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SPECIALIST MATHS TRIAL EXAM 2 2000 SOLUTIONS

Question 1

a. i. $\frac{1}{z_1} = \frac{1}{3-4i}$

$= \frac{1}{3-4i} \times \frac{3+4i}{3+4i}$

$= \frac{3+4i}{9+16}$

$= \frac{1}{25}(3+4i)$ **(1 mark)**

ii. $\text{Arg}\, z_1 = \tan^{-1}\left(\frac{-4}{3}\right)$

$= -53°8'$ to the nearest minute **(1 mark)**

iii. left side $= -4+2+i+(3-4i)(2-i)$

$= -2+i+6-3i-8i-4$

$= -10i$

= right side Have verified. **(1 mark)**

iv. Let $w^2 = 3-4i$ where $w = x+yi$, $x, y \in R$

So, $(x+yi)(x+yi) = 3-4i$

$x^2 + 2xyi - y^2 = 3-4i$

$x^2 - y^2 + 2xyi = 3-4i$

Equating real and imaginary parts, we have

$x^2 - y^2 = 3$ ________(A) and $2xyi = -4i$ **(1 mark)**

$x = \frac{-2}{y}$ ________(B)

(B) in (A) gives $\frac{4}{y^2} - y^2 = 3$

$4 - y^4 = 3y^2$

$y^4 + 3y^2 - 4 = 0$

$(y^2+4)(y^2-1) = 0$

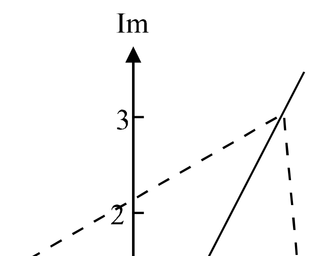
Since $y \in R$, $y^2 + 4 = 0$ has no solutions **(1 mark)**

So, $y = \pm 1$

When $y = 1$, $x = -2$ and when $y = -1$, $x = 2$

So the roots are $-2+i$ and $2-i$ **(1 mark)**

b. i. We are looking for the locus of points for which the distance from the complex number $2-i$ is equal to the distance from the complex number $-2+i$ Mark each of these two complex numbers on the Argand plane. Mark the midpoint of the line joining these two points. Draw a straight line which passes through this midpoint and runs at right angles to the line joining the two complex numbers. The diagram below shows this.



(1 mark)

ii. Let $z = x + yi$, so we have $|x + yi - 2 + i| = |x + yi + 2 - i|$

$$\sqrt{(x-2)^2 + (y+1)^2} = \sqrt{(x+2)^2 + (y-1)^2}$$
$$x^2 - 4x + 4 + y^2 + 2y + 1 = x^2 + 4x + 4 + y^2 - 2y + 1$$
$$-8x + 4y = 0$$
$$y = 2x \qquad \textbf{(1 mark)}$$

iii. For all points in S, except $0 + 0i$, $\theta = \tan^{-1} 2$ and $\tan^{-1} 2 \neq \frac{\pi}{3}$ so $z_4 \notin S$.

An alternative solution is as follows.

$$z_4 = 3(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$$
$$= \frac{3}{2} + \frac{3\sqrt{3}}{2}i$$

So, $x = \frac{3}{2}$ and $y = \frac{3\sqrt{3}}{2}$

For complex numbers belonging to the set of S the relationship between x and y is given by $y = 2x$. Clearly for z_4, $\frac{3\sqrt{3}}{2} \neq 2 \times \frac{3}{2}$ and so $z_4 \notin S$ **(1 mark)**

iv. Now, $z_4 = \frac{3}{2} + \frac{3\sqrt{3}}{2}i$ from part **iii.**

So the subset that we require is $\left\{ z : \left| z - \frac{3}{2} - \frac{3\sqrt{3}}{2}i \right| = |z - 3 + 4i| \right\}$

