Methods Unit 3 – Week 1 – Introduction to Functions and Graphs

By now you should be familiar with the two types of notation that we use in Mathematical Methods, set notation and interval notation. If you need a reminder of what they are:

- A **set** is a group of numbers or objects, with each item in the set being called an **element**. Sets are usually named after a letter and capitalised. For example, they can be written as A = {1, 3, 4}. A is the set, and 1, 3 and 4 are elements of set A.
- If an element is part of a set, we use ∈, and if an element is not part of a set, ∉. For the example above, 4 ∈ A and 5 ∉ A.
- If a set is contained in part of another set, we call it a **subset**, e.g. X ⊆ Y, the set X is contained in set Y.
- The set of elements common to two sets is called the **intersection**, ∩, whilst the set of elements that are in one set <u>or</u> another set is called the **union**, ∪.
- If a set is empty, then it is the **empty set**, ∅.
- The difference between two sets is \, e.g. X \ Y means all the elements of X that are not in Y.
- The dash ' refers to a set's **complement**, which lists all of the elements that are not in that set. For example, A' refers to all of the elements that are not in set A.

There are special names for different types of sets. Note that each set is a subset of the set below it, but not the other way round (e.g. N is a subset of Z, Q and R, Z is a subset of Q and R, and Q is a subset of R).

- ℕ describes the set of natural numbers (positive whole numbers or positive integers)
- ℤ describes the set of integers (all whole numbers, positive and negative)
- ℚ describes the set of rational numbers (numbers that can be written as a fraction, rarely used in MM)
- ℝ describes the set of real numbers, containing both rational and irrational numbers (such as π, e, $\sqrt{2}$, etc.)

Describing Intervals

Set notation can also be used to describe an interval of numbers. Instead of looking at just particular numbers, we may be looking at a range of numbers. This is when we use the {x : } notation, which simply means "The set of x such that…". For example, {x: 0 < x < 1} means the set of all real numbers between 0 and 1 (excluding 0 and 1), whilst {2n + 1: n ∈ ℕ} describes all positive odd numbers.

Instead of using set notation to describe intervals (which can be cumbersome at times), we can use the simpler interval notation method. For example, {x: 0 < x < 1}, as described above, turns to (0, 1). Rounded brackets are used when the number is non-inclusive, i.e. < or >, or goes towards infinity in either positive or negative direction. Square brackets, like [3, 5] is used when the number is inclusive, i.e. $\leq$ or $\geq$.

These can also be described on a number-line: an 'open' (unshaded) circle if the number is inclusive, and a 'closed' (shaded) circle if the number is not inclusive.

Testing Understanding

1. Let Set A be defined as A = {2, 3, 5, 7, 9, 13, 15, 19} and set B be defined as B = {2, 5, 6, 7, 8, 10, 13, 27}. Find the following:
   a) $A \cup B$ b) $A \cap B$ c) $A \backslash B$ d) $B \backslash A$
2. Use appropriate interval notation, i.e. [a, b] and illustrate these on a number line:
   a) $-3 \leq x \leq 2$ b) $0 \leq x < \infty$ c) $-\infty < x < 5$ d) $-2 < x \leq 3$
3. Draw up a number line for these intervals
   a) $[-3, -1)$ b) $[8, \infty)$ c) $(-\infty, 2] \cup (4, 6]$ d) $\mathbb{R} \setminus [-3, 6)$

Functions

An **ordered pair** is used to describe a pair of elements, x and y, which can also be described as a co-ordinate. A **relation** describes a set of ordered pairs. The set of all x values in a function is called the **domain** (abbreviated as dom), whilst the set of all y values in a function is called the **range** (ran).

A **function** is a special type of relation (all functions are relations, but not the other way around). It is when no two ordered pairs in a relation share the same first coordinate. In other words, if you sketched the graph out, and drew a vertical line anywhere on the graph and it cuts through the graph with a maximum of only one time, then it is a function. This is called the **vertical line test**.

The **maximal domain (implied domain)** of a function is the domain that still gives the function its meaning. For example:

- The function y = x^3 has a maximal domain of $\mathbb{R}$ because you can substitute any real number for x.
- The function y = $\sqrt{x-2}$ has a maximal domain of [2, ∞) as the number under the square root sign should be equal than or greater than zero (it should never be negative).
- The function y = $\frac{1}{x+3}$ has a maximal domain of R \ {-3} as the denominator cannot be zero.

Domains can be restricted as well. For example, if we had a function, e.g. s = 2 + 3t, where s = speed and t = time, then clearly we can't have negative time. Therefore, the domain of the function is t $\geq$ 0. In other questions, they may just randomly choose a restricted function, e.g. y = x^3, x ∈ [3, 5]. We need to know the domain of a function before we can write our functions using function notation.

Function notation

Instead of y, we use f(x) to describe a relation using function notation. If we wanted to substitute x = 3, then we say f(3) =_____, etc. To define a function, we need to know its domain (maximal, or restricted), and the rule. For the above example with the restricted function, we would write it as $f\colon [3,5] \to R, f(x) = x^3$. When writing functions, we must have both the **rule** and the **domain**. It is a common error to omit the domain when writing down a function. In other words, we always write: $f\colon domain \to R, f(x) = \ rule$.

Types of relations

Relations can come in different forms. For example, one x-value may map to multiple y-values like (1,3) and (1,5), or multiple x-values can map to one y-value, like (3, 5) and (4, 5). These are described as one/many to one/many. If described in this way, functions are one to one or many to one. To work out what kind of relation a list of ordered pairs is, you do the **horizontal-vertical line test**.

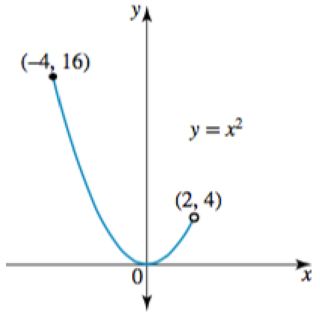
If a horizontal line, cuts through the graph at every point once, then the first word is one, otherwise, it is many. Then, if a vertical line cuts through the graph at every point once, then the second word is once, otherwise, it is many. For example, the graph of a circle is many-to-many because both horizontal and vertical lines cut through the graph more than once. This means that it is not a function because it is not one-to-one or many-to-one.

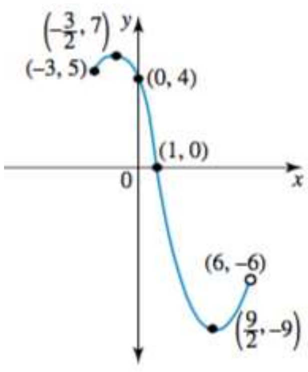
Testing Understanding

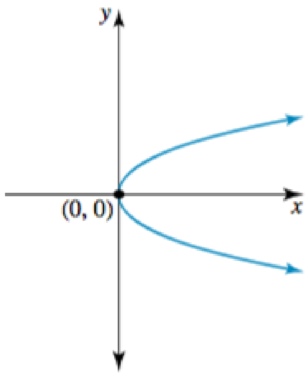
4. Find the domain and range for these relations. Also, next to each relation below, state whether or not each relation is also a function, and of those, which are also one-to-one.

a)

c)

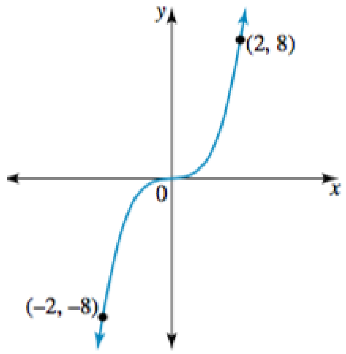


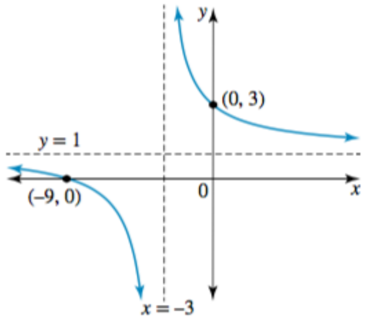


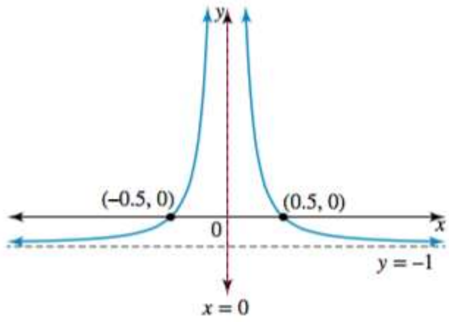


b)

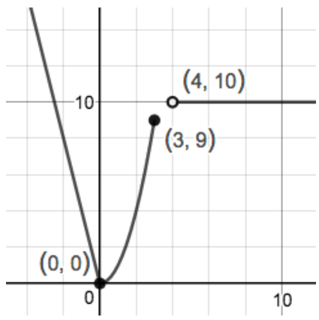
d)
e)
f)







5. Which of the following two relations (lists of ordered pairs) are functions? Explain why the one you chose is a function. State the domain and range for each. A = {(-1,1) (-1,2) (1,2) (3,4) (2,3)} and B = {(-2,0) (-1,-1) (0,3) (1,5) (2,-4)}
6. Determine the following if $f(x) = 3x^2 + x$
   a) f(-1) b) f(2a) c) f(3) d) f(a – 1)
7. Sketch the following functions (with open and closed circles) and state their range:
   a) $f: [2, \infty) \to R, f(x) = x + 1$
   b) $g: (-2, -4) \to R, g(x) = 5x - 1$
   c) $h: [-5, -1) \to R, h(x) = -3x - 1$
8. For $f(x) = x^2 - 4$, find the intervals/values of x when: (sketching a graph may be useful):
   a) {x: f(x) = 0} b) {x: f(x) > 0} c) {x: f(x) ≤ 0} d) {x: f(x) > 4}
9. Which of the following two functions are one-to-one? Explain why the one you chose is one-to-one. A = {(-1, -2) (-2, -2) (-3, 4) (-6, 7)} and B = {(1,2) (2,3) (3,4) (4,6)}
10. Keeping in mind that a number under a square root sign can never be negative, and that a denominator can never equal to zero, state the implied/maximal domain for the following functions:
    a) $y = x^2 + 3$ b) $y = \frac{1}{3-2x}$ c) $y = \sqrt{x+3}$ d) $y = \sqrt{25 - x^2}$



Piecewise (hybrid) functions

Piecewise functions are functions that have a different rule, depending on what value of x you have (i.e. There are different rules for different parts of the domain). You would sketch that rule for the required domain. For example, the function…

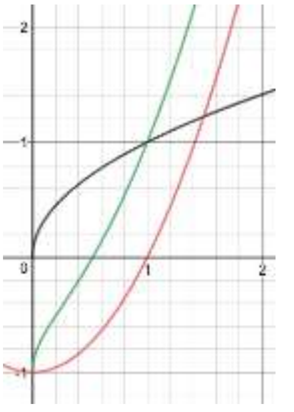
$$f(x) = \begin{cases} -4x & x < 0 \\ x^2 & 0 \le x \le 3 \\ 10 & x > 4 \end{cases}$$

Implies that when x < 0, we draw y = – 4x, between 0 and 3 inclusive we draw y = x^2, and when x > 4, we draw y = 10. This yields the following graph on the above right.

