

2000 Specialist Mathematics Exam 2

Suggested Answers and Solutions

Solutions:

Question 1 a i

$$f: R^+ \rightarrow R \text{ where } f(x) = \frac{6}{x} - 6 + 3 \log_e x$$

$$f'(x) = \frac{-6}{x^2} + \frac{3}{x}$$

$$= \frac{3x-6}{x^2}$$

Question 1 a ii

$$f'(x) = \frac{-6}{x^2} + \frac{3}{x}$$

$$f''(x) = \frac{12}{x^3} - \frac{3}{x^2}$$

$$= \frac{12-3x}{x^3}$$

Question 1 b

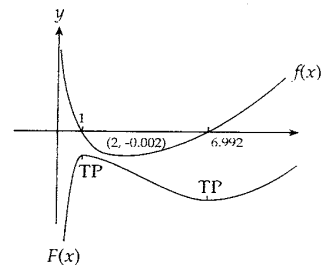
$$f(x) = \frac{6}{x} - 6 + 3 \log_e x$$

$$f(1) = \frac{6}{1} - 6 + 3 \log_e(1)$$

$$= 6 - 6 = 0$$

$\therefore f(1) = 0$ is an x -intercept

Question 1 c i



Eqn of graph not required:

$$F(x) = 6 \log_e x - 6x + 3(\log_e x - x)$$

Question 1 c ii

From a i

$$f'(x) = \frac{-6}{x^2} + \frac{3}{x}$$

$$= \frac{3x-6}{x^2}, \quad x \neq 0$$

Turning point occurs when $f'(x) = 0$

$$\frac{3x-6}{x^2} = 0$$

$$3x-6 = 0$$

$$x = 2$$

$$f(x) = \frac{6}{x} - 6 + 3 \log_e x$$

$$f(2) = \frac{6}{2} - 6 + 3 \log_e 2$$

$$= 3 \log_e 2 - 3$$

$$= 3(\log_e 2 - 1)$$

Turning point: $(2, 3(\log_e 2 - 1))$

Question 1 c iii

Using graphing calculator

x -intercept at $x = 4.92$

Question 1 d

Using result from a ii

$$f''(x) = \frac{12-3x}{x^3}, \quad x \neq 0$$

Max value of $f'(x)$ occurs when $f''(x) = 0$, i.e. $x = 4$

$$f'(4) = \frac{3 \times 4 - 6}{16}$$

$$= \frac{6}{16}$$

$$= \frac{3}{8}$$

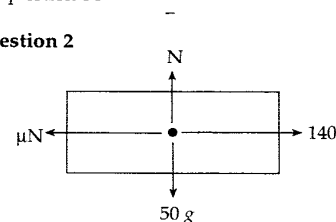
Max gradient at $x = 4$

Max gradient is at $\frac{3}{8}$

Question 1 e

See question c i

Question 2



$$140 = \mu N \quad (1)$$

$$N = 50g \quad (2)$$

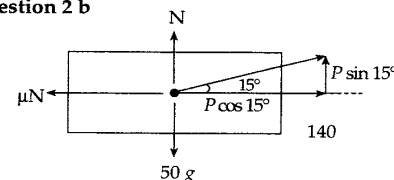
Substituting (2) into (1)

$$140 = \mu 50g$$

$$\mu = \frac{140}{50g}$$

$$\approx 0.286 \text{ (3 dp)}$$

Question 2 b



$$P \sin 15^\circ + N = 50g$$

$$N = 50g - P \sin 15^\circ \quad (3)$$

$$\mu N = P \cos 15^\circ \quad (4)$$

Using $\mu = \frac{14}{5g}$ from 2 a

and substituting (3) into (4)

$$\frac{14}{5g} (50g - P \sin 15^\circ) = P \cos 15^\circ$$

$$\frac{14}{5g} \times 50g - \frac{14}{5g} P \sin 15^\circ = P \cos 15^\circ$$

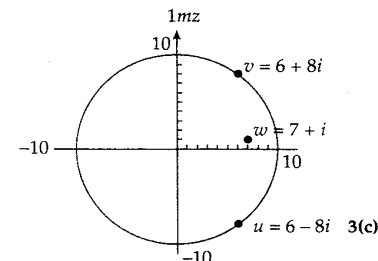
$$140 = \frac{14}{5g} P \sin 15^\circ + P \cos 15^\circ$$

