

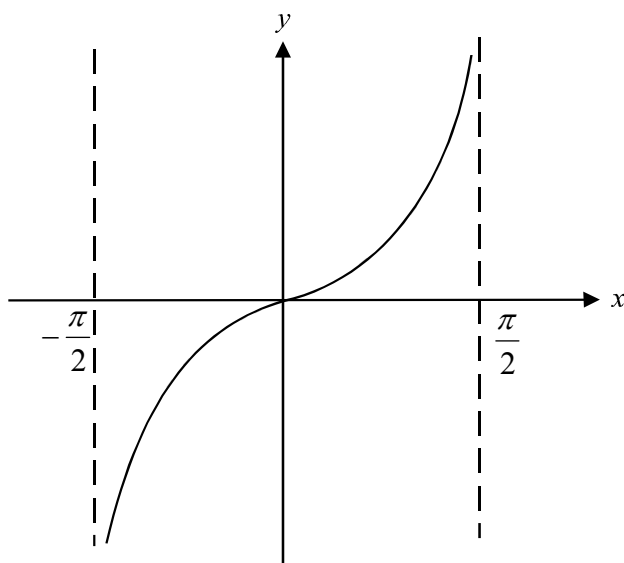
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**Question 1**

a. When  $x = 0$ ,  $y = 0$  So, the  $y$ -intercept is zero.

**(1 mark)**

b.



**(1 mark)** for asymptotes  
**(1 mark)** for shape of graph

c.  $f(x) = x \sec x$

$$= x(\cos x)^{-1}$$

So,  $f'(x) = x \times -1(\cos x)^{-2} \times -\sin x + (\cos x)^{-1}$

$$= \frac{x \sin x}{\cos^2 x} + \frac{1}{\cos x}$$

$$= \frac{x \sin x + \cos x}{\cos^2 x}$$

**(1 mark)**

d. A stationary point occurs when  $f'(x) = 0$ .

If  $f'(x) = 0$ , then  $x \sin x + \cos x = 0$

When  $x = 0$ , we have  $0 \times \sin 0 + \cos 0 = 1$

So,  $f'(0) \neq 0$  and so we do not have a stationary point at  $x = 0$ . **(1 mark)**

e. The gradient is a minimum when  $f''(x) = 0$

Now,  $\cos^3 x \neq 0$ , so  $x + x \sin^2 x + 2 \sin x \cos x = 0$  **(1 mark)**

Use a graphics calculator to show that this occurs when the graph of

$y = x + x \sin^2 x + 2 \sin x \cos x$  crosses the  $y$ -axis. This occurs when  $x = 0$ .

The gradient of the graph of  $y = f(x)$  is a minimum at  $(0,0)$ . **(1 mark)**

f. To verify:  $\frac{f''(x)}{\sec^3 x} - \frac{\cos x}{\operatorname{cosec}^2 x} f(x) = x + \sin(2x)$

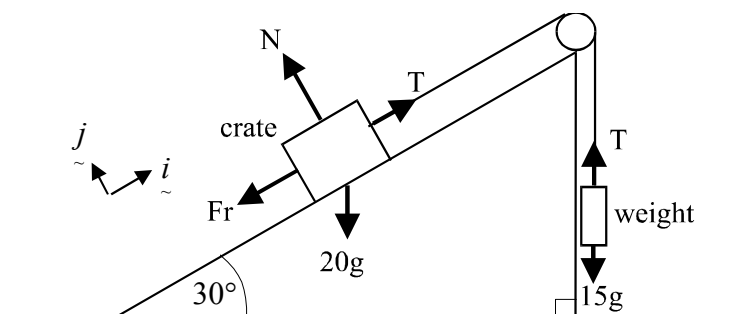
$$\begin{aligned} \text{Left side} &= \frac{f''(x)}{\sec^3 x} - \frac{\cos x}{\operatorname{cosec}^2 x} f(x) \\ &= \frac{x + x \sin^2 x + 2 \sin x \cos x}{\cos^3 x} \times \frac{\cos^3 x}{1} - \cos x \sin^2 x \times \frac{x}{\cos x} \quad (1 \text{ mark}) \\ &= x + x \sin^2 x + 2 \sin x \cos x - x \sin^2 x \\ &= x + 2 \sin x \cos x \\ &= x + \sin(2x) \\ &= \text{right side} \end{aligned}$$

Have verified (1 mark)

Total 9 marks

## Question 2

a.