Sums and products of functions

These are simply the addition of two functions and the multiplication of two functions respectively. Notation wise, $(f + g)(x) = f(x) + g(x)$ and $(fg)(x) = f(x)g(x)$. The domain of the new function is the intersection of the domains of the previous two functions.

Graphing sums of functions is also called **addition of ordinates**. Here are a few handy tricks in sketching sums of functions:



- If one of $f(x)\ or\ g(x)$ has an x-intercept at a point, then $(f+g)(x)$ will be equal to the point in the other graph.
- If $f(x)\ or\ g(x)$ are both positive or both negative, then the new graph will be more positive than both or more negative than both respectively.
- If one of $f(x)\ or\ g(x)$ is positive, and the other is negative, then $(f+g)(x)$ is somewhere in the middle.

For example, when sketching $y = \sqrt{x} + x^2 - 1$ (the upper graph starting at 0, -1), we can use these tricks to sketch it more easily. First, break it down by sketching $y = \sqrt{x}$ (starts at 0, 0) and $y = x^2 - 1$ (starts at -1, 0).

We can see the combined graph starts off at (-1, 0) because $y = \sqrt{x}$ is at its x-intercept. It continues onto (1, 1) because $y = x^2 - 1$ is at its x-intercept. The combined graph is between the old graphs when one is negative and the other is positive, and is more positive than both when the old graphs are positive.

11. Sketch the following two functions and state their domain and range:

a) $f(x) = \begin{cases} -2x-2 & x<0 \\ x^2 & 0 \le x < 2 \\ 3x-6 & x \ge 2 \end{cases}$ b) $f(x) = \begin{cases} 8 & x < -3 \\ 5-x & -3 \le x < 0 \\ x^2+5 & x \ge 0 \end{cases}$

12. Find the implied domain of $y = \sqrt{x+3} + \sqrt{9-x}$
13. If f(x) = x + 2, and g(x) = 4 – 3x, sketch both, and hence find (f+g)(x)
14. Sketch $f: R^+ \cup \{0\} \to R, f(x) = \sqrt{x} + 2x$ using addition of ordinates
15. Find (f+g)(x) and (fg)(x) for f(x) = x^2 and g(x) = $\sqrt{4-x}$, and state their domain for both

Methods Unit 3 – Week 2 – Further Functions

This week, we will be looking at further applications of functions. Questions involving these functions have a very specific method of doing them, so it is imperative that you understand how each kind of function works and what is required when answering these questions.

Composite functions

Simply put, composite functions are the result when functions are inserted into other functions. For example, given f(x) = x^2 and g(x) = 2x – 3, then f(g(x)) = $(2x-3)^2$. This new function is described as the composition of f with g. An alternative way of writing $f(g(x))$ is $f\ o\ g$.

The domain of the composite function is equivalent to the domain of the inner function, i.e. $dom\ (f\ o\ g) = \ dom\ g$. For a composite function to be defined, the *range of the inner function* must be a subset of the *domain of the outer function*. This means for $f(g(x))$ or $f\ o\ g, ran\ g \subseteq dom\ f$. This is a requirement that *must* be met, otherwise, it is not a composite function.

Example

For f(x) = $\sqrt{2-x}$ and g(x) = x^2, find which of f o g and g o f are defined. For the composite function(s) that are defined, state the domain and rule.

The domain of f(x) is (-∞, -2], the range of f(x) is [0, ∞). The domain of g(x) is **R**, the range of g(x) is [0, ∞).

For f o g, ran g $\not\subseteq$ dom f, so the composite function is not defined.
For g o f, ran f $\subseteq$ dom g, so the composite function is defined.

The rule for g o f or g(f(x)) is = $\left(\sqrt{2-x}\right)^2 = 2 - x$. The domain is (-∞, -2].

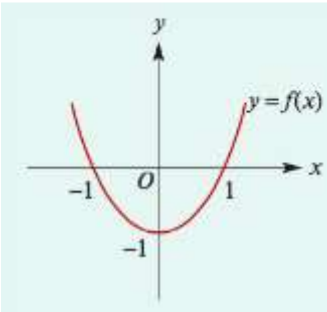
Note: If we determined f(g(x)) or f o g, our rule would be $\sqrt{2-x^2}$. If we tried to calculate f(g(3)), then we would get f(9), which would be impossible to solve, as we would have $f(9) = \sqrt{2-9} = \sqrt{-7}$. Composite functions need to work for the entire domain of g(x). This is why the range of the inner function must be a subset of the domain of the outer function.

Example

If f(x) = x^2 – 1, and g(x) = $\sqrt{x}$, explain why g o f is not defined, and restrict the function of f(x) so that it can be defined, finding g o f*.

g o f is not defined because the range of f(x) is not a subset of the domain of g(x), with [-1, ∞) $\not\subseteq$ [0, ∞).

g o f* will be defined when ran f* $\subseteq$ dom g. Therefore, we need to restrict the domain of f(x), so that its range is [0, ∞). When we sketch the graph of x^2 – 1, the range is [0, ∞) when $x \le -1$ and $x \ge 1$ (see diagram on the right).



$f^* : (-\infty, -1] \cup [1, \infty) \to R, f(x) = x^2 - 1$
$g\ o\ f^* = \sqrt{x^2 - 1}$, and domain = domain f*, so:
$g\ o\ f^* : (-\infty, -1] \cup [1, \infty) \to R, g(f^*(x)) = \sqrt{x^2 - 1}$

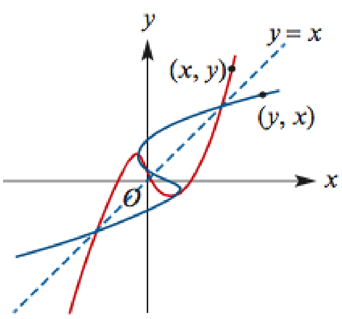
Note: As f* is a new function, we need to distinguish it from f with an asterisk.

Inverse functions

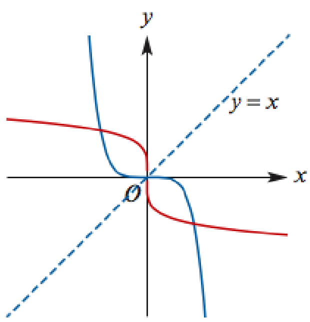
Inverse functions are functions that are essentially opposites of another function. You will have covered this in Year 11. It is notated as $f^{-1}(x)$.

Inverse functions have several properties. These include:

- They are only defined when the original function is one-to-one. If the original function is not one-to-one, then its domain needs to be restricted so that it is.
- The domains and ranges of the original functions are swapped, so dom $f^{-1}(x)$ = ran f(x), and ran $f^{-1}(x)$ = dom f(x).
- The graph of the inverse function is simply the graph of the old function reflected over the line y = x.
- Asymptotes and intercepts also switch axes. For example, an x-intercept of 2 in the original function becomes a y-intercept of 2 on the inverse function. An asymptote of y = -3 in the original function becomes an asymptote of x = -3 in the inverse function.
- All points switch their x and y values, so (2, 3) becomes (3, 2), etc.
- The points of intersection of the original and inverse functions occurs at f(x) = $f^{-1}(x)$. Another way of finding points of intersection is to solve f(x) = x or $f^{-1}(x)$ = x, but caution is advised when using this method because it does not always find all of the answers. When finding these points of intersection, it is better to use f(x) = $f^{-1}(x)$ on the CAS.



The diagram on the left illustrates how finding the point of intersection between a function and its inverse is nearly always the same as finding



the point of intersection with the line y = x. The diagram on the right illustrates how it does not always find all of the points of intersection.

Note that when solving for inverse equations, you must state that you're letting f(x) = y, and that you are swapping x and y to find inverse. At the end, you must state that y = $f^{-1}(x)$ as well. You will see how this is done in the next few examples.

Example
Find the inverse function of $h : [1, \infty) \to R,\ h(x) = x^2 - 2$, stating the domain and range.
(*Optional step*) An inverse function can be found if the function is one-to-one. We know it is.
ran $h^{-1}(x)$ = dom h(x) = $[1, \infty)$
dom $h^{-1}(x)$ = ran h(x) = $[-1, \infty)$

Let h(x) = y. To find inverse, swap x and y and solve for y. *(This must be written or you will be penalized)*
$x = y^2 - 2$
$x + 2 = y^2$
$\pm\sqrt{x + 2} = y$

We need to choose the correct branch, either y = $\sqrt{x+2}$ or $-\sqrt{x+2}$. We can tell this by looking at the range of $h^{-1}(x)$ which is $[1, \infty)$, so we are looking at the positive branch.
$\sqrt{x+2} = y$ as range = $[1, \infty)$.

Let y = $h^{-1}(x)$: *(Do not leave your equation as y =, restate it as h^{-1}, or you will be penalized)*
$h^{-1}: [-1, \infty) \to R, h^{-1}(x) = \sqrt{x+2}$

Example
Let f(x) = 1/x^2, define a suitable restriction for f(x) to make g(x), such that $g^{-1}(x)$ exists.
An inverse function is defined when the original function is one-to-one. f(x) is not one-to-one, so restrict the domain of f(x) to $(0, \infty)$. Therefore, g(x) = 1/x^2, x ∈ $(0, \infty)$. We use a new letter because it is a new function (it has a different domain to f(x)).

ran $g^{-1}(x)$ = dom g(x) = $(0, \infty)$
dom $g^{-1}(x)$ = ran g(x) = $(0, \infty)$

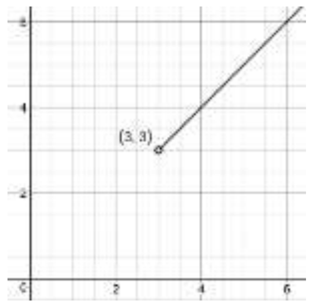
Let g(x) = y. To find inverse, swap x and y, and solve for y.
$x = \frac{1}{y^2}$
$y^2 = \frac{1}{x}$
$y = \pm\frac{1}{\sqrt{x}}$
We know the range of the function is $(0, \infty)$, so pick the positive branch.
$y = \frac{1}{\sqrt{x}}$
Let y = $g^{-1}(x)$. $g^{-1}(x) = \frac{1}{\sqrt{x}},\ x \in (0, \infty)$

Combining composite and inverse functions

Composite and inverse functions can be combined. The composite of a function and its inverse will always create the rule y = x. The one thing that distinguishes these functions from the function y = x is that the domain is different: it is equal to the domain of the inner function, so its graph is restricted.

Example

Let f(x) = $\frac{1}{x-2} + 3, x \in (2, \infty)$. Find f o f^{-1}, and its domain. Hence, sketch f o f^{-1}.



ran f^{-1} = dom f = $(2, \infty)$
dom f^{-1} = ran f = $(3, \infty)$

f o f^{-1} = x, dom(f o f^{-1}) = dom f^{-1} = $(3, \infty)$, remembering we use the domain of the inner function for a composition function. You would sketch the graph of y = x, between $(3, \infty)$.