(2 marks) 1 mark for each side of the equation

Total 11 marks

Question 2

a. When $y = 0$, we have $0 = \frac{16}{x^2} - 1$

So, $1 = \frac{16}{x^2}$

$$x^2 = 16$$
$$x = \pm 4$$

The x intercept of the function f is (4, 0). Note that $x = -4$ is not in the domain of f. **(1 mark)**

b. $f(x) = \frac{16}{x^2} - 1$

$= 16x^{-2} - 1$

So, $f'(x) = -32x^{-3}$

When $f'(x) = -1$, $\frac{-32}{x^3} = -1$

So, we need to solve the equation $x^3 - 32 = 0$ **(1 mark)**

To solve this analytically, we have

$$\left\{x - \sqrt[3]{32}\right\}\left\{x^2 + \sqrt[3]{32}x + 32^{\frac{2}{3}}\right\} = 0$$

Now, $\left\{x^2 + \sqrt[3]{32}x + 32^{\frac{2}{3}}\right\} = 0$ has no solution (a quick sketch on a graphics calculator will reveal this), and so we have

$x - \sqrt[3]{32} = 0$ and so $x = 3.1748$ (to 4 places)

Now, $f(3.1748) = 0.5874$ (to 4 places)

So, $(m, n) = (3.2, 0.6)$ correct to 1 decimal place **(1 mark)**

To solve this question using a graphics calculator, just graph $y = x^3 - 32$ and find the point where this graph crosses the x axis.

c. This can be evaluated numerically using a graphics calculator. Alternatively an analytical approach could be taken.

The corner points of the cross sectional area are (1, 15), (4, 0), (-4, 0) and (-1, 15)

area $= 2 \times 1 \times 15 + 2\int_1^4 f(x)dx$ **(1 mark)**

$= 30 + 2\int_1^4 (\frac{16}{x^2} - 1)dx$ **(1 mark)**

$= 30 + 2\left[\frac{16x^{-1}}{-1} - x\right]_1^4$

$= 30 + 2\left[\frac{-16}{x} - x\right]_1^4$

$= 30 + 2\{(-4-4) - (-16-1)\}$

$= 48$ units **(1 mark)**

d. Since we are rotating around the y axis, our terminals of integration must be y values.

volume required $= \pi \int_0^{15} x^2\, dy - \pi \int_1^{15} x^2 dy$ where the first integrand pertains to the function *f(x)* and the second integrand pertains to the function *g(x)* **(1 mark)**

Now, for *f(x)*, let $y = \frac{16}{x^2} - 1$ and for *g(x),* let $y = 14x^2 + 1$

so, $x^2 = \frac{16}{y+1}$ so, $x^2 = \frac{y-1}{14}$

So, volume required $= \pi \int_0^{15} \frac{16}{y+1} dy - \pi \int_1^{15} \frac{y-1}{14} dy$ **(2 marks)**

$$= 16\pi[\log_e(y+1)]_0^{15} - \frac{\pi}{14}\left[\frac{y^2}{2} - y\right]_1^{15}$$

$$= 16\pi\{\log_e 16 - \log_e 1\} - \frac{\pi}{14}\{(112.5 - 15) - (0.5 - 1)\}$$

$$= 16\pi \log_e 16 - 7\pi$$

$$= \pi(16\log_e 16 - 7) \text{ cubic units}$$ **(1 mark)**

Total 10 marks

Question 3

a. i. distance from origin $= \sqrt{(\sqrt{2}\sin(2t)+1)^2 + 2\cos^2(2t)}$

$$= \sqrt{2\sin^2(2t) + 2\sqrt{2}\sin(2t) + 1 + 2\cos^2(2t)}$$

$$= \sqrt{2(\sin^2(2t) + \cos^2(2t)) + 2\sqrt{2}\sin(2t) + 1}$$

$$= \sqrt{2 \times 1 + 2\sqrt{2}\sin(2t) + 1}$$

$$= \sqrt{3 + 2\sqrt{2}\sin(2t)}$$ **(1 mark)**

ii. Particle A is furtherest from the origin when $\sqrt{3 + 2\sqrt{2}\sin(2t)}$ is a maximum. This occurs when $\sin(2t) = 1, \quad t \ge 0$

$$2t = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}$$

$$t = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4} \ldots$$

$$= \frac{\pi}{4} + n\pi \qquad n \in J^+$$

So furtherest distance that particle A can be from the origin is $\sqrt{3 + 2\sqrt{2}}$ metres and this occurs when $t = \frac{\pi}{4} + n\pi \qquad n \in J^+$ **(2 marks)**

