

2001 Specialist Mathematics Exam 1

Suggested Answers and Solutions

Part I (Multiple-choice) Answers

1. A	2. D	3. C	4. D	5. C
6. C	7. D	8. A	9. D	10. A
11. B	12. D	13. D	14. B	15. C
16. A	17. C	18. C	19. E	20. C
21. D	22. D	23. B	24. B	25. A
26. A	27. E	28. A	29. D	30. E

Solutions — Part I (multiple choice)

Question 1

[A]

$$\begin{aligned}\vec{AB} &= \vec{AO} + \vec{OB} \\ &= \vec{OB} - \vec{OA} \\ &= \underset{\sim}{i} + 3\underset{\sim}{j} + 3\underset{\sim}{k}\end{aligned}$$

Question 2

[D]

$$\begin{aligned}|z-1| &= |z+3| \\ |x+yi-1| &= |x+yi+3| \\ (x-1)^2 + y^2 &= (x+3)^2 + y^2 \\ x^2 - 2x + 1 + y^2 &= x^2 + 6x + 9 + y^2 \\ 8x + 8 &= 0 \\ x &= -1\end{aligned}$$

Question 3

[C]

$$2 + \sin^{-1}\left(\frac{x}{2} + 1\right) = 2 + \sin^{-1}\left[\frac{1}{2}(x+2)\right]$$

$$f(x) = \sin^{-1}(x), \text{ dom } f = [-1, 1]$$

$$g(x) = \sin^{-1}\left(\frac{1}{2}x\right) \quad \text{Dilated by a factor of 2.}$$

$$\text{dom } g = [-2, 2]$$

$$h(x) = \sin^{-1}\left[\frac{1}{2}(x+2)\right] \text{ is } g(x) \text{ translated 2 units left.}$$

$$\text{dom } h = [-4, 0]$$

Question 4

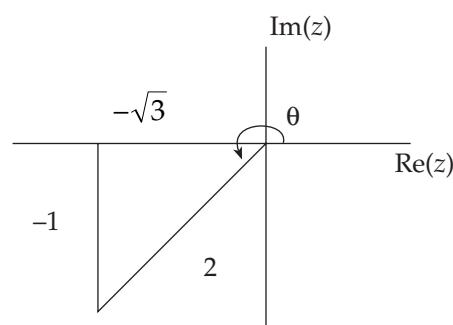
[D]

$$\begin{aligned}\sin^2 x &= 1 - \cos^2 x \\ &= 1 - \frac{4}{5} \\ &= \frac{1}{5}\end{aligned}$$

$$\sin x = \frac{1}{\sqrt{5}}, \text{ since } \frac{\pi}{2} < x < \pi$$

Question 5

[C]



$$\sin \theta = -\frac{1}{2}$$

$$\theta = \frac{-5\pi}{6}$$

$$-\sqrt{3} - i = 2\text{cis}\left(-\frac{5\pi}{6}\right)$$

Question 6**[C]**

$$\begin{aligned}\int \frac{2}{1+9x^2} dx &= 2 \int \frac{1}{9\left(\frac{1}{9} + x^2\right)} dx \\ &= \frac{2}{9} \times \frac{3}{1} \int \frac{1 \times \frac{1}{3}}{\left(\frac{1}{3}\right)^2 + x^2} dx \\ &= \frac{2}{3} \tan^{-1}(3x)\end{aligned}$$

Since question requests 'an antiderivative' '+c' is not required.

Question 7**[D]**

$$\begin{aligned}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin(3x) e^{\cos(3x)} dx & \quad \text{Let } u = \cos 3x \\ \frac{du}{dx} &= -3 \sin 3x \\ \text{Terminals:} \\ x = \frac{\pi}{2}, u &= 0 \\ x = \frac{\pi}{3}, u &= -1 \\ -\frac{1}{3} \int_{-1}^0 e^u du\end{aligned}$$

Question 8**[A]**

$$\begin{aligned}\vec{a} = \frac{\vec{F}}{m} &= \frac{6\vec{i} - 2\vec{j}}{2} = 3\vec{i} - \vec{j} \\ a = \left| \vec{a} \right| &= \sqrt{3^2 + (-1)^2} = \sqrt{10}\end{aligned}$$