Testing Understanding

1. Consider $f : R \to R,\ f(x) = x^2 - 4$ and $g : R^+ \cup \{0\} \to R,\ g(x) = \sqrt{x}$. State the ranges of f and g. Find f o g, stating its range. Explain why g o f doesn't exist.
2. The functions f and g are defined by: $f : R \to R,\ f(x) = x^2 - 2$ and $g : [0, \infty) \to R,\ g(x) = \sqrt{x}$. Explain why g(f(x)) doesn't exist. Find f(g(x)) and sketch its graph.
3. Consider $f : (-\infty, 3] \to R,\ f(x) = 3 - x$ and $g : R \to R,\ g(x) = x^2 - 1$. Show that f o g is not defined. Define a restriction g* of g such that f o g* is defined, and find f o g*.
4. Let $f : S \to R,\ f(x) = \sqrt{4 - x^2}$, where S is the set of all real values of x for which f(x) is defined and $g : R \to R,\ g(x) = x^2 + 1$. Find S. Find the range of f and the range of g. State whether or not f o g and g o f are defined, giving a reason.
5. Consider $g : [-1, \infty) \to R,\ g(x) = x^2 + 2x$. Find g^{-1}, stating the domain and range. Note: Completing the square is necessary to solve. Ask your teacher for help if you don't remember how to do it.
6. Let $f : [0,3] \to R,\ f(x) = 3 - 2x$. Find $f^{-1}(2)$ and the domain of f^{-1}.
7. Find the inverse function of the following: $f : R \setminus \{1\} \to R,\ f(x) = \frac{5}{x-1} - 1$
8. Let $g : [b, \infty) \to R,\ g(x) = x^2 + 4x$. If b is the smallest number such that g has an inverse function, find b and $g^{-1}(x)$. Hint: Inverse functions require one-to-one correspondence, so find the smallest b such that g(x) is still one-to-one. The method is similar to Q5 once 'b' is found.

Power Functions

Power functions are of the form f(x) = x^n, where n ∈ Q. Before we learn about them, there is some terminology that you need to know:

- A strictly increasing function is where as $x_2 > x_1$, then $f(x_2) > f(x_1)$, i.e. the further to the right you go, the graph will always go upwards
- A strictly decreasing function is where $x_2 > x_1$, then $f(x_1) > f(x_2)$, i.e. the further to the right you go, the graph will always go downwards.

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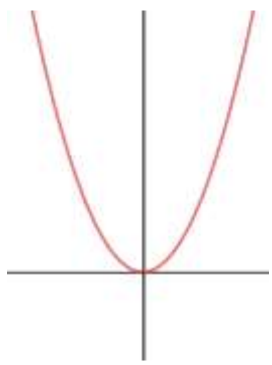
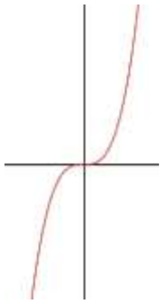
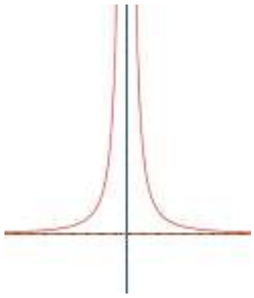
For f(x) = x^n, where n is an even positive integer, then graphs will have a similar behaviour to f(x) = x^2. It is strictly increasing when x > 0, strictly decreasing when x < 0, and when x approaches $\pm\infty$, f(x) approaches ∞. For example, the graph on the right could generally represent y = x^2, y = x^4, y = x^6, etc.



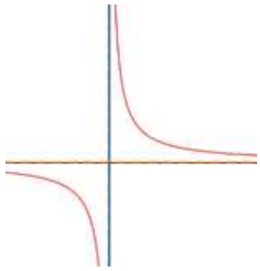
For f(x) = x^n, where n is an odd positive integer, then graphs will have a similar behaviour to f(x) = x^3. The graph is strictly increasing in all values, and the function is one-to-one. When x approaches $-\infty$, f(x) approaches $-\infty$, when x approaches ∞, f(x) approaches ∞. For example, the graph on the right could generally represent the behaviour of y = x, y = x^3, y = x^5, y = x^7, etc.



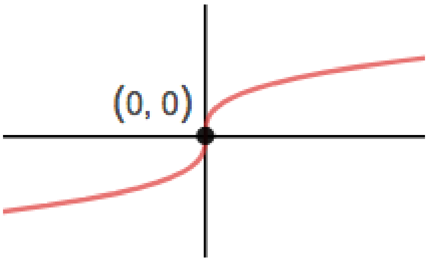
For f(x) = x^n, where n is an even negative integer, then graphs will have a similar behaviour to f(x) = x^{-2}. The maximal domain is R \ {0}, the range is R^+, and there are asymptotes at y = 0 and x = 0. This shape is called a truncus.

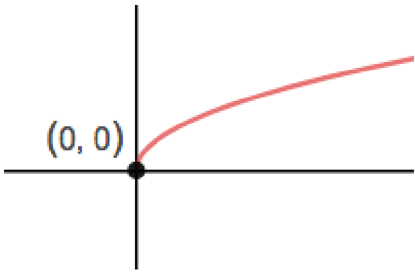


For f(x) = x^n, where n is an odd negative integer, then graphs will have a similar behaviour to f(x) = x^{-1}. The maximal domain is R \ {0}, the range is R \ {0}, and there are asymptotes at y = 0 and x = 0. This shape is called a hyperbola.



For f(x) = $x^{1/n}$, where n is a positive integer, the graph has a 'negative portion' only when n is odd. When n is even, it has a 'end point' at (0, 0), e.g. y = $x^{1/2}$.





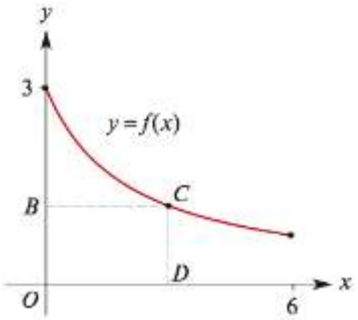
The graph on the left has the same behaviour as f(x) = $x^{1/n}$, where n is odd, such as $x^{1/3}$, $x^{1/5}$, $x^{1/7}$, etc. Note how it extends past the y-axis, with negative x-values, and intersects both axes at (0, 0).

The graph on the right have the same behaviour as f(x) = $x^{1/n}$, where n is even, such as $x^{1/2}$, $x^{1/4}$, etc. Note how it only has positive x-values, and its 'end-point' is at (0, 0).

Testing Understanding

9. Let $f : R \setminus \{0\} \to R,\ f(x) = \frac{1}{x^2}$ and $g : R \setminus \{0\} \to R,\ g(x) = \frac{1}{x^4}$. Find the values of x for which f(x) = g(x). Sketch the graphs of f(x) and g(x) using a CAS calculator on the same set of axes.

10. Let $f : R \to R,\ f(x) = \frac{1}{x^3}$ and $g : R \cup \{0\} \to R,\ g(x) = \frac{1}{x^4}$. Find the values of x for which f(x) = g(x). Sketch the graphs of f(x) and g(x) using a CAS calculator on the same set of axes.
11. Let $f : [0, 6] \to R,\ f(x) = \frac{6}{x+2}$. Rectangle OBCD is formed so that the coordinates of C are (a, f(a)).
    a) Find an expression for the area-of-rectangle function A.
    b) Rewrite this expression for A in the form $j + \frac{k}{x+2}$
    c) State the maximum value of A(x) for $x \in [0, 6]$, use a calculator if necessary



12. A man walks at the speed of 2km/h for 45 minutes and then runs 4km/h for 30 minutes. Let S be the distance the man has travelled (in km) after *t* minutes. The distance travelled can be described by:

$$S(t) = \begin{cases} at & 0 \le t \le c \\ bt + d & c < t \le e \end{cases}$$

    a) Find, a, b, c, d and e.
    b) Sketch the graph of S(t) against t
    c) State the range of the function.

Methods Unit 3 – Week 3 – Algebra Revision and Coordinate Geometry

The focus on this week's lesson will be revising your algebra skills. These include working with linear and quadratic equations, as well as determining distance, midpoints and gradients between two points. Remember, the key points of any graph include:

- **Basic shape of graph**
- **y-intercept (found by setting x = 0)**
- **x-intercept (found by setting y = 0)**
- **Turning/inflection points**

Coordinate Geometry

Whenever we want to find the length of the hypotenuse of a right angled triangle, we use Pythagoras' Theorem. We can use the same technique to find the distance between two points: the straight line connecting the two points is the hypotenuse of a right-angled triangle. As long as you have two different coordinates we can easily determine the shortest, straight line distance.

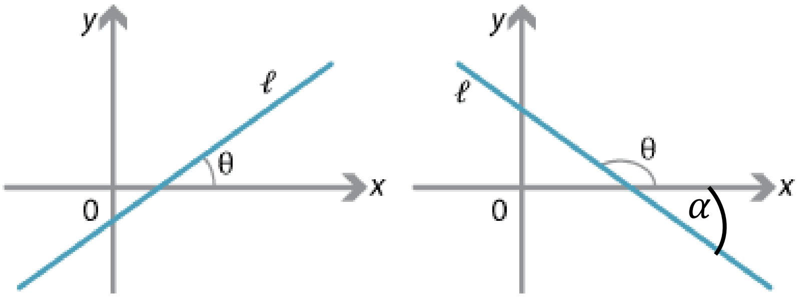
Let's say we have points (-1, 3) and (3, -2). Algebraically this can be rewritten as (x_1, y_1) and (x_2, y_2). Applying Pythagoras' Theorem, $distance = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. For the above coordinates:

$d = \sqrt{(3 - (-1))^2 + ((-2) - 3)^2}$
$d = \sqrt{16 + 25}$
$d = \sqrt{41}$ units

The midpoint between two points is calculated by adding the two coordinates and halving (taking their average). The general formula is $Midpoint = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$. For the above coordinates: $MP = \left(\frac{-1+3}{2}, \frac{3+(-2)}{2}\right) = \left(1, \frac{1}{2}\right)$

The gradient or slope between two points is calculated by finding $\frac{rise}{run}$ which is the same in finding the change in y-values over the change in x-values, so gradient/$m = \frac{y_2-y_1}{x_2-x_1}$. Note that for a vertical line, the gradient is undefined, and for a horizontal line, the gradient is 0. For the above coordinates: $m = \left(\frac{-2-3}{3-(-1)}\right) = -\frac{5}{4}$

One further way of calculating the gradient is in terms of its **angle** to the positive direction of the x-axis (the part of the x-axis that is to the right of the line). We call this θ. If the line had a **positive gradient**, then $m = \tan\theta$, where θ is acute. If the line had a **negative gradient**, then $m = \tan\alpha$ and $\theta = \alpha + 180^o$, where θ is obtuse. Let's say we had to find the angle made (θ) with the positive direction of the x-axis for both m = 3/2 and m = -5/4. These are illustrated below, the left shows m = 3/2, and the right shows m = -5/4. Note that for the negative gradient graph, we are finding α, the angle **below** the x-axis.