b. To Show: speed $= \left|\dot{\underset{\sim}{r}}\right| = k$ where k is a constant

$$\dot{\underset{\sim}{r}}_A = 2\sqrt{2}\cos(2t)\,\underset{\sim}{i} - 2\sqrt{2}\sin(2t)\,\underset{\sim}{j}$$ **(1 mark)**

$$\begin{aligned}\left|\dot{\underset{\sim}{r}}_A\right| &= \sqrt{8\cos^2(2t) + 8\sin^2(2t)} \\ &= \sqrt{8(\cos^2(2t) + \sin^2(2t))} \\ &= \sqrt{8} \\ &= 2\sqrt{2} \quad \text{which is a constant}\end{aligned}$$ **(1 mark)**

Have shown

c. i. $x = \frac{4}{\sqrt{3}}\cos(2t)$ and $y = 4\sin t\cos t$ **(1 mark)**

$$= 2\sin(2t)$$

So, $x^2 = \frac{16}{3}\cos^2(2t)$ and $y^2 = 4\sin^2(2t)$

So, $\frac{3x^2}{16} = \cos^2(2t)$ and $\frac{y^2}{4} = \sin^2(2t)$

So, $\frac{3x^2}{16} + \frac{y^2}{4} = \cos^2(2t) + \sin^2(2t)$

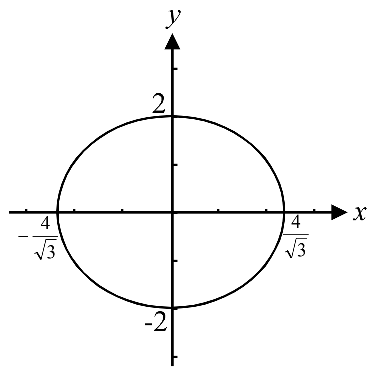
So, $\frac{3x^2}{16} + \frac{y^2}{4} = 1$ is the required Cartesian equation **(1 mark)**

ii. Now $x = \frac{4}{\sqrt{3}}\cos(2t)$ and $t \geq 0$, so, $x \in [-\frac{4}{\sqrt{3}}, \frac{4}{\sqrt{3}}]$ **(1 mark)**

Also, $y = 4\sin t\cos t$

$= 2\sin(2t)$ and $t \geq 0$, so, $y \in [-2, 2]$ **(1 mark)**

iii.



(1 mark)

d. To show: $\ddot{\underset{\sim}{r}}_B = k\underset{\sim}{r}_B$ where k is a constant

Now, $\underset{\sim}{r}_B = \dfrac{4}{\sqrt{3}}\cos(2t)\underset{\sim}{i} + 4\sin t\cos t\,\underset{\sim}{j}$

$= \dfrac{4}{\sqrt{3}}\cos(2t)\underset{\sim}{i} + 2\sin(2t)\underset{\sim}{j}$

So, $\dot{\underset{\sim}{r}}_B = \dfrac{-8}{\sqrt{3}}\sin(2t)\underset{\sim}{i} + 4\cos(2t)\underset{\sim}{j}$

So, $\ddot{\underset{\sim}{r}}_B = \dfrac{-16}{\sqrt{3}}\cos(2t)\underset{\sim}{i} - 8\sin(2t)\underset{\sim}{j}$ **(1 mark)**

Now, left side $= \ddot{\underset{\sim}{r}}_B$

$= \dfrac{-16}{\sqrt{3}}\cos(2t)\underset{\sim}{i} - 8\sin(2t)\underset{\sim}{j}$

$= -4\left(\dfrac{4}{\sqrt{3}}\cos(2t)\underset{\sim}{i} + 2\sin(2t)\underset{\sim}{j}\right)$

$= k\underset{\sim}{r}_B$ where $k = -4$

$=$ right side

Have shown **(1 mark)**

e. Particle A and B collide iff $\sqrt{2}\sin(2t) + 1 = \dfrac{4}{\sqrt{3}}\cos(2t)$ AND $\sqrt{2}\cos(2t) = 4\sin t\cos t$