$$140 = P \left[\frac{14}{5g} \sin 15^\circ + \cos 15^\circ \right]$$

$$P = \frac{140}{\frac{14 \sin 15^\circ}{5g} + \cos 15^\circ}$$

$$= 135 \text{ to the nearest integer}$$

Question 3 a



Question 3 b i

$$v = 6 + 8i$$

$$|v| = \sqrt{6^2 + 8^2}$$

$$= \sqrt{36 + 64}$$

$$= \sqrt{100}$$

$$= 10$$

Question 3 b ii

See above—it is a circle with radius 10 and centre $(0, 0)$

Question 3 c

$$u + i\bar{w} = \bar{w}$$

$$w = 7 + i$$

$$\bar{w} = 7 - i$$

$$u + i\bar{w} = \bar{w}$$

$$u + i(7 - i) = 7 - i$$

$$u = 6 - 8i$$

Question 3 d

Given on above graph as perpendicular bisector of UV, i.e. $y = 0$

Question 3 e

$$\text{Let } \underline{OV} = 6\mathbf{i} + 8\mathbf{j}$$

$$\text{and } \underline{OW} = 7\mathbf{i} + \mathbf{j}$$

Question 7 [A]

$$z = \sqrt{3} + i$$

$$|z| = 2$$

$$\arg z = \frac{\pi}{6}$$

In its simplest form, z can be written as $2 \operatorname{cis} \left(\frac{\pi}{6} \right)$

polar form. This is equivalent to $2 \operatorname{cis} \left(\frac{-11\pi}{6} \right)$

Question 8 [B]

$|z| = |z + 2|$ defines the set of points which are equidistant from the origin, 0, and the point $-2 + 0i$. In other words, the set of points will lie on a line parallel to $\operatorname{Im}(z)$ and passing through the point $-1 + 0i$, that is, a line $\operatorname{Re}(z) = -1$.

Question 9 [E]

$$\frac{-1}{\sqrt{1-2x^2}} \text{ can be transformed to: } \frac{-1}{\sqrt{1-(\sqrt{2}x)^2}}$$

Which is an antiderivative of $\frac{1}{\sqrt{2}} \cos^{-1}(\sqrt{2}x)$

Question 10 [C]

$$\begin{aligned} & \int_0^1 (\cos^2 x - \sin^2 x) dx \\ &= \int_0^1 \cos 2x dx \\ &= \left[\frac{\sin 2x}{2} \right]_0^1 \\ &= \frac{\sin(2)}{2} = 0.4546 \end{aligned}$$

Question 11 [C]

$$\int_1^3 \frac{1}{x^2} e^{\frac{3}{x}} dx$$

$$\text{Let } u = \frac{3}{x}, \text{ then } \frac{du}{dx} = \frac{-3}{x^2}$$

$$\text{So, } \frac{1}{x^2} = \frac{-1}{3} \cdot \frac{du}{dx}$$

$$\begin{aligned} \text{When } x = 1, u &= 3 \\ x = 3, u &= 1 \end{aligned}$$

So the above integral can be changed to

$$\frac{-1}{3} \int_3^1 e^u \cdot \frac{du}{dx} dx$$

$$\frac{1}{3} \int_1^3 e^u du$$

Question 12 [E]

$$\frac{4}{x^2 - 4x}$$

$$= \frac{4}{x(x-4)}$$

$$= \frac{A}{x} + \frac{B}{x-4}$$

$$\text{Hence, } A(x-4) + B(x) \equiv 4$$

$$A = -1, B = 1$$

$$= \frac{1}{x-4} - \frac{1}{x}$$

So an antiderivative of the above function would be:

$$\log_e(x-4) - \log_e(x)$$

$$\log_e \left(\frac{x-4}{x} \right)$$

Which is defined for $x > 4$

Question 13 [E]

The required solid of revolution is given by:

$$\int_0^{\sqrt{3}} \pi y^2 dx - \int_0^{\sqrt{3}} \pi (1)^2 dx$$

$$\pi \int_0^{\sqrt{3}} (y^2 - 1) dx$$

$$\pi \int_0^{\sqrt{3}} \left(\frac{4}{1+x^2} - 1 \right) dx$$

Question 14 [D]

$y = f(x)$ is the derivative function corresponding to $y = F(x)$

i.e. $F'(x) = 0$ at $x = -1$ where

$F'(x) > 0$ for $x < -1$ and

$F'(x) < 0$ for $x > -1$, implying that

$F'(x)$ has a local maximum at $x = -1$.