(1 mark)

b. Resolving around the weight, we get  $T = 15g$  \_\_\_\_\_ (A) (1 mark)

Resolving around the crate, we get

$$\underline{\underline{R}} = (T - 20g \sin 30^\circ - Fr) \underline{\underline{i}} + (N - 20g \cos 30^\circ) \underline{\underline{j}} = 0 \underline{\underline{i}} + 0 \underline{\underline{j}}$$

$$\text{So, } T = 10g + Fr \quad \text{and} \quad N = \frac{20g\sqrt{3}}{2} \quad (1 \text{ mark})$$

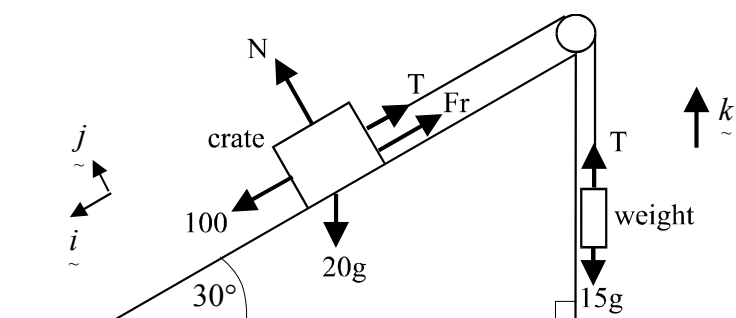
$$= 10g + \mu N \quad = 10\sqrt{3}g \quad \text{_____ (B)}$$

Substitute (A) and (B) into  $T = 10g + \mu N$

$$\text{We obtain} \quad 15g = 10g + \mu 10\sqrt{3}g$$

$$\begin{aligned} \mu &= \frac{5g}{10\sqrt{3}g} \\ &= \frac{\sqrt{3}}{6} \quad (1 \text{ mark}) \end{aligned}$$

c. Draw a diagram showing all the forces involved.



Resolving around the crate, we obtain

$$\vec{R} = (100 + 20g \sin 30^\circ - T - \mu N) \vec{i} + (N - 20g \cos 30^\circ) \vec{j}$$

$$\text{Also, } \vec{R} = m \vec{a}$$

$$= 20a \times \vec{i}$$

So, equating components in the  $\vec{i}$  direction, we obtain

$$100 + 10g - T - \mu N = 20a \quad \text{_____ (A)} \quad \text{(1 mark)}$$

Equating components in the  $\vec{j}$  direction, we obtain

$$N - 20g \cos 30^\circ = 0$$

$$\text{so, } N = 10\sqrt{3}g$$

$$\text{In (A), this gives } T = 100 + 10g - \frac{\sqrt{3}}{6}(10\sqrt{3}g) - 20a$$

$$T = 5g + 100 - 20a \quad \text{_____ (C)} \quad \text{(1 mark)}$$

Resolving around the weight, we obtain

$$\vec{R} = (T - 15g) \vec{k}$$

$$\text{Also, } \vec{R} = m \vec{a}$$

$$\vec{R} = 15a \vec{k}$$

$$\text{So, } T - 15g = 15a$$

$$T = 15a + 15g \quad \text{_____ (D)} \quad \text{(1 mark)}$$

Equating (C) and (D) we obtain

$$5g + 100 - 20a = 15a + 15g$$

$$-35a = 10g - 100$$

$$a = \frac{10(g - 10)}{-35}$$

$$= 0.06 \text{ (to 2 decimal places)}$$

So the weight accelerates upwards at  $0.06 \text{ m/s}^2$ . (1 mark)

**Total 8 marks**

**Question 3**

a.  $u = 0 + 5i$

So,  $r = \sqrt{0^2 + 5^2} = 5$  and  $\text{Arg } u = \frac{\pi}{2}$  since on an Argand diagram,  $u$  is located on the imaginary axis.