$m = \tan\theta \rightarrow \frac{3}{2} = \tan\theta \rightarrow \theta = \tan^{-1}\left(\frac{3}{2}\right) \rightarrow \theta = \tan^{-1}\left(\frac{3}{2}\right) = 56.31^o$
$m = \tan\alpha \rightarrow -\frac{5}{4} = \tan\alpha \rightarrow \alpha = \tan^{-1}\left(-\frac{5}{4}\right) \rightarrow \alpha = \tan^{-1}\left(-\frac{5}{4}\right) = -51.34^o$.
$\therefore \theta = \alpha + 180^o = 128.66^o$

Finding equations of straight lines

There are four common forms of expressing a linear relation. These are the gradient-intercept form, the point-gradient form, the intercept form and the general form.

- The most common form, and the form you should be most familiar with is the **gradient-intercept** form where m = gradient and c = y-intercept, y = mx + c. If the form isn't specified in the question, it is generally expected you convert to this form.
- The **point-gradient** form is used if the information that was given was one coordinate (x_1, y_1) and a gradient m, such that $y - y_1 = m(x - x_1)$.
- The next is **intercept** form, where given an x-intercept a, and a y-intercept b, then $\frac{x}{a} + \frac{y}{b} = 1$. This does not work if a and/or b are zero.
- The last is **general** form, which is found by rearranging the other forms. It is given as mx + ny + p = 0, where m, n and p can be any number, except both m and n can't be 0.

For horizontal lines, i.e. lines parallel to the x-axis, it has the equation of y = c, where c is the y-intercept.
For vertical lines, i.e. lines parallel to the y-axis, it has the equation y = a, where a is the x-intercept.

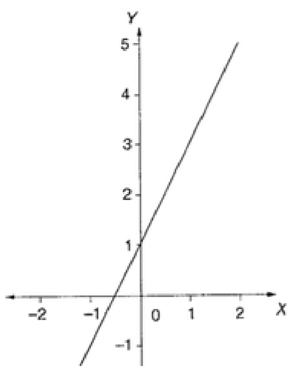
Graphing straight lines

You should also be familiar with graphing straight lines from previous study. Given any linear line, to graph it, find the x and y intercepts, plot them, and put a line through them. If the line goes through the origin, then you **must** plot another point other than the origin.

There is a simple trick in drawing horizontal and vertical lines. If for example, you were asked to sketch x = 3, then draw a x-intercept at 3, and draw a vertical line. If you were asked to sketch y = -2, draw a y-intercept at -2 and draw a horizontal line.

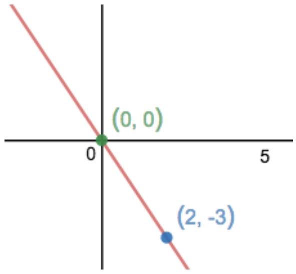
Example
Sketch the line y – 2x – 1 = 0.

To find the x-intercept, let y = 0: $-2x - 1 = 0 \rightarrow x = -\frac{1}{2}$
To find the y-intercept, let x = 0: $y - 1 = 0 \rightarrow y = 1$



Ensure you label all of the points.

Example
Sketch the line $y = -\frac{3x}{2}$.



As y-intercept is equal to zero, we know that this goes through the origin. As a result, we plot another point. If we substitute x = 2, we get y = -3, so we plot that point to complete the sketch.

Factorising and Solving Quadratics

When factorising and solving quadratics, it is important that we remember the following rules

- When factorising, the first step is **always** to find the highest common factor and factorise out
- Difference of perfect squares (DOPS): $(a+b)(a-b) = a^2 - b^2$
- $(x+a)^2 = x^2 + 2ax + a^2$ and similarly, $(x-a)^2 = x^2 - 2ax + a^2$
- When factorising y = x^2 + bx + c, find the two factors that multiply to make c and add to b. These are the ones that pair up with x in your factors.
- When factorising y = ax^2 + bx + c, find the two factors that multiply to make ac and add to b. Once you find those two factors, split b up and then factorise in pairs.
- Finding **factors** means finding the factorised form in the form (x + c)(x + d). Solving, or finding **roots, solutions** or **zeros** means setting those factors equal to zero and using the null factor law.

Example: Factorise and solve 2x² – 18 = 0
$2x^2 - 18 = 0$
$2(x^2 - 9) = 0$
$2(x+3)(x-3) = 0$ (*Stop here if it says factorise only*)
$x + 3 = 0$ or $x - 3 = 0$ → x = -3 or 3.

Example: Solve x² – x – 6 = 0
Factors of 'c' (-6) that add to 'b' (-1) are -3 and 2.
$(x-3)(x+2) = 0$
$x - 3 = 0$ or $x + 2 = 0$ → x = 3 or -2.

Example: Solve 12x² – 13x + 3 = 0
Factors of 'ac' (36) that add to 'b' (-13) are -4 and -9.
$12x^2 - 4x - 9x + 3 = 0$ (split b up into the two factors, the order of the factors doesn't matter)
$4x(3x-1) - 3(3x-1) = 0$ (factorise each pair separately, ensuring you get the same factor in both)
$(3x-1)(4x-3) = 0$ → x = 1/3 or 3/4.

Testing Understanding

1. Solve for the missing variable.
   a) $2x - 4 = x$ b) $3x - 6 = 8 - 4x$ c) $5(y-3) = 2(2y+4)$ d) $\frac{x}{3} + \frac{x}{5} = 10$
2. For MaxiCab taxis, to get in the taxi costs $8 and there is a charge of $1.20 per kilometre. Meanwhile, for SilverTop taxis, it costs $6 to call the taxi and a charge of $1.50 per kilometre.

Determine the distance travelled, in kilometres, so that travelling on SilverTop is more expensive than MaxiCab.

3. Find the midpoint, distance and gradient of the following pairs of points: (-2, 3) and (5, 6). Then find the line that goes through those points. Hence, graph this line, showing key features.
4. If the midpoint of AB is (4, 8), and we know that A is (6, -3), find B.
5. A line of gradient 5 passes through coordinates (2, -13) and (7, d). Find d.
6. Find the angle, in degrees, to two decimal points, that the lines joining the following pairs of points with the positive direction of the x-axis:
   a) (0,2) and (-2, 0) b) (0, -4) and (4, 0)
7. Find the gradient of a straight line inclined at an angle of 120 degrees from the positive direction of the x-axis.
8. Find the gradient of a straight line inclined at an angle of 30 degrees from the positive direction of the x-axis.
9. Find the equation of the straight line with a gradient of -1 and passes through (2, 7).
10. Find the equation of the straight line that passes through (2, -3) and has a y-intercept of -11.
11. Find the equation of the straight line that has an x-intercept of 1 and a y-intercept at 4.
12. Express your answers from questions 9-11 in general form. Also, graph these lines too.
13. Factorise and solve:
    a) $x^2 - 5x - 36 = 0$ b) $2x^2 + 12x + 18 = 0$ c) $6x^2 - 30 = 0$ d) $6x^2 - 15x + 9 = 0$

Literal equations are equations where variables represent known values. In other words, there are less numbers to work with. Solving these literal equations is exactly the same as solving any other equation.

14. Solve the following literal equations for x:
    a) $\frac{1}{x+a} = \frac{b}{x}$ b) $\frac{x}{m} + n = \frac{x}{n} + m$ c) $-b(ax + b) = a(bx - a)$ d) $\frac{x}{a-b} + \frac{2x}{a+b} = \frac{1}{a^2-b^2}$

Functional equations are used to describe properties of functions. This is covered in much greater detail in a later chapter, however we can start to do some work with them. These functional equations are used to prove **identities**, i.e. no matter which numbers you substitute for a variable, the left hand side is always equal to the right hand side. To prove them, you just need to substitute the given variables and show that they are equal.

15. Prove the following functional equations:
    a) For f(x) = 2x, prove f(x – y) = f(x) – f(y)
    b) For f(x) = x – 3, verify f(x – y) $\neq$ f(x) – f(y)
    c) For f(x) = 3/x, show that f(x) + f(y) = (x+y)f(xy) for all non-zero real numbers x and y.
    d) For f(x) = $1/x^2$, show that f(x) + f(y) = ($x^2 + y^2$) f(xy)

Methods Unit 3 – Week 4 – Simultaneous Equations and Matrices

Parallel and perpendicular lines

Note: This content on perp. lines is not necessarily in the study design, but is useful to know in verifying answers and can be used as an alternative method to solve questions. It is also useful for the next section.

When a line is parallel to another line, it means both lines have the same gradient. When a line is perpendicular to another line, it means the lines intersect each other at right angles (90°). The gradients of two perpendicular lines, when multiplied with each other, should equal -1. In other words, $m_1m_2 = -1$. In order to find a parallel or perpendicular line, we first need an equation to work from.

Example

A line that is parallel to the equation y = -x + 2 passes through the point (3, -2). Find the equation of this parallel line. In addition, find the equation of the perpendicular line that passes through the same point.

Because we are looking for a parallel line, the gradient of our equation must be the same as the original line, so we know m = -1. Then, substituting our point:

$y - y_1 = m(x - x_1)$
$y + 2 = -(x - 3)$
$y = -x + 1$ is the parallel line.

To find the perpendicular line, recall that $m_1m_2 = -1$. If our original line had a gradient of -1, then:
$(-1)m_2 = -1 \rightarrow m_2 = 1$

Substitute our point as before:
$y - y_1 = m(x - x_1)$
$y + 2 = 1(x - 3)$
$\mathrm{y = x - 5}$ is the perpendicular line.

1. Find the perpendicular bisector of (1, 5) and (-3, 7). Note: A perpendicular bisector is the perpendicular line that goes through the midpoint of a line segment.

Simultaneous linear equations

By now, you should be experienced in solving pairs of linear equations using both substitution and elimination methods to find a unique solution. Double check that you know how to do so by solving for x and y in the following equations using substitution and elimination respectively.

Example 1: -4x + 3y = -2 and y = x - 1

Example 2: -4x + 7y = 10 and x - 3y = -10

What all of these questions have in common with the ones you did in previous years was that there was only <u>one</u> pair of solutions. In reality, simultaneous linear equations can have three different cases. Remember, when we are solving these simultaneous linear equations, we are looking for the <u>points of intersection</u> between two different lines. As a result, there are three different cases:

- There is only one (unique) solution, where the two lines have different gradients (y-intercepts can be the same or different), i.e. the two lines intersect at one point
- There are infinitely many solutions, where the two lines have the same gradients and y-intercept, i.e. the two lines are exactly the same. To list the solutions in this case, we use a parameter, as it would be impossible to write down an infinitely long list of solutions.
- There is no solution, where the two lines have the same gradient, but the y-intercepts are different i.e. the two lines are parallel with each other

Example: Determine how many possible solutions there are for this pair of lines: 2x + 5y = 10 and 4x + 10y = 15

If these equations were rewritten in gradient-intercept form, then we get $y = -\frac{2}{5}x + 2$ and $y = -\frac{2}{5}x + \frac{3}{2}$. The two lines have the same gradient and different y-ints, so there are no solutions at all as they're parallel.