Also $F'(x) = 0$ for $x = 2$ but since the sign of $F'(x)$ is the same either side of $x = 2$, this point will be a stationary point of inflexion.

Question 15 [D]

$$y = \cos(2x - 3)$$

$$\frac{dy}{dx} = -2\sin(2x - 3)$$

$$\frac{d^2y}{dx^2} = -4\cos(2x - 3)$$

Options A, B, C cannot be correct, since

$$4y + \frac{d^2y}{dx^2} = 0$$

$$\text{since } -2 \frac{dy}{dx} = +4 \sin(2x - 3)$$

D is correct and not E.

Question 16 [B]

$$g'(x) = x\sqrt{x^2 + 1}$$

$$= x(x^2 + 1)^{\frac{1}{2}}$$

$$g(x) = \frac{1}{3}(x^2 + 1)^{\frac{3}{2}} + c$$

$$g(0) = \frac{1}{3} + c = 1, \quad c = \frac{2}{3}$$

$$g(x) = \frac{1}{3}(x^2 + 1)^{\frac{3}{2}} + \frac{2}{3}$$

Question 17 [B]

No of steps n	x_n	y_n
0	1	2
1	1.1	$2 + (\log_e 1) \times 0.1 = 2$
2	1.2	$2 + (\log_e 1.1) \times 0.1$ $= 2.00953$ $= 2.0095$ (to 4 decimal places)

Question 18 [D]

$$a = \frac{dv}{dt} = 5e^{-0.1t}$$

$$v = -50e^{-0.1t} + c$$

when $t = 0, v = 0$

$$0 = -50 + c, \quad c = 50$$

$$v = 50 - 50e^{-0.1t}$$

when $t = 1$

$$v = 50(1 - e^{-0.01})$$

$$v = 50(0.9516)$$

$v = 4.8$ correct to 2 significant figures.

Question 19 [C]

$v = 15 - 10t$ is the relationship between v and t . This is found by considering the v -intercept (0, 15) and the gradient of the line which is -4 .

Defining $x = 0$ as ground level,

$$x = 15t - 5t^2 + c$$

$$x = 0 \text{ and } t = 4$$

$$0 = 60 - 80 + c, \text{ giving } c = 20$$

$$x = 15t - 5t^2 + 20$$

when $t = 0, x = 20\text{m}$ which is the height of the balcony above the ground.

$$\vec{OW} = 7\vec{i} + \vec{j}$$

$$\begin{aligned}\vec{WV} &= 7\vec{i} + \vec{j} - 6\vec{i} + 8\vec{j} \\ &= \vec{i} - 7\vec{j}\end{aligned}$$

$$\begin{aligned}\vec{OW}, \vec{WV} &= \begin{pmatrix} 7\vec{i} + \vec{j} \end{pmatrix} + \begin{pmatrix} \vec{i} - 7\vec{j} \end{pmatrix} \\ &= 7 - 7 = 0\end{aligned}$$

$\vec{OW}$ perpendicular to $\vec{WV}$

Question 4 a i

$$\frac{d(x \cos x)}{dx} = \cos x - x \sin x$$

Question 4 a ii

$$\frac{d(x \cos x)}{dx} = \cos x - x \sin x$$

$$x \sin x = \cos x - \frac{d(x \cos x)}{dx}$$

$$\begin{aligned}\int x \sin x \, dx &= \int \cos x \, dx - \int \frac{d(x \cos x)}{dx} \, dx \\ &= \sin x - x \cos x\end{aligned}$$