So,  $u = 5\text{cis } \frac{\pi}{2}$  (1 mark)

b.  $v = \bar{u} + |u| - 1 + 6i + \text{Re } u$   
 $= -5i + 5 - 1 + 6i + 0$   
 $= 4 + i$

(1 mark)

c. We have  $|z - 5i| = |z - 4 - i|$

so,  $|x + yi - 5i| = |x + yi - 4 - i|$  (1 mark)

$$|x + i(y - 5)| = |x - 4 + i(y - 1)|$$

$$\sqrt{x^2 + (y - 5)^2} = \sqrt{(x - 4)^2 + (y - 1)^2} \quad (1 \text{ mark})$$

$$x^2 + y^2 - 10y + 25 = x^2 - 8x + 16 + y^2 - 2y + 1$$

$$-8y = -8x - 8$$

$$y = x + 1 \quad (1 \text{ mark})$$

d.

Using our result from part c., we can draw the line with Cartesian equation  $y = x + 1$ .

Note that the boundary is not included since we have a less than sign and not a less than or equal to sign.

To decide which side of the line is required, choose a point say  $0 + 0i$ .

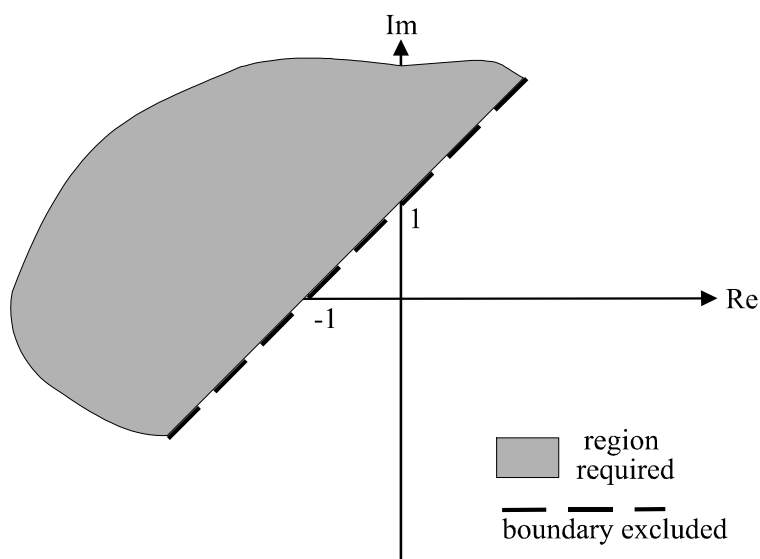
Substitute into  $|z - 5i| < |z - 4 - i|$

We obtain  $|0 + 0i - 5i| < |0 + 0i - 4 - i|$

$$|-5i| < |-4 - i|$$

$$\sqrt{25} < \sqrt{16 + 1} \quad \text{Clearly this statement is not true and so the side from}$$

which we chose the point,  $0 + 0i$  is not the required side.



(1 mark) for correct required region (1 mark) for correct boundary

e. Let  $z^3 = 8$   
 so,  $z^3 - 8 = 0$   
 $(z - 2)(z^2 + 2z + 4) = 0$   
 $z = 2$  or  $z = \frac{-2 \pm \sqrt{4 - 4 \times 1 \times 4}}{2}$   
 $= \frac{-2 \pm \sqrt{-12}}{2}$   
 $= \frac{-2 \pm 2\sqrt{3}i}{2}$   
 $= -1 \pm \sqrt{3}i$

Alternatively, use De Moivre's theorem.

Let  $z^3 = 8\text{cis}0$

So,  $z_1 = 2\text{cis}0$

and  $z_2 = 2\text{cis}\frac{2\pi}{3}$  and  $z_3 = 2\text{cis}\frac{-2\pi}{3}$

since the three roots are equally spaced.

Only  $z_2 = 2\text{cis}\frac{2\pi}{3} = -1 + \sqrt{3}i$  lies in S.

Remember that the answer is required in Cartesian form.