(1 mark

Now,

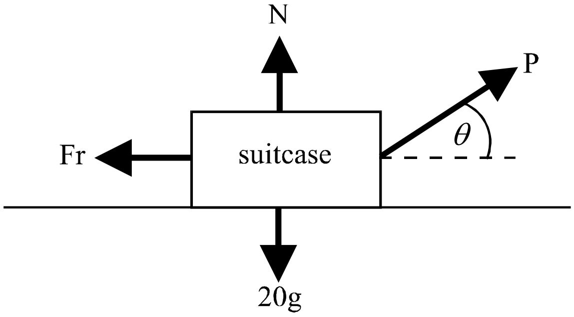
$$\sqrt{2}\cos(2t) = 2\sin(2t)$$
$$\tan(2t) = \frac{\sqrt{2}}{2}$$
$$2t = 0.6154$$
$$t = 0.3077$$

It is not necessary to solve the other equation since for a collision to occur, both the equations must have the same value of t. Since $t = 0.3077$ occurs outside the required interval, a collision cannot occur in the first one tenth of a second of the particles' motion. **(1 mark)**

Total 14 marks

Question 4

a.



(1 mark)

b. Using part **a.**, we have, resolving horizontally, $Fr = \frac{20\sqrt{2}g}{3}\cos\theta$ ________(A)

and resolving vertically we have $N + \frac{20\sqrt{2}g}{3}\sin\theta = 20g$ ____________(B)

Since the suitcase is on the point of moving, $Fr = \mu N$, so $Fr = 0.5N$ _________(C) **(1 mark)**

Using (B), we have $N = 20g - \frac{20\sqrt{2}g}{3}\sin\theta$ ________(D)

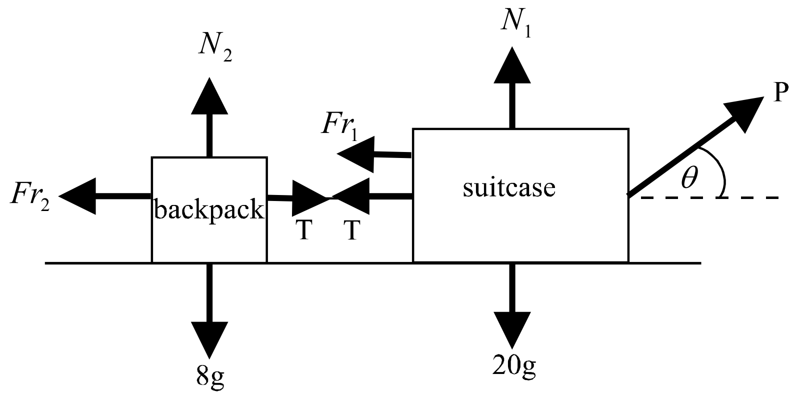
In (C), we have $Fr = 0.5(20g - \frac{20\sqrt{2}g}{3}\sin\theta)$ ________(E)

To show: $0.5(20g - \frac{20\sqrt{2}g}{3}\sin\theta) = \frac{20\sqrt{2}g}{3}\cos\theta$ where $\theta = 45°$

$$\text{Left side} = 0.5(20g - \frac{20\sqrt{2}g}{3} \times \frac{1}{\sqrt{2}})$$
$$= \frac{20g}{3}$$
$$\text{Right side} = \frac{20\sqrt{2}g}{3} \times \frac{1}{\sqrt{2}} = \frac{20g}{3}$$ **(1 mark)**

c. The suitcase is on wheels and therefore has smoother contact with the ground. **(1 mark)**

d. i.



(1 mark) for forces around the backpack **(1 mark)** for the forces around the suitcase