Question 4 b

$$y = \frac{-x}{50} (8 + \sin x)$$

$$\text{at } x = 10$$

$$y = \frac{-10}{50} (8 + \sin 10)$$

$$= -1.5 \text{ metres}$$

Depth is 1.5 metres

Question 4 b ii

$$= \int_0^{60} \frac{x}{50} (8 + \sin x) \, dx \quad (\text{treating area as positive})$$

$$= \frac{1}{50} \int_0^{60} 8x + x \sin x \, dx$$

$$= \frac{1}{50} \left[4x^2 + \sin x - x \cos x \right]_0^{60}$$

$$= \frac{1}{50} \left[(4 \times 60^2 + \sin 60 - 60 \cos 60) - (0) \right]$$

$$= 289 \text{ m}^2$$

Question 4 c i

$$\begin{aligned}\text{Volume} &= 289.14 \times 150 \\ &= 43371 \text{ m}^3/\text{sec}\end{aligned}$$

$$\begin{aligned}\text{Nutrient} &= \frac{\text{Vol}}{500} \\ &= \frac{43371}{500} \\ &= 87.74\end{aligned}$$

Nutrient required 87 units to nearest unit

Question 4 c ii

$$\begin{aligned}\text{Units required} &= \frac{150}{500} \int_0^{60} \frac{x^2}{1600} (8 + \sin x) \, dx \quad (\text{treating area as positive}) \\ &= \frac{3}{1600} \int_0^{60} x^2 (8 + \sin x) \, dx \\ &= 108.6353\end{aligned}$$

109 units require to the nearest unit

Question 5 a

Javelin reaches the ground when the $\vec{k}$ component is zero

$$\vec{k} \Rightarrow z = 2 + 19.5t - 5t^2$$

$$\begin{aligned}\text{at } t = 4 \quad z &= 2 + 19.5 \times 4 - (5 \times 16) \\ &= 0\end{aligned}$$

Question 5 b

$$r(t) = 19.5t\vec{i} + \left(\frac{\pi t}{2} - 4\sin\left(\frac{\pi t}{8}\right)\right)\vec{j} + (2 + 19.5t - 5t^2)\vec{k}$$

$$r(4) = 78\vec{i} + (2\pi - 4)\vec{j} + 0\vec{k}$$

$$\begin{aligned}r(4) &= \sqrt{78^2 + (2\pi - 4)^2} \\ &= 78.03 \text{ metres (to nearest cm)}\end{aligned}$$

Question 5 c

$$r(t) = 19.5\vec{i} + \left(\frac{\pi t}{2} - 4\sin\left(\frac{\pi t}{8}\right)\right)\vec{j} + (2 + 19.5t - 5t^2)\vec{k}$$

$$\dot{r}(4) = 19.5\vec{i} + \left(\frac{\pi}{2} - \frac{4 \times \pi}{8} \cos\left(\frac{\pi t}{8}\right)\right)\vec{j} + (19.5 - 10t)\vec{k}$$

$$\begin{aligned}\dot{r}(4) &= 19.5\vec{i} + \left(\frac{\pi}{2} - \frac{\pi}{2} \cos\left(\frac{\pi}{2}\right)\right)\vec{j} + (19.5 - 40)\vec{k} \\ &= 19.5\vec{i} + \frac{\pi}{2}\vec{j} - 20.5\vec{k}\end{aligned}$$

Question 5 d

The angle with which the javelin strikes the ground is given by the angle made by the velocity vector $\dot{r}(4)$

$$\tan \theta = \frac{20.5}{\sqrt{19.5^2 + \frac{\pi^2}{4}}}$$

$$\theta = 46.3^\circ \text{ (or } 133.7^\circ)$$

A longer method would be to find the angle between $\dot{r}(4)$ and $r(4)$ using scalar product