The cube roots of 8 are 2 and  $-1 \pm \sqrt{3}i$ . **(1 mark)**

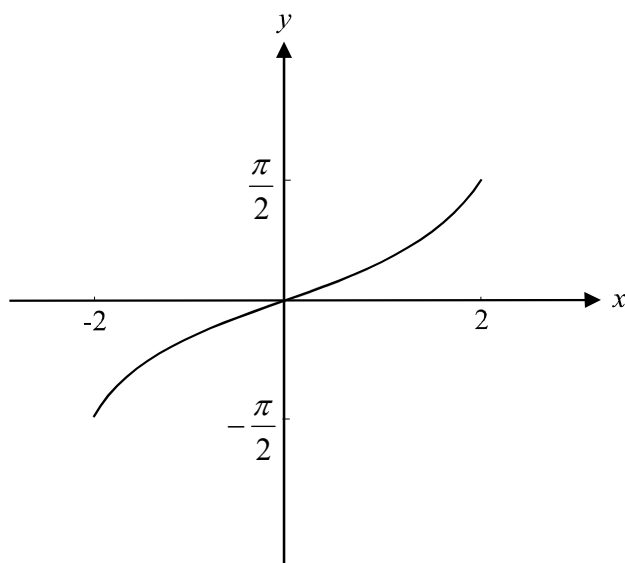
Looking at the diagram in part d., we see that the root 2 does not lie in S and the root  $-1 - \sqrt{3}i$  doesn't lie in S. The root  $-1 + \sqrt{3}i$  is the only root which lies in S. **(1 mark)**

**Total 9 marks**

#### Question 4

a.  $\sin^{-1} \frac{x}{a}$  is defined for  $x \in [-a, a]$ . So in this case,  $a = 2$ . **(1 mark)**

b. i.



**(1 mark)**

ii.  $f(x) = \sin^{-1} \frac{x}{2}$

$f'(x) = \frac{1}{\sqrt{4-x^2}}$  **(1 mark)**

Now, stationary points occur when  $f'(x) = 0$ , that is  $\frac{1}{\sqrt{4-x^2}} = 0$ . Now  $\sqrt{4-x^2} \neq 0$  or

the function is undefined. A fraction can only equal zero if the numerator equals zero.

Clearly  $1 \neq 0$  and so the function  $f$  can have no stationary point.

**(1 mark)**

$$\begin{aligned} \text{c. } \frac{d}{dx} \left( x \sin^{-1} \frac{x}{2} \right) &= x \times \frac{1}{\sqrt{4-x^2}} + 1 \times \sin^{-1} \frac{x}{2} && \text{(product rule)} \\ &= \frac{x}{\sqrt{4-x^2}} + \sin^{-1} \frac{x}{2} && \text{(1 mark)} \end{aligned}$$

$$\text{d. i. Now from part c., } \frac{d}{dx} \left( x \sin^{-1} \frac{x}{2} \right) = \frac{x}{\sqrt{4-x^2}} + \sin^{-1} \frac{x}{2}$$

$$\text{So, } \int \frac{d}{dx} \left( x \sin^{-1} \frac{x}{2} \right) dx = \int \frac{x}{\sqrt{4-x^2}} dx + \int \sin^{-1} \frac{x}{2} dx$$

$$\text{So, } x \sin^{-1} \frac{x}{2} + c_1 = \int \frac{x}{\sqrt{4-x^2}} dx + \int \sin^{-1} \frac{x}{2} dx$$

$$\text{Rearranging, we obtain, } \int \sin^{-1} \frac{x}{2} dx = x \sin^{-1} \frac{x}{2} + c_1 - \int \frac{x}{\sqrt{4-x^2}} dx \quad \text{(1 mark)}$$

$$\begin{aligned} \text{Now, } \int \frac{x}{\sqrt{4-x^2}} dx &= \int \frac{1}{\sqrt{u}} \times -\frac{1}{2} \frac{du}{dx} dx && \text{where } u = 4-x^2 \text{ and } \frac{du}{dx} = -2x \\ &= -\frac{1}{2} \int u^{-\frac{1}{2}} du \\ &= -\frac{1}{2} \times 2 \times u^{\frac{1}{2}} + c_2 \\ &= -\sqrt{4-x^2} + c_2 \end{aligned}$$

$$\text{So, } \int \sin^{-1} \frac{x}{2} dx = x \sin^{-1} \frac{x}{2} + \sqrt{4-x^2} \quad \text{(1 mark)}$$

Note that we equate the constants of antidifferentiation to zero since we were asked for “an” antiderivative.