Question 9**[D]**

$$\begin{aligned}h'(x) &= 2x\sqrt{1-x^2} & \text{Let } u = 1-x^2 \\ \frac{du}{dx} &= -2x \\ h(x) &= \int -u^{\frac{1}{2}} du \\ &= -\frac{2}{3}(1-x^2)^{\frac{3}{2}} + c\end{aligned}$$

$$\text{Since } h(0) = 1, \quad 1 = -\frac{2}{3} + c, \quad c = \frac{5}{3}$$

$$\therefore h(x) = \frac{5}{3} - \frac{2}{3}(1-x^2)^{\frac{3}{2}}$$

Question 10**[A]**

$$\begin{aligned}\vec{r}(t) &= 2e^{-3t} \vec{i} + \frac{5}{2} \sin(3t) \vec{j} \\ \vec{r}'(t) &= -6e^{-3t} \vec{i} + \frac{5}{2} \times 3 \cos t \vec{j} \\ \left| \vec{r}'(t) \right| &= \sqrt{36e^{-6t} + \frac{225}{4} \cos^2 3t} \\ \left| \vec{r}'(t) \right| &= \sqrt{36 + \frac{225}{4}} \\ &= \frac{3\sqrt{41}}{2}\end{aligned}$$

Question 11**[B]**

$$\begin{aligned}f(x) &= 2x^2 + x - 15 \\ &= (2x-5)(x+3)\end{aligned}$$

Asymptotes of $\frac{1}{f(x)}$ occur where $f(x) = 0$

Hence asymptotes occur at $x = -3$, and $x = \frac{5}{2}$

Question 12**[D]**

$$\begin{aligned} & \int \sin^3(2x) dx \\ &= \int \sin(2x) \sin^2(2x) dx \\ &= \int \sin(2x) \times (1 - \cos^2(2x)) dx \end{aligned}$$

Let $u = \cos(2x)$ $\frac{du}{dx} = -2\sin(2x)$

$$\int -\frac{1}{2}(1 - u^2) du$$

Question 13**[D]**

$$\begin{aligned} A &= \frac{1}{2}(2 + \sqrt{2})2 + \frac{1}{2}(\sqrt{2} + 0)2 \\ &= 2 + 2\sqrt{2} \\ &= 4.8284 \end{aligned}$$

Question 14**[B]**

$$\int \frac{3x}{2x^2 + 3} dx$$

Let $u = 2x^2 + 3$

$$= 3 \int \frac{1}{4} \frac{1}{u} \frac{du}{dx} dx \quad \frac{du}{dx} = 4x$$

$$= \frac{3}{4} \log_e(2x^2 + 3) + c$$

Since question requests 'an antiderivative' 'c' is not required.

Question 15**[C]**

Inflow: $\frac{dQ}{dV} = 3 \text{ gL}^{-1}$, $\frac{dV}{dt} = 5 \text{ Lmin}^{-1}$

$$\frac{dQ}{dt_{IN}} = \frac{dQ}{dV} \frac{dV}{dt} = 15 \text{ gmin}^{-1}$$

Given inflow of 5 Lmin^{-1} and outflow of 2 Lmin^{-1} , then the volume at any time t is $(40 + 3t) \text{ L}$.

Outflow: $\frac{dQ}{dV} = \frac{Q}{40 + 3t} \text{ gL}^{-1}$, $\frac{dV}{dt} = 2 \text{ Lmin}^{-1}$

$$\frac{dQ}{dt_{OUT}} = \frac{dQ}{dV} \frac{dV}{dt} = \frac{2Q}{40 + 3t}$$

$$\begin{aligned} \frac{dQ}{dt} &= \frac{dQ}{dt_{IN}} - \frac{dQ}{dt_{OUT}} \\ &= 15 - \frac{2Q}{40 + 3t} \end{aligned}$$

Question 16**[A]**

Volume of revolution about y -axis given by

$$V = \int_{y=a}^{y=b} \pi x^2 dy$$

$$\begin{aligned} V &= \int_0^2 \pi(2y)^2 dy - \int_0^2 \pi(y^2)^2 dy \\ &= \pi \int_0^2 (4y^2 - y^4) dy \end{aligned}$$