ii. At the point of moving, $Fr_1 = \mu N_1$

Now, resolving around the suitcase we have $N_1 + P\sin\theta = 20g$

$$N_1 = 20g - \frac{90}{\sqrt{2}}$$

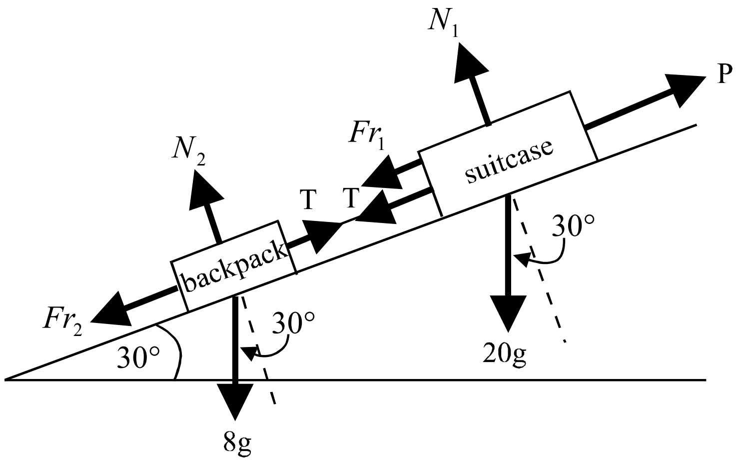
So, $$Fr_1 = 0.5(20g - \frac{90}{\sqrt{2}})$$
$$= 66.2 \text{ correct to 1 decimal place}$$

Resolving around the suitcase, we have $Fr_1 + T = P\cos\theta$

$$Fr_1 = 90 \times \frac{1}{\sqrt{2}} - 20$$
$$= 43.6 \text{ to 1 decimal place}$$

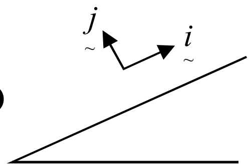
Since $43.6 < 66.2$, we see that the friction is not at a maximum and so sliding is not about to happen. **(1 mark)**

e.



(1 mark) for forces around the backpack and **(1 mark)** for forces around the suitcase

f. Using the diagram from part **e.**, and resolving around the suitcase, we have,



$$\underset{\sim}{R} = m\underset{\sim}{a}$$

So, $(300 - 20g\sin 30° - T - Fr_1)\underset{\sim}{i} + (N_1 - 20g\cos 30°) = 20a\underset{\sim}{i}$ **(1 mark)**

So, equating components in the $\underset{\sim}{i}$ direction, we have

$$300 - 10g - T - Fr_1 = 20a \quad \_\_\_\_\_\_\_\_(A)$$

and equating components in the $\underset{\sim}{j}$ direction, we have

$$N_1 - 20g \times \frac{\sqrt{3}}{2} = 0$$

So, $\qquad N_1 = 10\sqrt{3}g$

Also, $\qquad Fr_1 = \mu N_1$

$$= 0.5 \times 10\sqrt{3}g$$

$$= 5\sqrt{3}g$$

So, we have from ____(A), $\quad T = 300 - 10g - 5\sqrt{3}g - 20a \quad \_\_\_\_\_\_\_(B)$ **(1 mark)**

Resolving around the backpack we have

$$(T - Fr_2 - 8g\sin 30°)\underset{\sim}{i} + (N_2 - 8g\cos 30°)\underset{\sim}{j} = 8a\underset{\sim}{i}$$ **(1 mark)**

So, equating components in the $\underset{\sim}{i}$ direction, we have

$$T - Fr_2 - 4g = 8a \quad \_\_\_\_\_\_\_\_\_(C)$$

and $\qquad N_2 - 8g \times \frac{\sqrt{3}}{2} = 0$

$$N_2 = 4\sqrt{3}g$$

Also, $\qquad Fr_2 = \mu N_2$

$$= 0.6 \times 4\sqrt{3}g$$

$$= 2.4\sqrt{3}g$$

Substituting this into (C) gives

$$T = 2.4\sqrt{3}g + 4g + 8a \qquad \underline{\qquad\qquad}(D) \qquad \textbf{(1 mark)}$$

Equating (B) and (D) gives

$$2.4\sqrt{3}g + 4g + 8a = 300 - 10g - 5\sqrt{3}g - 20a$$
$$28a = 300 - 14g - 7.4\sqrt{3}g$$
$$a = 1.32827$$

So, $a = 1.3283 \text{ m/s}^2$ **(1 mark)**

ii. Since the pulling force is constant, the acceleration will be constant.