$$\text{ii. So area required} = \int_0^1 \sin^{-1} \frac{x}{2} dx \quad \text{(1 mark)}$$

$$\begin{aligned} &= \left[ x \sin^{-1} \frac{x}{2} + \sqrt{4-x^2} \right]_0^1 \\ &= \left\{ \left( \sin^{-1} \frac{1}{2} + \sqrt{3} \right) - (0 + \sqrt{4}) \right\} \\ &= \sin^{-1} \frac{1}{2} + \sqrt{3} - 2 \\ &= \frac{\pi}{6} + \sqrt{3} - 2 \text{ square units} && \text{(1 mark)} \end{aligned}$$

e. Volume required  $= \pi \int_0^{\frac{\pi}{2}} x^2 dy$  **(1 mark)**

Now,  $y = \sin^{-1} \frac{x}{2}$

So,  $\frac{x}{2} = \sin y$  where  $x \in [-2, 2]$  and  $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$x = 2 \sin y$$

$$x^2 = 4 \sin^2 y$$

So, volume required  $= \pi \int_0^{\frac{\pi}{2}} 4 \sin^2 y dy$  **(1 mark)**

$$= \pi \int_0^{\frac{\pi}{2}} (2 - 2 \cos(2y)) dy \quad \text{since } 2 \sin^2 \theta = 1 - \cos(2\theta)$$

$$= \pi [2y - \sin(2y)]_0^{\frac{\pi}{2}}$$

$$= \pi \{(\pi - \sin \pi) - (0 - \sin 0)\}$$

$$= \pi(\pi - 0 - 0 + 0)$$

$$= \pi^2 \text{ cubic units} \quad \textbf{(1 mark)}$$

**Total 12 marks**

### Question 5

a. distance from origin  $= \left| \tilde{r}(t) \right|$

$$= \sqrt{\left(t + \frac{1}{t}\right)^2 + \left(t - \frac{1}{t}\right)^2}$$

$$= \sqrt{t^2 + 1 + 1 + \frac{1}{t^2} + t^2 - 1 - 1 + \frac{1}{t^2}}$$

$$= \sqrt{2t^2 + \frac{2}{t^2}}$$

**(1 mark)**

b. Now,  $\vec{v}(t) = (1 - \frac{1}{t^2})\vec{i} + (1 + \frac{1}{t^2})\vec{j}$

So,  $\vec{v}(5) = \frac{24}{25}\vec{i} + \frac{26}{25}\vec{j}$  **(1 mark)**

Speed at time  $t = 5$  is given by  $|\vec{v}(5)| = \sqrt{(\frac{24}{25})^2 + (\frac{26}{25})^2}$

$$= \sqrt{\frac{576 + 676}{625}}$$

$$= \frac{\sqrt{1252}}{25}$$

$$= \frac{2\sqrt{313}}{25} \quad \textbf{(1 mark)}$$

c. i. Given that  $\vec{r}(t) = (t + \frac{1}{t})\vec{i} + (t - \frac{1}{t})\vec{j}$

$$x = t + \frac{1}{t} \quad \text{and} \quad y = t - \frac{1}{t} \quad \textbf{(1 mark)}$$

So,  $x^2 = (t + \frac{1}{t})^2$  and  $y^2 = (t - \frac{1}{t})^2$

$$= t^2 + 2 + \frac{1}{t^2} \quad \text{and} \quad = t^2 - 2 + \frac{1}{t^2}$$

So,  $x^2 - 2 = t^2 + \frac{1}{t^2}$  and  $y^2 + 2 = t^2 + \frac{1}{t^2}$

So,  $x^2 - 2 = y^2 + 2$

So,  $y^2 = x^2 - 4$

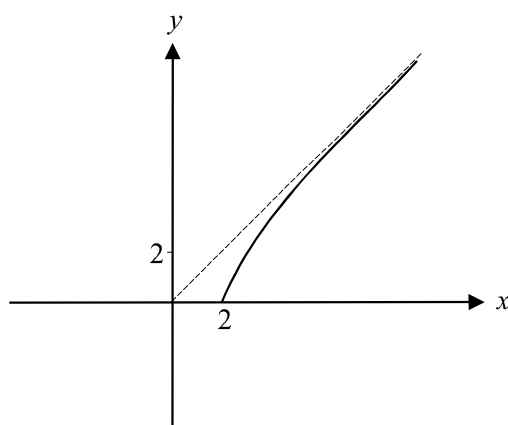
$$y = \pm\sqrt{x^2 - 4}$$

Note that since  $t \geq 1$ , and  $x = t + \frac{1}{t}$ , then  $x \geq 2$ . So, the domain is  $x \in [2, \infty)$  **(1 mark)**

