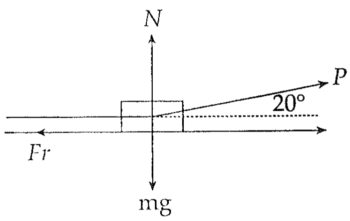
Acceleration is

$\frac{v_2 - v_1}{t_2 - t_1}$

$\frac{8\,\underset{\sim}{j} - (-6\,\underset{\sim}{i})}{2}$

$4\,\underset{\sim}{j} + 3\,\underset{\sim}{i}$ **[B]**

29.



The body is on the point of moving:

Horizontally,

$P\cos(20°) = Fr$

$Fr = \mu N$

$P\cos(20°) = \mu N$

[Vertically, $P\sin 20° + N = mg$] **[D]**

30. The caravan is being pulled forward by the towbar. This force T newtons must therefore be shown acting to the right of the diagram. Resistance force R_2 newtons must be shown acting to the left. R_1 is irrelevant to the motion of the caravan. **[A]**

Part 2

1. $\sin(4x) = 2\sin(2x)\cos(2x)$

$2\sin(2x)\cos(2x) - \cos(2x) = 0$

$\cos(2x)[2\sin(2x) - 1] = 0$

$\cos(2x) = 0$

$2x = \frac{\pi}{2}, \frac{3\pi}{2}$ hence $x = \frac{\pi}{4}, \frac{3\pi}{4}$

or

$\sin(2x) = \frac{1}{2}$

$2x = \frac{\pi}{6}, \frac{5\pi}{6}$ hence $x = \frac{\pi}{12}, \frac{5\pi}{12}$

Solutions are $\frac{\pi}{12}, \frac{\pi}{4}, \frac{5\pi}{12}, \frac{3\pi}{4}$

2. a. $\frac{2x-1}{x^2+6x+9} = \frac{2x-1}{(x+3)^2}$

$\frac{2x-1}{(x+3)^2} \equiv \frac{A}{x+3} + \frac{B}{(x+3)^2}$

$2x - 1 \equiv A(x+3) + B$

let $x = -3$

$-7 = B$

let $x = 0$

$-1 = 3A + B$

$-1 = 3A - 7$

$A = 2$

So $\frac{2x-1}{x^2+6x+9} = \frac{2}{x+3} - \frac{7}{(x+3)^2}$

b. $\int \frac{2x-1}{x^2+6x+9} = \frac{7}{x+3} + 2\log_e(x+3)$

for $x > -3$

c. $\int_{-2}^{4} \frac{2x-1}{x^2+6x+9}\,dx$

$= \frac{7}{7} + 2\log_e(7) - \frac{7}{1} - 2\log_e\ (1)$

$1 + 2\log_e(7) - 7 - 0$

$= -6 + 2\log_e(7)$

$= -6 + 3.892$

$= -2.108$

$= -2.11$ (correct to 3 significant figures)

3. $-2 + 2i$ needs to be expressed in polar form. Its polar form is:

$$(r cis\theta)^3 = 2\sqrt{2}cis\left(\frac{3\pi}{4}\right)$$

$$r cis\theta = \sqrt{2}cis\left(\frac{\pi}{4}\right), \sqrt{2}cis\left(\frac{2\pi}{3} + \frac{\pi}{4}\right), \sqrt{2}cis\left(\frac{4\pi}{3} + \frac{\pi}{4}\right)$$

$$= \sqrt{2}cis\left(\frac{\pi}{4}\right), \sqrt{2}cis\left(\frac{11\pi}{2}\right), \sqrt{2}cis\left(\frac{19\pi}{12}\right)$$

$\sqrt{2}cis\left(\frac{\pi}{4}\right) = 1 + i$

$\sqrt{2}cis\left(\frac{11\pi}{2}\right) = -1.366 + 0.366i$

$\sqrt{2}cis\left(\frac{19\pi}{12}\right) = 0.366 - 1.366i$

4. $\underline{AD} = \underline{u} + \underline{v}$

$\underline{AB} = \underline{u} - \underline{v}$

$\underline{AD}.\underline{AB} = (\underline{u} + \underline{v}).(\underline{u} - \underline{v})$

$= |\underline{u}|^2 - |\underline{v}|^2$

$= 0$

As $|\underline{u}| = |\underline{v}|$

$\underline{AD}.\underline{AB} = 0$

So $\angle BAD$ is a right angle.

5. Let Q kg be the amount of salt dissolved in the tank after t minutes.
Volume of water in the tank after t min is $(500 + 2t)$ litres.
Concentration after t minutes is therefore:

$\frac{Q}{500 + 2t}$ kg/litre

Outflow per minute will be: $\frac{3Q}{500 + 2t}$ kg/min

Inflow per minute will be: 5×0.05 kg/min (constant)

Rate of change at t mins will be:

$\frac{dQ}{dt} = 0.25 - \frac{3Q}{500 + 2t}$

6. The required solid of revolution is the difference of two solids of revolution.

$$\pi\int_0^{\frac{\pi}{2}} \sin^2(x)dx - \pi\int_0^{\frac{\pi}{2}} [1-\cos(x)]^2 dx$$

$$\pi\int_0^{\frac{\pi}{2}} \left[\sin^2(x) - (1 - 2\cos(x) + \cos^2(x))\right]dx$$

$$\pi\int_0^{\frac{\pi}{2}} \left[2\cos(x) + 1 - \cos^2(x) - 1 - \cos^2(x)\right] dx$$

$$\pi\int_0^{\frac{\pi}{2}} \left[2\cos(x) - 2\cos^2(x)\right] dx$$

Note:

$$2\cos^2(x) - 1 = \cos(2x)$$

$$-2\cos^2(x) = -1 - \cos(2x)$$

$$\pi\int_0^{\frac{\pi}{2}} [2\cos(x) - 1 - \cos(2x)]dx$$

$$\pi\left[2\sin(x) - x - \frac{1}{2}\sin(2x)\right]_0^{\frac{\pi}{2}}$$

$$\pi\left[2 - \frac{\pi}{2}\right]$$