For constant acceleration, $s = ut + \frac{1}{2}at^2$

$$40 = 0 + \frac{1}{2} \times 1.3283t^2$$

$t = 7.8$ secs (correct to 1 decimal place) **(1 mark)**

Total 15 marks

Question 5

Now, $\dfrac{dC}{dt} = (C-3)(C+2)$

So, $\dfrac{dt}{dC} = \dfrac{1}{(C-3)(C+2)}$

And $\displaystyle\int \frac{dt}{dC}\,dC = \int \frac{1}{(C-3)(C+2)}\,dC$ **(1 mark)**

Let $\dfrac{1}{(C-3)(C+2)} \equiv \dfrac{A}{(C-3)} + \dfrac{B}{(C+2)}$

$$\equiv \frac{A(C+2) + B(C-3)}{(C-3)(C+2)}$$

True iff $1 = A(C+2) + B(C-3)$

Put $C = -2, \quad 1 = -5B$

$$B = -\frac{1}{5}$$

Put $C = 3, \quad 1 = 5A$

$A = \dfrac{1}{5}$ **(1 mark)**

So, $\dfrac{1}{(C-3)(C+2)} \equiv \dfrac{1}{5(C-3)} - \dfrac{1}{5(C+2)}$

So, $\displaystyle\int \frac{dt}{dC}\,dC = \int \frac{1}{5(C-3)}\,dC - \int \frac{1}{5(C+2)}\,dC$

$= \dfrac{1}{5}\log_e(C-3) - \dfrac{1}{5}\log_e(C+2) + k$ where k is a constant

so, $t = \dfrac{1}{5}\log_e \dfrac{C-3}{C+2} + k$ **(1 mark)**

Now, when $C = -3, \quad t = \frac{1}{5}\log_e 6$

So, $\qquad \frac{1}{5}\log_e 6 = \frac{1}{5}\log_e 6 + k$

So, $\qquad k = 0$

And so $\qquad t = \frac{1}{5}\log_e \frac{C-3}{C+2}$ **(1 mark)**

b. $\qquad 5t = \log_e \frac{C-3}{C+2}$

$$e^{5t} = \frac{C-3}{C+2}$$

$$(C+2)e^{5t} = C-3$$

$$Ce^{5t} + 2e^{5t} = C-3$$

$$Ce^{5t} - C = -2e^{5t} - 3$$

$$C(e^{5t} - 1) = -2e^{5t} - 3$$

$$C = \frac{-2e^{5t} - 3}{e^{5t} - 1}$$

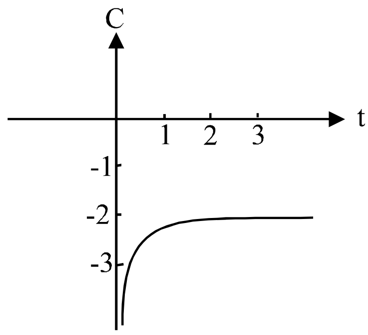
So, $\qquad C = \frac{2e^{5t} + 3}{1 - e^{5t}}$ **(1 mark)**

c. Now, $\qquad C = \frac{2e^{5t} + 3}{1 - e^{5t}}$

So at $t = 4$, $\qquad C = -2.00000001$

So at $t = 4$, the pathology sample is $0.00000001°$ below $-2°$ Celsius **(1 mark)**

d. Use a graphics calculator to graph the function.
We see that initially, say for $t \in (0, 1]$ the temperature increases at a rapid pace. **(1 mark)**
The rate at which the temperature increases then slows dramatically and the temperature approaches $-2°$ Celsius
(1 mark)



e. Now, $C = \frac{2e^{5t} + 3}{1 - e^{5t}}$ $\qquad$ and so,

$$\begin{array}{r} -2 \\ 1 - e^{5t}\overline{)\,2e^{5t} + 3} \\ 2e^{5t} - 2 \\ \hline 5 \end{array}$$

So, $\qquad C = -2 + \frac{5}{1 - e^{5t}}$ **(1 mark)**

As $t \to \infty, \; C \to -2$

(since as $t \to \infty$, $e^{5t} \to \infty$ so, $1 - e^{5t} \to -\infty$, so, $\frac{5}{1 - e^{5t}} \to 0$ from below

and so $-2 + \frac{5}{1 - e^{5t}} \to -2$ **(1 mark)**

Total 10 marks