Also,  $y = t - \frac{1}{t}$ , and  $t \geq 1$ , so  $y \geq 0$ . So, the range is  $y \in [0, \infty)$  **(1 mark)**

So, the required Cartesian equation is  $y = \sqrt{x^2 - 4}$  **(1 mark)**

ii.



**(1 mark)**



d. Since  $\vec{v}_B(t) = \left(2 - \frac{2}{t^2}\right)\vec{i} + \left(2 + \frac{2}{t^2}\right)\vec{j}, \quad t > 0$

$$\begin{aligned}\vec{r}_B(t) &= \int \left( \left(2 - \frac{2}{t^2}\right)\vec{i} + \left(2 + \frac{2}{t^2}\right)\vec{j} \right) dt \\ &= \left(2t + \frac{2}{t}\right)\vec{i} + \left(2t - \frac{2}{t}\right)\vec{j} + \vec{c} \quad (1 \text{ mark})\end{aligned}$$

When  $t = 1$ ,  $\vec{r}_B(t) = 4\vec{i}$

So,  $4\vec{i} = 4\vec{i} + \vec{c}$

So,  $\vec{c} = \vec{0}$

So,  $\vec{r}_B(t) = \left(2t + \frac{2}{t}\right)\vec{i} + \left(2t - \frac{2}{t}\right)\vec{j} \quad (1 \text{ mark})$

The position vector of the first particle is given by

$$\vec{r}(t) = \left(t + \frac{1}{t}\right)\vec{i} + \left(t - \frac{1}{t}\right)\vec{j}$$

So,  $\vec{r}_B(t) = 2\vec{r}(t)$  and hence the 2 position vectors are parallel.

(1 mark)

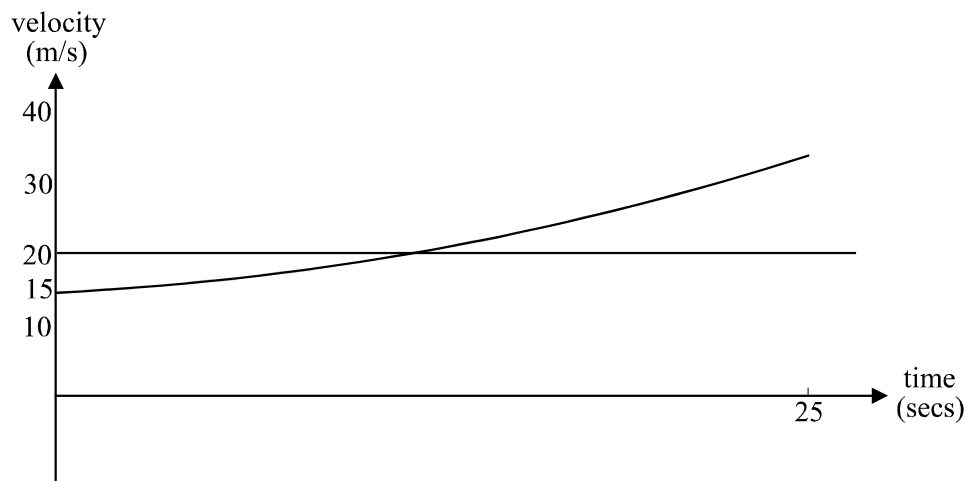
**Total 11 marks**

### Question 6

a. At time  $t = 0$ , Julie gets onto the freeway and so her entry speed is given by  $v(0) = 15$  m/s

(1 mark)

b.



(1 mark)

c. Let Julie catch up to Tom at time  $T$  seconds.

We require that the area under the velocity-time graph of Julie and of Tom are the same.