$$\dot{r}(4) \cdot r(4) = |\dot{r}(4)| |r(4)| \cos \theta$$

Question 6 a

$$v = u + at$$

$$v = 20 \quad a = -9.8 \quad u = 0$$

$$-20 = 0 - 9.8t$$

$$t = \frac{20}{9.8} = 2.04 \text{ seconds}$$

$$s = ut + \frac{1}{2}at$$

$$s = 0 + \frac{1}{2} \times 9.8 \times \frac{-20}{9.8} \times \frac{20}{9.8}$$

$$s = -20.4 \text{ metres}$$

Time in free fall 2.04 second

Distance in free fall 20.4 metres

Question 6 b

$$ma = gm - 0.04 \, mg \, v^2$$

$$a = g - 0.04 \, g \, v^2$$

$$a = -0.04 \, g \left(-\frac{1}{0.4} + v^2 \right)$$

$$= -0.04 \, g (v^2 - 25)$$

$$\frac{dv}{dt} = -0.04 \, g (v^2 - 25)$$

Question 6 c i

$$\begin{aligned}\frac{dt}{dv} &= \frac{1}{-0.04 \, g (v^2 - 25)} \\ &= \frac{-25}{g(v^2 - 25)}\end{aligned}$$

$$\text{Consider } \frac{1}{v^2 - 25} = \frac{A}{v - 5} + \frac{B}{v + 5}$$

$$1 = A(v + 5) + B(v - 5)$$

using $v = 5$, $1 = 10A$

$$A = \frac{1}{10}$$

$$\therefore \frac{1}{v^2 - 25} = \frac{1}{10} \left(\frac{1}{v - 5} \right) - \left(\frac{1}{v + 5} \right)$$

$$\therefore \frac{dt}{dv} = \frac{-2.5}{g} \left(\frac{1}{v - 5} \right) - \left(\frac{1}{v + 5} \right)$$

$$t + c = \frac{-2.5}{g} \ln \frac{v - 5}{v + 5}$$

at $t = 0$

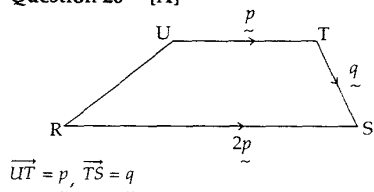
$$v = 20$$

$$c = \frac{-2.5}{g} \ln(0.6)$$

$$t = \frac{-2.5}{g} \ln \left(\frac{v - 5}{v + 5} \right) + \frac{-2.5}{g} \ln(0.6)$$

$$t = \frac{-2.5}{g} \ln \left(\frac{v - 5}{0.6(v + 5)} \right)$$

$$\text{or } t = \frac{-1}{0.4g} \ln \left[\frac{5(v - 5)}{3(v + 5)} \right]$$

Question 20 [A]

$$\text{So } \vec{RS} = 2\vec{p}$$

$$\vec{RS} + \vec{ST} + \vec{TU} + \vec{UR} = 0$$

$$2\vec{p} - \vec{q} - \vec{p} + \vec{UR} = 0$$

$$\text{hence } \vec{UR} = \vec{q} - \vec{p}$$

Question 21 [A]

The given vector is:

$$-3\vec{i} + 2\vec{j} + 6\vec{k}$$

it has magnitude of 7 units $= \sqrt{9 + 4 + 36} = \sqrt{49} = 7$

So a vector with magnitude of 14 units and parallel to the given vector could be

$$2(-3\vec{i} + 2\vec{j} + 6\vec{k})$$

$$\text{or } -2(-3\vec{i} + 2\vec{j} + 6\vec{k})$$

The second one corresponds to A.

Question 22 [C]

$$\vec{m} = -3\vec{i} + 4\vec{j}$$

$$\vec{n} = -\vec{i} + 2\vec{j} + 2\vec{k}$$

$$\vec{n} = \frac{1}{3}(-\vec{i} + 2\vec{j} + 2\vec{k})$$

The scalar resolute of $\vec{m}$ in the direction of $\vec{n}$ is given by

$$\vec{m} \cdot \hat{n} = m \cos \theta$$

$$= (-3\vec{i} + 4\vec{j}) \cdot \frac{1}{3}(-\vec{i} + 2\vec{j} + 2\vec{k})$$

$$= \frac{1}{3}(3 + 8)$$

$$= \frac{11}{3}$$

Question 23 [E]

A rhombus has all four sides equal and two pair of parallel sides.

- A. Would be sufficient to define a trapezium not a rhombus.
- B. Would be sufficient to define a parallelogram only.
- C. Would be sufficient to define a rectangle.
- D. Would be sufficient to define a rhombus or a square.
- E. Is sufficient to define a rhombus only.