Example: The simultaneous equations 2x + 5y = 10 and 4x + 10y = 20 have an infinite number of solutions. Describe these infinite solutions through a parameter.
Use λ as the parameter. If y = λ, then:
$2x + 5\lambda = 10$
$2x = 10 - 5\lambda$
$x = \dfrac{10 - 5\lambda}{2}$
So the solutions are (x, y) or $\left(\frac{10-5\lambda}{2}, \lambda\right)$ where λ can be any real number.

Example: Find k such that the simultaneous equations (m-2)x + y = 2 and mx + 2y = k such that they have a) infinitely many solutions b) no solution c) a unique solution
Rewrite both of them in gradient-intercept form: $y = -(m-2)x + 2$ and $y = -\frac{mx}{2} + \frac{k}{2}$.

There are infinite solutions if both the gradients and the y-intercept is the same. Equating gradients: $-\frac{m}{2} = -(m-2) \rightarrow \frac{m}{2} = m - 2 \rightarrow m = 2m - 4 \rightarrow m = 4$. Equating y-intercepts: $2 = \frac{k}{2} \rightarrow k = 4$. So for infinite solutions, m = 4 and k = 4.

There are no solutions if the gradient is the same but the y-intercept is different. So for no solutions, m = 4 and k ≠ 4.

There is a unique solution if the gradient is different. Therefore, m ≠ 4 and k is any number.

Remember to state what both of the variables are, in the last case, it is easy to omit 'k', but you still need to state that k is any number.

Simultaneous equations with more than two unknowns

There are two possible ways to solve simultaneous equations with more than two unknowns. These are:

- Using technology: e.g. on CAS: Menu > Algebra > Solve System of Equations
- By hand: e.g. if there were three missing variables and three equations, eliminate one variable using one pair, eliminate the same variable using another pair, and use those two new equations to solve for two variables. Substitute back to find the last variable.
- In general, to find a unique solution, we require there to be as many equations as there are unknowns

Example: Solve the following trio of equations: x – y + z = 6, 2x + z = 4 and 3x + 2y – z = 6.
Equation 2 has only two variables, so we can use that to eliminate variables.
When we perform [1] – [2] we obtain: -x - y = 2. Call this equation [4].
When we perform [3] + [2] we obtain: 5x + 2y = 10. Call this equation [5].
Now we can solve [4] and [5] simultaneously for x and y. This leads us to x = 14/3 and y = -20/3.
Substituting this fact into any one of the 3 equations above yields z = -16/3.

2. Solve the following simultaneous equations for x and y
   a) y = x + 3 and 4x + y = -7
   b) 2x – 6y = -4 and -3x – 9y = 0
   c) ax + y = c and x + by = d

d) $ax + by = p$ and $bx - ay = q$
e) mn = 14 and m^5n = 224
f) pq = -6 and p^3q = -54
g) a(5 – b) = 8 and a(2 + b) = 20
h) j(k + 3) = -45 and j(k – 4) = -10

3. Explain how many possible solutions there are for 2x + 3y = 6 and 4x + 6y = 18
4. Firstly, prove that 3x + 4y = 8 and 9x + 12y = 24 have infinitely many solutions. Then, describe these solutions using a parameter.
5. Firstly, prove that x + y = 3 and 3x + 3y = 9 have infinitely many solutions. Then, describe these solutions using a parameter.
6. Find the values of m such that the simultaneous equations 3x + my = 5 and (m + 2) x + 5y = m have:
   a) infinitely many solutions
   b) no solutions
   c) a unique solution
7. Find the values of k such that the simultaneous equations kx + 3y = 1 and 4x + 3ky = 0 have:
   a) infinitely many solutions
   b) no solutions
   c) a unique solution
8. Find the values of m such that the simultaneous equations -2x + my = 1 and (m + 3) x – 2y = –2m have:
   a) infinitely many solutions
   b) no solutions
   c) a unique solution
9. Find the values of b and c for which the equations x + 5y = 4 and 2x + by = c have:
   a) a unique solution
   b) infinitely many solutions
   c) no solution
10. Solve this system of equations manually: 2x + 3y – z = 12, 2y + z = 7, 2y – z = 5
11. Solve this system of equations manually: x – y – z = 0, 5x + 20z = 50, 10y – 20z = 30
12. Consider x + 2y – 3z = 4 and x + y + z = 6. Subtract the second equation from the first to find y in terms of z. If z = λ, solve the equations to give the solution in terms of λ.
13. The system of equations: x + y + z + w = 4, x + 3y + 3z = 2, x + y + 2z – w = 6 has infinitely many solutions. Use the CAS to solve, and give the unique solution when w = 6.
14. Find all solutions for the following systems of equations: 3x – y + z = 4, x + 2y – z = 2 and –x + y –z = -2, using the CAS.

Introduction to matrices

Matrices are arrays of numbers, arranged into **rows and columns**. The size, or **dimension** of a matrix is described by how many rows and columns it has (in that order). The notation of matrices is also based on rows first and columns second. If a matrix is fully comprised of zeros, it is called a **zero matrix**.

A 2 x 2 matrix can be described as: $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$
Note: Mathematical Methods VCE only requires you to consider 2 x 2 and 2 x 1 matrices only.

If two matrices have the same dimension, then you can add or subtract the two matrices. This is done by adding or subtracting the corresponding entries from both matrices.

Example

$A = \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -3 \\ 6 & 2 \end{bmatrix}$, find A + B.
$A + B = \begin{bmatrix} -3 & 0 \\ 9 & -2 \end{bmatrix}$

You can also multiply a matrix by a real number. This is also called **multiplication by a scalar**.
$A = \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix}$, find 3A.
$3A = \begin{bmatrix} -6 & 9 \\ 9 & -12 \end{bmatrix}$

Multiplying matrices is the most difficult operation that you need to remember how to do. There are a few rules you need to remember:

- The order of the matrix multiplication is important, i.e. A x B is not the same as B x A.
- If the first matrix has dimensions a x b, and another matrix has dimensions c x d, then multiplication only works if b and c are the same number. The result will then have dimensions a x d.
- For VCE Methods this means you can only multiply a 2 x 2 and a 2 x 1 to make a 2 x 1 matrix, or multiply a 2 x 2 and a 2 x 2 matrix to make a 2 x 2 matrix.
- To find the entry in row i and column j of **AB**, find row i in the first matrix, and column j in the second matrix. Multiply the corresponding entries from the row and column, and add the resulting products.

Example
$A = \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, find **AB**.
$AB = \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \times 2 + 3 \times -1 \\ 3 \times 2 + -4 \times -1 \end{bmatrix} = \begin{bmatrix} -7 \\ 10 \end{bmatrix}$

Example
$A = \begin{bmatrix} 2 & -2 \\ 5 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 4 \\ 7 & -6 \end{bmatrix}$, find **AB and BA**.

$$AB = \begin{bmatrix} 2 & -2 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} -1 & 4 \\ 7 & -6 \end{bmatrix} = \begin{bmatrix} 2 \times -1 + -2 \times 7 & 2 \times 4 + -2 \times -6 \\ 5 \times -1 + 3 \times 7 & 5 \times 4 + 3 \times -6 \end{bmatrix} = \begin{bmatrix} -16 & 20 \\ 16 & 2 \end{bmatrix}$$

$$BA = \begin{bmatrix} -1 & 4 \\ 7 & -6 \end{bmatrix}\begin{bmatrix} 2 & -2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 18 & 14 \\ -16 & -32 \end{bmatrix}$$

15. If A = $\begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix}$ and B = $\begin{bmatrix} 1 & -4 \\ -3 & 6 \end{bmatrix}$, find the matrix **X** such that 2**A** + **X** = B.
16. Let X = $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$, A = $\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$ and B = $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$. Find:
    a) AX
    b) BX
    c) AB
    d) BA
    e) A^2
    f) B^2

Methods Unit 3 – Week 5 – Transformations
Introduction

Transformations are the rules that are used to move each ordered pair of a relation to a new ordered pair. In other words, as a result of a transformation, such as translation, reflection or dilation, each point on a plane is mapped to a second unique point, also called the **image**. There are three kinds of transformations you will learn in VCE Maths Methods: Translations, Dilations and Reflections.

Say we wanted to move (**translate**) a point one unit to the right, and two units up. This means we are mapping (x, y) to (x + 1, y + 2). The point (-2, 3) would therefore map to (-1, 5). Every individual point (x, y) is mapped to the new point/the image (x', y'). In other words, for this translation, x' = x + 1, y' = y + 2.

Translations

A translation moves every point the same distance in the same direction, i.e. the function is shifted vertically or horizontally.

Translation	Notation (from (x, y)
a units to the right (positive direction parallel to x-axis)	(x + a, y)
a units to the left (negative direction parallel to x-axis)	(x – a, y)
a units up (positive direction parallel to y-axis)	(x, y + a)
a units down (positive direction parallel to y-axis)	(x, y – a)

Example

Find the function which is translated 2 units in the negative direction of the x-axis and 3 units in the positive direction of the y-axis, with the original function being $y = x^2$.

x' = x – 2 and y' = y + 3 *(The new x-values are 2 units to the left of the old ones, and y-values are 3 units up)*
Hence, x = x' + 2 and y = y' – 3 *(Rearrange for x and y in terms of x' and y')*
From before, $y = x^2$, substitute in the two equations above:
$y' - 3 = (x' + 2)^2$
$y' = (x' + 2)^2 + 3$ *(you are free to remove the dashes).*
$y = (x + 2)^2 + 3$ *(Determining translations in this manner works for all power functions)*

Dilations

A dilation has the effect of the graph 'stretching away' or 'shrinking towards' the x or y axes. Note for this section, we are only interested in the number, not whether or not it is negative.

Dilation	Notation (from (x, y)
Dilation of factor *a* from the x-axis (a > 1) – stretching	(x, ay)
Dilation of factor *a* from the x-axis (0 < a < 1) – shrinking	(x, ay)
Dilation of factor *a* from the y-axis (a > 1) – stretching	(ax, y)
Dilation of factor *a* from the y-axis (0 < a < 1) – shrinking	(ax, y)

When we dilate from the x-axis, what we are really doing is affecting the y-values, which is why we multiply a with y, not with x. Vice versa when dilating from the y-axis.

Example

Find the function that is dilated by a factor of 2 from the x-axis, with the original function being $y = x^{1/2}$.
x' = x and y' = 2y
Hence, x = x' and y = y'/2.
From before, $y = x^{1/2}$, substitute in the two equations above:
$y'/2 = x'^{1/2}$
$y = 2x^{1/2} = 2\sqrt{x}$ *(Just like translations, this method works for every power function)*

Note: that in certain types of graphs, dilations can be expressed from either the y-axis or x-axis.

Example

Determine the dilation factor when the graph y = $\sqrt{x}$ is transformed into y = $\sqrt{3x}$ by dilation from:
a) The y-axis, b) The x-axis.