That is, we require that  $20T = \int_0^T (0.03t^2 + 15)dt$  **(1 mark)**

$$20T = \left[ \frac{0.03t^3}{3} + 15t \right]_0^T$$

$$20T = 0.01T^3 + 15T$$

$$0.01T^3 - 5T = 0$$

$$T(0.01T^2 - 5) = 0$$

$$T = 0 \text{ or } T = \pm 22.36$$

Now  $t \geq 0$  so we reject the negative value. The result  $T = 0$  relates to the time when Julie enters the freeway, and so the required value of  $T$  correct to 2 decimal places is  $T = 22.36$  seconds. **(1 mark)**

d. i.  $\frac{dv}{dt} = -0.02(v^2 - 625)$

So,  $\frac{dt}{dv} = \frac{1}{-0.02(v^2 - 625)}$

$$= \frac{-50}{(v-25)(v+25)} \quad \textbf{(1 mark)}$$

Now let  $\frac{-50}{(v-25)(v+25)} = \frac{A}{(v-25)} + \frac{B}{(v+25)}$

$$= \frac{A(v+25) + B(v-25)}{(v-25)(v+25)}$$

True iff  $-50 \equiv A(v+25) + B(v-25)$

Put  $v = -25$   $-50 \equiv -50B$

So,  $B = 1$

Put  $v = 25$   $-50 \equiv 50A$

So,  $A = -1$

So,  $\frac{dt}{dv} = \frac{-1}{(v-25)} + \frac{1}{(v+25)}$

So  $\int \frac{dt}{dv} dv = -\int \frac{1}{(v-25)} dv + \int \frac{1}{(v+25)} dv$

$$t = -\log_e(v-25) + \log_e(v+25) + c \quad \textbf{(1 mark)}$$

$$t = \log_e \frac{(v+25)}{(v-25)} + c$$

Given that  $v = 35$  when  $t = 0$

We have  $0 = \log_e \frac{60}{10} + c$

So  $c = -\log_e 6$

So,  $t = \log_e \frac{(v+25)}{(v-25)} - \log_e 6$

$$t = \log_e \frac{v+25}{6(v-25)} \quad \textbf{(1 mark)}$$

ii. Now,  $t = \log_e \frac{v+25}{6(v-25)}$

$$\begin{aligned} \text{So, } e^t &= \frac{(v+25)}{6(v-25)} && \begin{array}{r} 1 \\ v-25 \overline{) v+25} \end{array} \\ 6e^t &= 1 + \frac{50}{v-25} && \begin{array}{r} v-25 \\ \hline 50 \end{array} \quad (1 \text{ mark}) \\ 6e^t - 1 &= \frac{50}{v-25} \\ v-25 &= \frac{50}{6e^t - 1} \\ v &= \frac{50}{6e^t - 1} + 25 \\ &= \frac{50 + 25(6e^t - 1)}{6e^t - 1} \\ &= \frac{50 + 150e^t - 25}{6e^t - 1} \\ \text{So } v &= \frac{25(6e^t + 1)}{6e^t - 1} \text{ as required } (1 \text{ mark}) \end{aligned}$$

Alternatively

$$\begin{aligned} e^t &= \frac{(v+25)}{6(v-25)} \\ 6e^t(v-25) &= v+25 \\ 6ve^t - 150e^t &= v+25 \\ 6ve^t - v &= 150e^t + 25 \\ v(6e^t - 1) &= 25(6e^t + 1) \\ v &= \frac{25(6e^t + 1)}{6e^t - 1} \end{aligned}$$

$$\begin{aligned} \text{f. Since } v &= \frac{25(6e^t + 1)}{6e^t - 1} && \begin{array}{r} 25 \\ 6e^t - 1 \overline{) 150e^t + 25} \end{array} \\ \text{we have, } v &= 25 + \frac{50}{6e^t - 1} && \begin{array}{r} 150e^t - 25 \\ \hline 50 \end{array} \quad (1 \text{ mark}) \end{aligned}$$

As  $t \rightarrow \infty$ ,  $6e^t - 1 \rightarrow \infty$  and so  $\frac{50}{6e^t - 1} \rightarrow 0$ . Therefore  $25 + \frac{50}{6e^t - 1} \rightarrow 25$

So, Julie's limiting velocity is 25m/s. (1 mark)

**Total 11 marks**