Question 17**[C]**

$$y = \log_e(\sin x)$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{\cos x}{\sin x} \quad \frac{d^2y}{dx^2} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} \\ &= -\frac{1}{\sin^2 x} \end{aligned}$$

$$\sin 2x = 2 \sin x \cos x$$

$$\frac{1}{\sin x \cos x} \frac{\cos x}{\sin x} - \frac{1}{\sin^2 x} = 0$$

Hence $\frac{2}{\sin 2x} \frac{dy}{dx} + \frac{d^2y}{dx^2} = 0$

Question 18**[C]**

$$wz = 2 \times 3 \text{cis}\left(\frac{3\pi}{4} + \frac{\pi}{3}\right)$$

$$= 6 \text{cis}\left(\frac{13\pi}{12}\right)$$

$$= 6 \text{cis}\left(-\frac{11\pi}{12}\right)$$

Question 19**[E]**

$$x = 3 \cos(2t)$$

$$y = 4 \sin(2t)$$

$$\frac{x}{3} = \cos(2t)$$

$$\frac{y}{4} = \sin(2t)$$

Since $\sin^2(2t) + \cos^2(2t) = 1$

then $\frac{x^2}{9} + \frac{y^2}{16} = 1$

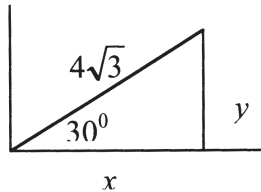
$$16x^2 + 9y^2 = 144$$

$$-3 \leq x \leq 3$$

Question 20**[C]**

$$\frac{dy}{dx} = 3x^2 - 1, \quad y_{n+1} = y_n + hf(x_n), \quad h = 0.1$$

x	y
1	3
1.1	$3 + 0.1(3(1)^2 - 1) = 3.2$
1.2	$3.2 + 0.1(3(1.1)^2 - 1) = 3.463$

Question 21**[D]**

$$x = 4\sqrt{3} \cos(30^\circ)$$

$$= 6$$

$$\therefore \underline{\underline{v}} = \underline{\underline{6i}} + \underline{\underline{2\sqrt{3}j}}$$

$$y = 4\sqrt{3} \sin(30^\circ)$$

$$= 2\sqrt{3}$$

Question 22**[D]**

$$\underline{\underline{b}} = \sqrt{2} \quad \underline{\underline{a}} \cdot \underline{\underline{b}} = \frac{1}{\sqrt{2}}(5 - 3) = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$(\underline{\underline{a}} \cdot \underline{\underline{b}}) \underline{\underline{b}} = \underline{\underline{i}} + \underline{\underline{j}}$$

$$\underline{\underline{a}} - (\underline{\underline{a}} \cdot \underline{\underline{b}}) \underline{\underline{b}} = \underline{\underline{4i}} - \underline{\underline{4j}}$$

Question 23**[B]**

$$\frac{dt}{dv} = \frac{v}{4}$$

$$t = \frac{v^2}{8} + c$$

$$t = 0, v = -2, 0 = \frac{1}{2} + c$$

$$t = \frac{v^2}{8} - \frac{4}{8}$$

$$v^2 = 8t + 4$$

$$v = \pm\sqrt{8t + 4}$$

$$v = -2\sqrt{2t + 1}, \text{ since when } t = 0, v = -2$$

Question 24**[B]**

$$\underline{\underline{a}} = 3, \underline{\underline{b}} = 5, \cos \theta = \frac{\underline{\underline{a}} \cdot \underline{\underline{b}}}{\underline{\underline{a}} \times \underline{\underline{b}}}$$

$$\cos \theta = \frac{11}{15}, \quad \theta = 42.83^\circ$$

Question 25**[A]**

$$y = \cos^{-1}\left(\frac{3}{2x}\right) \quad \text{Let } u = \frac{3}{2x} = \frac{3}{2}x^{-1}$$

$$\frac{du}{dx} = -\frac{3}{2}x^{-2}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \frac{-1}{\sqrt{1 - \frac{9}{4x^2}}} \times \frac{-3}{2x^2}$$

$$= \frac{3}{2\sqrt{x^4 - \frac{9}{4}x^2}}$$

$$= \frac{3}{2x\sqrt{\frac{4x^2}{4} - \frac{9}{4}}}$$

$$= \frac{3}{x\sqrt{4x^2 - 9}}$$

Question 26**[A]**

Since accelerating upwards, resultant force is upwards.

Hence (force up) - (force down) = ma

$$T - 75g = 75(2.5)$$

$$T = 75(g + 2.5)$$

Question 27**[E]**

Since object moves with constant speed,
forces up plane = forces down plane

$$F + 5g \sin 25^\circ = T \cos 40^\circ$$

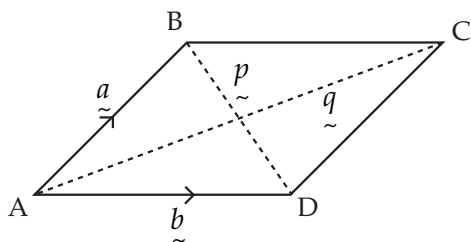
$$\text{Frictional force } F = \mu N = 0.2N$$