The dilation has already occurred, so the given function is already an image, i.e. y' = $\sqrt{3x'}$. As we are looking for the dilation from the y-axis, this means the x-values have changed. By comparing y' = $\sqrt{3x'}$ and y = $\sqrt{x}$, we obtain x = 3x' and y = y'. Rearrange to get x' = x/3 and y' = y. This means that the graph of y = $\sqrt{3x}$ is simply the graph of y = $\sqrt{x}$ dilated by a factor of 1/3 from the y-axis.

As we are looking for the dilation from the x-axis, this means the y-values have changed. We need to rearrange slightly as currently; x is the one that has changed (see above). y' = $\sqrt{3x'}$ is rearranged to $\frac{y'}{\sqrt{3}} = x'$. By comparing $\frac{y'}{\sqrt{3}} = x'$ y = $\sqrt{x}$, we obtain x = x' and y = $\frac{y'}{\sqrt{3}}$. Rearrange to get x' = x and y' = $\sqrt{3}$y. This means that the graph y = $\sqrt{3x}$ is also the graph of of y = $\sqrt{x}$ dilated by a factor of $\sqrt{3}$ from the x-axis.

In other words, for the graph y = $\sqrt{3x}$, can be interpreted as a dilating by a factor of 1/3 from the y-axis, OR by a factor of $\sqrt{3}$ from the x-axis for the original function y = $\sqrt{x}$.

Reflections

Reflections are simply when functions are 'flipped' across a mirror line (either the x-axis or the y-axis). Note that inverse functions are technically reflections, 'flipped' across the line y = x.

Reflection	Notation (from (x, y)
Reflection about the x-axis (x-axis is the mirror line)	(x, -y)
Reflection about the y-axis (y-axis is the mirror line)	(-x, y)
Reflection across the line y = x (finding the inverse)	(y, x)

When we reflect in the x-axis, what we are really doing is affecting the y-values (as the function is 'upside-down', which is why we use -y, not -x. Vice versa when reflecting in the y-axis. By now you should be comfortable with using transformation notation, so it is left as an exercise to the reader to test the above table for accuracy.

Combining Transformations

It is important to note that transformations do not happen in isolation: they are often combined in combination. It is important to note that the order of transformations is extremely important: performing a dilation then a translation is not the same as performing a translation and then a dilation.

Example

Find the equation of the image of y = $\sqrt{x}$, when it is first translated 6 units in the negative direction of the x-axis, reflected in the y-axis and then dilated by a factor of 2 from the x-axis.

Combining these transformations: (x, y) → (x – 6, y) → (6 – x, y) → (6 – x, 2y) = (x', y').
We get x' = 6 – x and y' = 2y, rearranged to get x = 6 – x' and y = y'/2. Replacing these in our original function we get $\frac{y'}{2} = \sqrt{6 - x'}$, so the image function is $y = 2\sqrt{6 - x}$.

Determining Transformations

To determine the transformations applied to a function, all you need to do is reverse the steps in the previous example. Like most things this week you can determine transformations by inspection, once you are used to manipulating transformations. A simple way to remember the order of most transformations is **DRT**, dilation and/or reflection comes first, then translation.

Example

Find the sequence of transformations needed to get the graph y = $\sqrt{x}$ to $y = 2\sqrt{6-x}$.

Rearrange to get $\frac{y'}{2} = \sqrt{6-x'}$ *(The image, the second equation, needs to be rewritten, and dashes readded)*

Comparing with y = $\sqrt{x}$, $\frac{y'}{2} = y$ and $6 - x' = x$

Rearranging, 2y = y' and – x + 6 = x'.

So: (x, y) → (x, 2y) → (-x, 2y) → (-x + 6, 2y) *(We did the transformations in DRT order)*

Hence, the sequence is: Dilate by a factor of 2 from the x-axis, reflect in the y-axis, and translate 6 units to the positive direction of the x-axis. *(Note, this leads to a slightly different order of transformations as well as a different translation, but due to the nature of transformations, multiple answers are possible)*
Sometimes, we have to find the sequence of transformation of a graph 'in the opposite direction,' i.e. from something that looks complicated to something simple.

Example

Find the sequence of transformations from $y = \frac{3}{(x-1)^2} + 6$ to $y = \frac{1}{x^2}$.

The end result is $y = \frac{1}{x^2}$, so $y' = \frac{1}{x'^2}$.

We need to manipulate the original function so that it resembles the end result.

By rearranging, we get $\frac{y-6}{3} = \frac{1}{(x-1)^2}$. *(This allows us to compare the two)*

Hence, $y' = \frac{y-6}{3}$ and $x' = x - 1$.

We can write this sequence as (x, y) → (x – 1, y – 6) → (x – 1, $\frac{y-6}{3}$).

This gives us a sequence of: translate one unit in the negative direction of the x-axis and six units in the negative direction of the y-axis. Then, dilate by a factor of 1/3 from the x-axis.

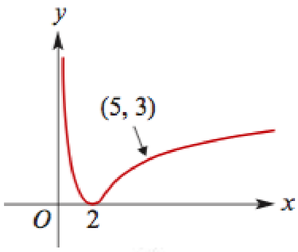
Testing Understanding

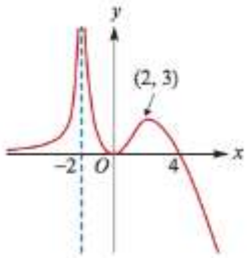
1. Find the equation for the image of the curve y = f(x), where f(x) = $\frac{1}{x}$, under:
   a) Translation 2 units in the positive direction of the x-axis and 3 units in the negative direction of the y-axis.
   b) Translation ½ unit in the positive direction of the x-axis and 4 units in the positive direction of the y-axis.
2. State a sequence of transformation which maps the first function onto the second function
   a) $f_1(x) = x^2$, $f_2(x) = (x + 5)^2$
   b) $f_1(x) = (1/x^2) – 3$, $f_2(x) = 1/x^2$
   c) $f_1(x) = \sqrt{x}$, $f_2(x) = \sqrt{x+4} + 2$
3. Find the equation for the image of the graph of each of the following under the following translations:
   a) $y = 1/(x-2)^2 + 3$, translation: (x, y) → (x + 4, y – 2)
   b) $y = 2(x+3)^2 + 3$, translation: (x, y) → (x + 3, y – 3)
4. Determine the rule of the image when the graph $y = x^3$ is dilated by a factor of 2
   a) from the x-axis
   b) from the y-axis

5. Determine the factor of dilation when the graph of y = $\sqrt{5x}$, is obtained by dilating y = $\sqrt{x}$:

a) from the x-axis
b) from the y-axis

6. State a transformation which maps the first function onto the second
   a) $f(x) = 1/x^2$, $f_2(x) = 5/x^2$
   b) $f(x) = \sqrt{\frac{x}{3}}$, $f_2(x) = \sqrt{x}$
   c) $f(x) = 1/4x^2$, $f_2(x) = 1/x^2$
7. Find the equation of the image when the graph $y = (x - 1)^2$ is reflected
   a) in the x-axis
   b) in the y-axis
8. Sketch the following image of the graphs shown undergoing the following sequence of transformations
   a) Reflection in the x-axis, dilation of a factor 2 from the x-axis, translation 3 units in the positive direction of the x-axis and translation 4 units in the positive direction of the y-axis
   b) Reflection in the y-axis, a translation 2 units in the negative direction of the x-axis, a translation 3 units in the negative direction of the y-axis, and a dilation of factor 2 from the y-axis.



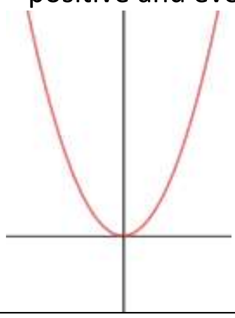


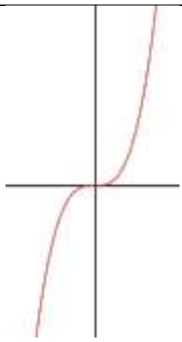
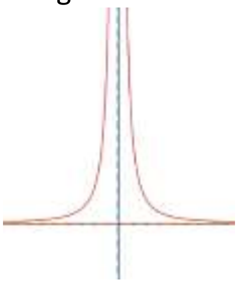
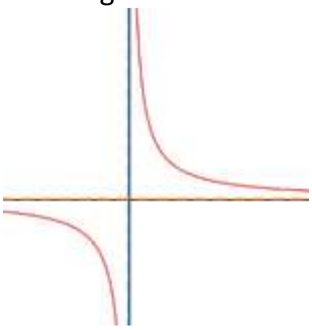
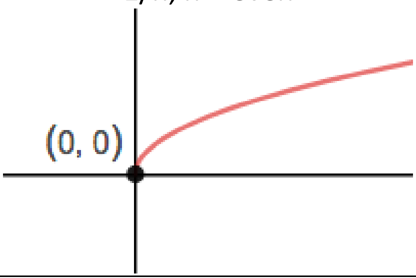
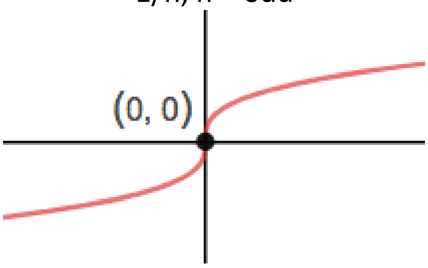
9. State the rule for the image of the graph $y = 3/x^2$, when it is transformed by having a dilation of factor 2 from the x-axis, a translation of 2 units to the left, and one unit down, and then reflected in the x-axis.
10. State the rule for the image of the graph $y = x^{1/3}$, when it is transformed by having a reflection in the y-axis, translated 1 unit to the right and 2 units down, then dilated by a factor of ½ from the y-axis.
11. Find a sequence of transformations that takes the graph $y = x^3$ to $y = -(x+1)^3 + 2$.
12. Find a sequence of transformations that takes the graph $y = \frac{2}{(x-1)^2} - 5$ to the graph $y = 1/x^2$
13. Find a sequence of transformations that takes the graph $y = 2\sqrt{4 - x} + 3$ to $y = -\sqrt{x} + 6$. (Hard: Ask your teacher for the opening steps before solving)

Methods Unit 3 – Week 6 – Graphing and Further Transformations

Review of Graphs

For this week's lesson, we are going to use transformations to help us draw graphs in a certain form. This form is $y = a\left(b(x-h)\right)^n + k$, where n is a rational number, Q (except n = 1, which are linear graphs that was covered in Week 3). To sketch these graphs, we need to remember what we learnt in previous weeks, especially the topic on Power Functions in week 2. Here is a short summary of that section:

Power and graph	Key properties
n = positive and even 	Quadratic shape Turning point at (h, k) As x approaches positive and negative infinity, y approaches infinity Up to two x-intercepts
n = positive and odd (n > 1)	Cubic shape **Point of inflection** at (h, k)