Question 24 [A]

$$\vec{r} = (\sin^2 t)\vec{i} - (2\cos^2 t)\vec{j} \quad 0 \leq t \leq \frac{\pi}{2}$$

$$\vec{r} = x\vec{i} + y\vec{j}$$

$$x = \sin^2 t$$

$$y = -2\cos^2 t$$

$2x - y = 2$ which is a straight line passing through (0, -2) and (1, 0)

Question 25 [D]

$$\vec{r} = \left(\frac{3}{2}\sin(2t)\right)\vec{i} - (2e^{-2t})\vec{j}$$

$$\dot{\vec{r}} = (3\cos(2t))\vec{i} - (4e^{-2t})\vec{j}$$

at $t = 0$

$$\dot{\vec{r}} = 3\vec{i} - 4\vec{j}$$

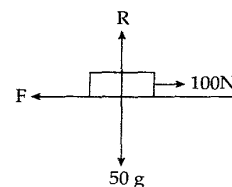
speed is $|\dot{\vec{r}}|$, so at $t = 0$, speed is 5 units.

Question 26 [B]

Initial momentum is $5 \times 10 \text{ kg m/s}$

Subsequent momentum is $5 \times 6 \text{ kg m/s}$

Change in momentum is a loss of 20 kg.m/s in the direction of the motion.

Question 27 [C]

Since the coefficient of friction is 0.5, the limiting value of friction would be $0.5R$ where $R = 50g$. A horizontal force of 100 Newton is less than the limiting value of friction. So the body will remain at rest with $F = 100N$.

Question 28 [D]

Since the body, B, is in equilibrium the vectors $\vec{P}$ and $\vec{W}$ together will be equivalent to a vector equal in magnitude but opposite in direction to the vector $\vec{T}$. This is expressed by writing

$$\vec{P} + \vec{W} + \vec{T} = 0$$

This relationship is independent of the angle AB makes with the wall, so long as the body is in equilibrium.

Question 29 [C]

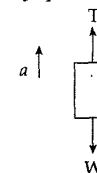
The resultant force acting on the car, F , is given by $F = 900 - 750cv$

(Note: that the total resistance is dependent upon the mass of the car)

Also $F = ma$

$$= 750 \frac{dv}{dt}$$

$$\text{So } 750 \frac{dv}{dt} = 900 - 750cv$$

Question 30 [A]

Resultant force acting on the lift is $F = T - W$.

For constant acceleration upwards $F > 0$, so $T > W$ and T is also constant.

For constant velocity, $F = 0$, so $T = W$

For constant retardation, $F < 0$, so $T < W$ and T is also a constant.

Diagram A meets all of these criteria.

Question 6 c ii

From 6 c i

$$t = \frac{-2.5}{g} \ln\left(\frac{v-5}{0.6(v+5)}\right)$$

$$-0.4gt = \ln\frac{v-5}{0.6(v+5)}$$

$$e^{-0.4gt} = \frac{v-5}{0.6(v+5)}$$

$$0.6ve^{-0.4gt} = \frac{v-5}{v+5}$$

$$0.6ve^{-0.4gt} + 3e^{-0.4gt} = v - 5$$

$$3e^{-0.4gt} + 5 = v - 0.6ve^{-0.4gt}$$

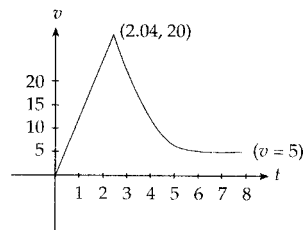
$$5(1 + 0.6e^{-0.4gt}) = v(1 - 0.6e^{-0.4gt})$$

$$v = \frac{5(1 + 0.6e^{-0.4gt})}{1 - 0.6e^{-0.4gt}}$$

Question 6 dLimiting velocity occurs for large values of t

$$\text{when } t \rightarrow \infty \quad e^{-0.4gt} \rightarrow 0$$

$$\begin{aligned} V_{\text{limit}} &= \frac{5(1+0)}{1-0} \\ &= 5 \text{ m/s} \end{aligned}$$

Question 6 enote discontinuity at $t = 8$ when $v = 0$