$$\text{Hence } T \cos 40^\circ = 0.2N + 5g \sin 25^\circ$$

Question 28**[A]**

$$\vec{p} = (\vec{a} + \vec{b})$$

$$\vec{q} = (\vec{a} - \vec{b})$$



need to prove $\vec{p} \cdot \vec{q} = 0$

$$\text{hence } (\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 0$$

Question 29**[D]**

$$2 \times T \sin \theta = 4g$$

$$T = \frac{2g}{\sin \theta}$$

Question 30**[E]**

$$2 \text{ kg mass: } T - 2g = 2a \quad (\text{I})$$

$$3 \text{ kg mass: } 3g - T = 3a \quad (\text{II})$$

$$(\text{I}) + (\text{II}) \quad g = 5a$$

$$a = \frac{g}{5}$$

Answers — Part II: Short answers**Question 1a**

$$\vec{AC} = \vec{c} - \vec{a}$$

[A1]

$$\vec{BC} = \vec{c} + \vec{a}$$

[A1]**Question 1b**

If $\vec{AC} \cdot \vec{BC} = 0$ then ACB is a right angle

$$\vec{AC} \cdot \vec{BC} = (\vec{c} - \vec{a}) \cdot (\vec{c} + \vec{a})$$

[M1]

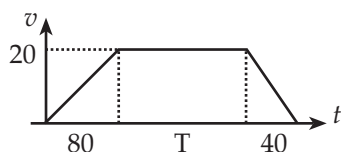
$$= \vec{c} \cdot \vec{c} + \vec{a} \cdot \vec{c} - \vec{a} \cdot \vec{c} - \vec{a} \cdot \vec{a}$$

$$= |\vec{c}|^2 - |\vec{a}|^2$$

[M1]

$|\vec{c}| = |\vec{a}|$ because both are radii of a circle

$$\therefore \vec{AC} \cdot \vec{BC} = 0$$

[A1]**Question 2a****[M1]** shape**[A1]** values

Note: Steepness in third section must be greater than in first interval for “shape” mark. 80 & 40 may be implicit in the working for (2b).

Question 2b

$$1600 = 800 + 20T + 400$$

[M1]

$$20T = 400$$

$$T = 20$$

$$\text{Total time} = 80 + 20 + 40 = 140 \text{ seconds}$$

[A1]**Question 3a**

$$(z - 2)(z - \sqrt{3}i)$$

$$= z^2 - \sqrt{3}iz - 2z + 2\sqrt{3}i$$

$$= z^2 - (\sqrt{3}i + 2)z + 2\sqrt{3}i$$

[A1]**Question 3b**

$$(z - 2)(z - \sqrt{3}i)(z + \sqrt{3}i)$$

[M1]

$$= (z - 2)(z^2 + 3)$$

$$= z^3 - 2z^2 + 3z - 6$$

[A1]

Question 4

$$\int_{-\frac{1}{2}}^{\frac{3}{2}} x\sqrt{1+2x} dx$$

$$\text{Let } u = 1 + 2x$$

$$x = \frac{u-1}{2}$$

$$\frac{du}{dx} = 2$$

$$dx = \frac{du}{2}$$

$$x = \frac{3}{2}, u = 4$$

$$u = \frac{-1}{2}, u = 0$$

[M1]

$$\int_0^4 \frac{(u-1)\sqrt{u}}{2} \times \frac{du}{2}$$

[M1]

$$\int_0^4 \frac{u^{\frac{3}{2}} - u^{\frac{1}{2}}}{4} du$$

$$\frac{1}{2} \left[\frac{u^{\frac{5}{2}}}{5} - \frac{u^{\frac{3}{2}}}{3} \right]_0^4$$

[A1]

$$= \frac{56}{30} = \frac{28}{15}$$

[A1]

Question 5

$$\cos 2x = 1 - 2\sin^2 x$$

$$\text{using } x = \frac{\pi}{8}$$

$$\cos \frac{\pi}{4} = 1 - 2\sin^2 \frac{\pi}{8} \quad \text{[M1]}$$

$$\frac{1}{\sqrt{2}} = 1 - 2\sin^2 \frac{\pi}{8}$$

$$\sin \frac{\pi}{8} = \frac{\sqrt{2 - \sqrt{2}}}{2} \quad \text{[A1]}$$

$$\cos \frac{15\pi}{8} = \cos \frac{\pi}{8} \quad \text{[A1]}$$

$$= \frac{\sqrt{2 + \sqrt{2}}}{2} \quad \text{[A1]}$$