	As x approaches negative infinity, y approaches negative infinity, and as x approaches infinity, y approaches infinity. Up to three x-intercepts
n = negative and even 	Truncus shape Asymptotes at x = h and y = k As x approaches positive and negative infinity, y approaches 0 As x approaches zero from both sides, y approaches infinity.
n = negative and odd 	Hyperbola shape Asymptotes at x = h and y = k As x approaches negative infinity, y approaches 0. As x approaches positive infinity, y approaches 0. As x approaches 0 from the negative direction, y approaches negative infinity As x approaches 0 from the positive direction, y approaches positive infinity.
1/n, n = even 	Square root graph shape End point at (h, k) No negative x section
1/n, n = odd 	Cube root graph shape Special point at (h, k) Does have a negative x section that extends past the y-axis

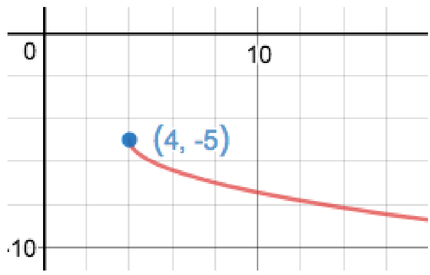
Graphing Transformed Functions

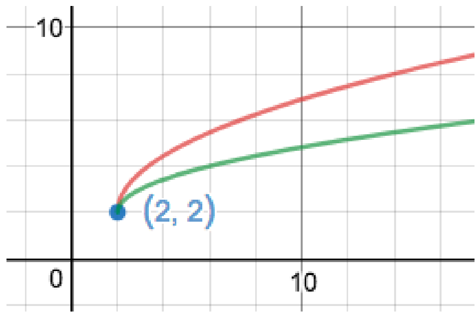
Now that we have learnt the various different types of transformations that can be performed, we can now move on towards actually sketching those types of graphs. We first start by relying on the normal power function, then successively apply transformations until we get the graph we want.

These 'normal' power functions follow the form f(x) = x^n, where n is ½, 1/3, -1, -2, 2, 3, etc.
Transformations can be rewritten in the form $y = af\left(b(x-h)\right) + k$, yielding $y = a\left(b(x-h)\right)^n + k$.

- h represents the horizontal translation. You will move h units to the left or right. For example, if there was a movement 3 to the right, then h = 3, so the bracket would be f(x – 3). If there was a movement 2 to the left, then h = -2, so this yields f(x + 2).
- k represents the vertical translation. If k is positive, then you are moving upwards, if negative, then downwards.
- b represents the dilation to the factor 1/b from the y-axis. So if b = 3, then it is a dilation of 1/3 from the y-axis. This dilation is reciprocated.
- a represents the dilation to the factor a from the x-axis.
- If a is negative, reflect in the x-axis. If b is negative, reflect in the y-axis.
- **Note:** Whenever there is a coefficient of x, you <u>must</u> start by isolating the x first, so that instead of (bx - g) it becomes b(x - h). This is known as the lonely x rule. If the coefficient is negative, then you also need to do this step to get rid of the negative. x should have a coefficient of +1 at all times.

Example
Sketch y = $-\sqrt{x-4} - 5$. The base function is obviously f(x) = $\sqrt{x}$. In terms of y, y = $-f(x-4) - 5$, which indicates to sketch the $\sqrt{x}$ graph with a translation of four units to the right, five units downwards, and to then reflect about the x-axis. Looking at the review of graphs above, that means the end point is translated to (4, -5), and the graph is now upside down.





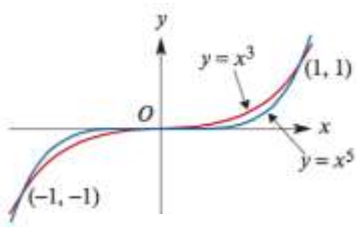
Example
Sketch y = $\sqrt{3x-6} + 2$. Start by isolating the x (lonely x rule): y = $\sqrt{3(x-2)} + 2$. The base function is f(x) = $\sqrt{x}$. Therefore, we now know that y = $f(3(x-2)) + 2$, so to sketch the new function, translate the $\sqrt{x}$ graph two units to the right and up, and dilate by 1/3 from the y-axis. The end point is translated to (2, 2), and the graph hugs closer to the y-axis than usual.

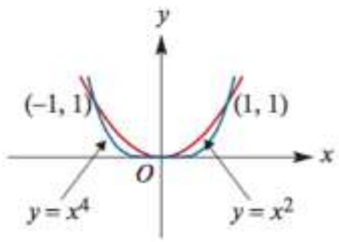
On the diagram on the right, the lower graph has no y-axis dilation, whereas the upper graph has the dilate 1/3 from the y-axis transformation. As you can see, it has shrunk closer to the y-axis.

Note: To assist with graphing, it helps to determine the x-intercepts (set y = 0) and y-intercepts (set x = 0). You will need to know the translations to determine any possible changes to asymptotes as well.

Comparing Power Functions

Briefly, flip back to Question 9 and 10 in Week 2. When we are sketching two of the same type of graph (i.e. when y = x^n, n ∈ Z) on the one plane, they intersect at (1, 1) and either (-1, 1) if n is even or (-1, -1) if n is odd. Notice how they have different behaviours based on whether or not they are less than 1 or greater than 1.





The higher the integer n, the shallower it is when it is between (-1, 1), and the steeper it is when it is less than -1 or greater than 1. This intuitively makes sense: compare y = x^2 and y = x^4. When x = ½ (less than 1), clearly y = x^2 is greater. When x is a number greater than 1, say, x = 2, y = x^4 is greater. This explains the behaviour of power functions of *different n* on the same plane. Generally, you don't need to worry about this distinction if you're only drawing one graph on the plane.

Determining Rules

Determining rules for functions require us to use the information provided and using that information to determine the rule of a function. Some general guidelines are:

- Determine the type of graph you are dealing with. For example, for a square root graph, you would start with $y = a\sqrt{x-h} + k$, or when working with a quadratic with two known x-intercepts you would start with $y = a(x-e)(x-f)$ or $y = a(x-h)^2 + k$ if you knew its turning point.
- If you are given one point, substitute it in and solve for the missing variable
- If you are given multiple points, such as intercepts or other points, substitute them in and solve simultaneously, if required.
- If you are given a relevant point such as a turning point, point of inflection, endpoint or asymptotes, substitute these in for (h, k).

The best way to do these questions is to practice them.

Matrix Transformations

We can also represent transformations of functions and points using matrix notation. Mappings using dilation and reflection can be defined by x' = ax + by and y' = cx + dy. In matrix form, $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$. In other words, (x, y) → (ax + by, cx + dy). These mappings will be listed below (put this in your reference book). These mappings are also called linear transformations (dilations and reflections).

Mapping	Rule	Matrix Representation
Reflection in the x-axis	$x' = x \text{ and } y' = -y$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Reflection in the y-axis	$x' = -x \text{ and } y' = y$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Reflection in y = x (inverse)	$x' = y \text{ and } y' = x$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
Dilation of *a* from y-axis	$x' = ax \text{ and } y' = y$	$\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$
Dilation of *a* from x-axis	$x' = x \text{ and } y' = ay$	$\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}$

Translations are simpler. When x' = x + a, and y' = y + b, then $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \end{bmatrix}$.

Note: The order of transformations is vital. When combining transformations of matrices (especially dilation and reflection), let the one that occurs first to be right most, the one that occurs at the end the left most.

At the end of the transformations, once you have solved for x and y in terms of x' and y', substitute into the old function just like in last week to find the new image function. You can use last weeks method or this weeks method, but questions may specify that one method should be used (i.e. they supply a matrix), or you may need to identify matrices in a multiple choice question.

Example

Find the image of a point (4, 3) when it is first reflected in the y-axis, dilated by a factor of 1/3 from the x-axis, and then reflected in the x-axis. It is then translated 2 units up and 1 unit left.

$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$ *(Write down the sequence of transformations from L to R)*

$\begin{bmatrix} -1 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$ *(Multiply the matrices step by step, as learnt in Week 4)*

$\begin{bmatrix}-1 & 0\\ 0 & -1/3\end{bmatrix}\begin{bmatrix}4\\3\end{bmatrix}+\begin{bmatrix}2\\-1\end{bmatrix}=\begin{bmatrix}x'\\y'\end{bmatrix}$

$\begin{bmatrix}-4\\-1\end{bmatrix}+\begin{bmatrix}2\\-1\end{bmatrix}=\begin{bmatrix}x'\\y'\end{bmatrix}$

$\begin{bmatrix}-2\\-2\end{bmatrix}=\begin{bmatrix}x'\\y'\end{bmatrix}$. Therefore, the new point is (-2, -2). Double checking with the method from last week, (4, 3) → (-4, 3) → (-4, 1) → (-4, -1) → (-2, -2), as shown.

Example

A linear transformation occurs such that (1, 0) → (3, -1) and (0, 1) → (-2, 4). Find the image of (-3, 5).

$\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}3\\-1\end{bmatrix}$, so a = 3, c = -1. $\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}-2\\4\end{bmatrix}$, so b = -2 and d = 4. Therefore: $\begin{bmatrix}3 & -2\\ -1 & 4\end{bmatrix}\begin{bmatrix}-3\\5\end{bmatrix}=\begin{bmatrix}-19\\23\end{bmatrix}$, so the new image of (-3, 5) is (-19, 23).

Example

A transformation is described by the matrix equation **A**(**X + B**) = **X'**, such that $A=\begin{bmatrix}0 & -3\\ 2 & 0\end{bmatrix}$ and $B=\begin{bmatrix}1\\2\end{bmatrix}$. Find the image of the straight line with the equation y = 2x + 5 under this transformation.

Note: X is usually taken to mean $\begin{bmatrix}x\\y\end{bmatrix}$, the original function, and X', $\begin{bmatrix}x'\\y'\end{bmatrix}$, the image.

$\begin{bmatrix}0 & -3\\ 2 & 0\end{bmatrix}\left(\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}1\\2\end{bmatrix}\right)=\begin{bmatrix}x'\\y'\end{bmatrix}$

$\begin{bmatrix}0 & -3\\ 2 & 0\end{bmatrix}\begin{bmatrix}x+1\\y+2\end{bmatrix}=\begin{bmatrix}x'\\y'\end{bmatrix}$

$\begin{bmatrix}-3y-6\\2x+1\end{bmatrix}=\begin{bmatrix}x'\\y'\end{bmatrix}$, so $-3y-6=x'\rightarrow y=\frac{-x'}{3}-2$ and $2x+1=y'\rightarrow x=\frac{y'-1}{2}$. Substituting:

$$\frac{-x'}{3}-2=2\left(\frac{x'+6}{-3}\right)+5\rightarrow y'=-\frac{x'}{3}-5$$

Example

A transformation $T:\mathbb{R}^2\rightarrow\mathbb{R}^2$ is defined by $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}5 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}5\\2\end{bmatrix}$. Find the image of $y=\frac{1}{x}$ under this transformation.

$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}5x\\y\end{bmatrix}+\begin{bmatrix}5\\2\end{bmatrix}=\begin{bmatrix}5x+5\\y+2\end{bmatrix}=\begin{bmatrix}x'\\y'\end{bmatrix}$.

So: $5x+5=x'\rightarrow x=\frac{x'-5}{5}$ and $y+2=y'\rightarrow y=y'-2$.

Substituting:

$$y'-2=\frac{5}{x'-5}\rightarrow y'=\frac{5}{x'-5}+2$$

Testing Understanding

1. Sketch the following graphs, showing all relevant features.
   a) $y=\frac{2}{x-1}+3$
   b) $y=\frac{-1}{(x+2)^2}+1$
   c) $y=-\sqrt{x-3}+2$
   d) $y=\sqrt{2-x}-4$
   e) $y=\frac{5}{2x}+5$
   f) $y=\frac{-2}{(x+2)^2}-4$
   g) $y=\frac{3x+2}{x+1}$
   h) $y = 2(3 – x)^2 + 1$

i) y = $-2(3 - x)^3 + 2$

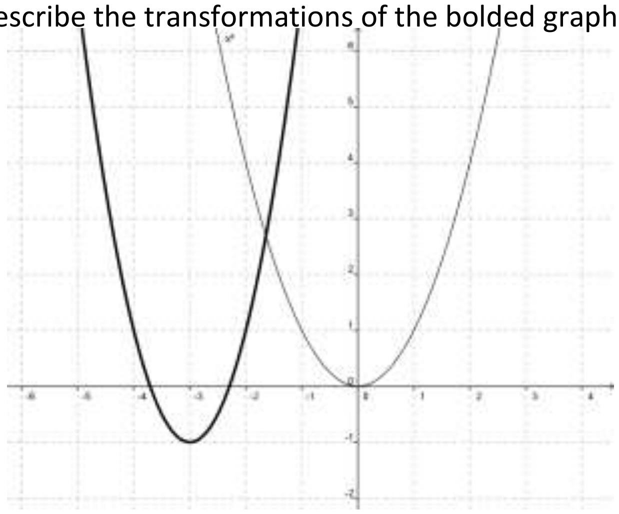
j) y = $\frac{4x-5}{2x+1}$

k) $f(x) = (x+1)^5 - 32$

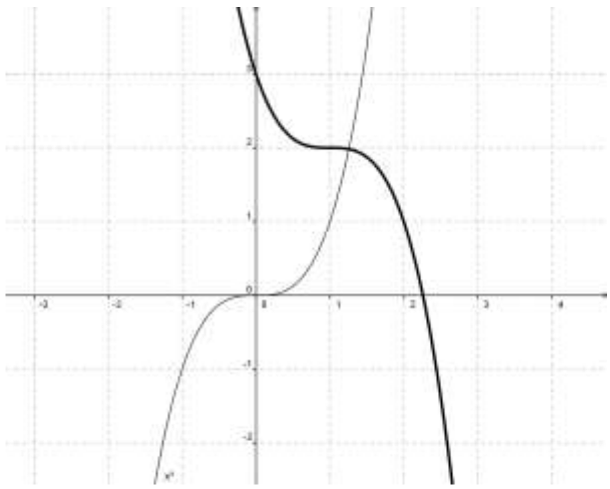
l) $g(x) = 1 - (x+4)^4$

2. A cubic has a point of inflection of (0, 4) and passes through (1, 1). Find the rule.
3. A quartic of form $y = a(x-h)^4 + k$ has turning point (-2, 3) and passes through (0, -6). Find the rule.
4. Find a sequence of transformations from y = x^4 to y = $5 - 3(x + 1)^4$
5. Find the rule of a hyperbola with asymptotes x = 1 and y = 2, with a y-intercept of (0, 1)
6. The points (1, 5) and (16, 11) lie on a curve with a rule in the form of $y = A\sqrt{x} + B$. Find the rule.
7. Find the rule of a truncus with asymptotes x = -2 and y = -3, with y-intercept of (0, -1)
8. The points (1, -1) and (2, ¾) lie on a curve with a rule of form of $y = \frac{a}{x^3} + b$. Find the rule.
9. A linear transformation occurs such that (1, 0) → (2, -1) and (0, 1) → (4, -2). Find the image of (-3, 5).
10. Find the matrix that is equivalent to reflection in the x-axis and dilation of a factor 2 from the x-axis (in that order), and hence find the image of the point (-3, 2) under this transformation.
11. Describe the transformations of the bolded graph from its original unbolded posistion

a)



b)



12. Write the equation of the bolded graph in the form $a(x - h)^2 + k$ based on your transformation responses to question 11.
13. A transformation is defined by the matrix $\begin{bmatrix} 0 & 3 \\ -2 & 0 \end{bmatrix}$. Find the equation of the image with equation y = 2x + 3 under the transformation.
14. A transformation is described by **AX + B = X'**, where A = $\begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix}$ and B = $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Hence find the image of the curve with equation y = $-2x^3 + 6x^2$.
15. A transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. Find the image of $y = 3(x-2)^2 - 4$ under this transformation.

Methods Unit 3 – Week 7 – Polynomials

Quadratics

Quadratics have a degree of 2, that is, the highest power of x in the function is 2, and it is a polynomial.

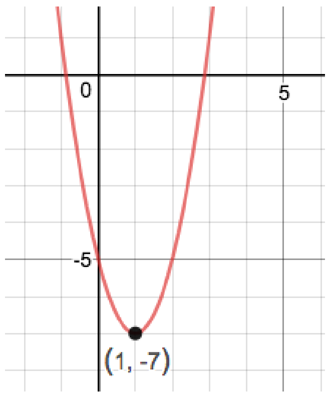
Turning-point form

Recalling our previous work on transformations, quadratics can be written in turning point form of $y = a(x-h)^2 + k$, where (h, k) is the turning point and a affects x-axis dilation and reflection. The axis of symmetry is at x = h as well. To graph, sketch the TP, find the axes intercepts and connect the points.

Polynomial form
Quadratics can also be written in the polynomial form of of $y = ax^2 + bx + c$. Its axis of symmetry, as well as the x-value of the turning point occurs at x = -b/2a. To sketch, find the y (set x = 0) and x-intercepts (by factorising, as covered in Week 3 and setting y = 0), as well as the turning point, then connect the points.

To convert from this form to T-P form, you must complete the square. To complete the square of $y = x^2 + bx + c$, take half of the coefficient of x, then add and subtract its square. To complete the square of $y = ax^2 + bx + c$, first take a as a factor from all terms, then proceed as normal.

Example
Write f(x) = $2x^2 - 4x - 5$ in turning-point form, and hence sketch it:



$f(x) = 2\left(x^2 - 2x - \frac{5}{2}\right)$

$f(x) = 2\left(x^2 - 2x + 1 - 1 - \frac{5}{2}\right)$

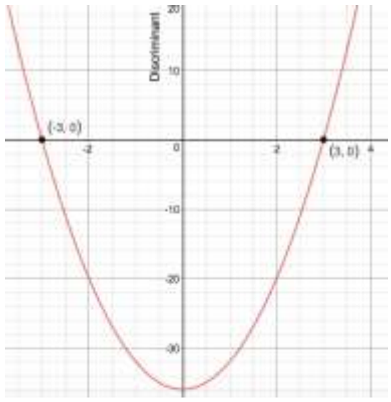
$f(x) = 2((x-1)^2 - \frac{7}{2})$

$f(x) = 2(x-1)^2 - 7$. Turning point at (1, -7).

When x = 0, f(x) = -5 (y-intercept).

When f(x) = 0, $\frac{7}{2} = (x-1)^2 \rightarrow \pm\sqrt{\frac{7}{2}} = x - 1 \rightarrow x = 1 \pm \frac{\sqrt{14}}{2}$ (x-intercepts)

The quadratic equation
To solve all quadratic equations, $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$, where $0 = ax^2 + bx + c$. From the formula, we can say that the discriminant, $\Delta = b^2 - 4ac$. If $\Delta > 0$, there are two solutions, if $\Delta = 0$, there is only one solution, and if $\Delta < 0$, there are no real solutions.



Example
Find the values of m for which $3x^2 - 2mx + 3 = 0$ has a) one solution, b) no solution, c) two distinct solutions.

$\Delta = 4\text{m}^2 - 36$ (see graph on the right for analysis)

a) For one solution: $0 = 4\text{m}^2 - 36 \rightarrow 36 = 4m^2 \rightarrow 9 = m^2 \rightarrow m = \pm 3$
b) For no solution: $4\text{m}^2 - 36 < 0 \rightarrow -3 < m < 3$
c) For two distinct solutions: $4\text{m}^2 - 36 > 0 \rightarrow m < -3 \text{ and } m > 3$

Determining Rules of Quadratics
Just like last week, to determine the rules of quadratics, you need to use the information that is provided, and substitute them into the appropriate form and solve to find the rule:

- If you have two x-intercepts and a point, then $y = a(x-e)(x-f)$
- If you know the turning point and one other point, then $y = a(x-h)^2 + k$
- If you know three points, then $y = ax^2 + bx + c$

Higher Order Polynomials
Polynomials are functions which consist of entirely x's raised to a positive whole number. There can be no fraction or negative powers. The leading term is the term of the polynomial with the highest power. The degree of a polynomial is the highest power in a polynomial. A monic polynomial is a polynomial whose leading term has coefficient 1. There are also constants (numbers only, terms not involving x).

Polynomials can be added, subtracted and multiplied in an easy manner, just note that when multiplying or subtracting, to take into any negatives that can arise. The degree when two functions are multiplied is equal to the sum of the two degrees, e.g. $(x+a)^3 \times (x-b)^2$ will form a polynomial of degree 5.

Sum and Difference of Cubes
$x^3 + a^3 = (x + a)(x^2 - ax + a^2)$
$x^3 - a^3 = (x - a)(x^2 + ax + a^2)$

Example
Express $x^4 - 8x$ in the form $x(x - a)((x + b)^2 + c)$
$x^4 - 8x = x(x^3 - 8)$
$= x(x - 2)(x^2 + 2x + 4)$
$= x(x - 2)(x^2 + 2x + 1 - 1 + 4)$
$= x(x - 2)((x + 1)^2 + 3)$

Equating Coefficients
If two polynomials are equal to each other, then the coefficients of each of the terms must be the same. To solve these problems, expand all brackets and equate the various powers of x.

Example
If $a(x + c)^3 + b = -2x^3 + 18x^2 - 54x + 52$, for all $x \in R$, find the values of a, b and c.
$-2x^3 + 18x^2 - 54x + 52 = a(x^3 + 3x^2c + 3xc^2 + c^3) + b$
$-2x^3 + 18x^2 - 54x + 52 = ax^3 + 3acx^2 + 3ac^2x + ac^3 + b$
Equating x^3: -2 = a
Equating x^2: 18 = 3ac → 18 = -6c → c = -3
Equating x^1: $-54 = 3ac^2$ → $-54 = 3(-2)(-3)^2 = -54$ *(This step is added for completeness, to confirm values)*
Equating x^0/constants: $52 = ac^3 + b$ → $52 = -2(-3)^3 + b$ → 52 = 54 + b → b = -2.
Hence, $-2(x - 3)^3 - 2 = -2x^3 + 18x^2 - 54x + 52$

Polynomial Factorisation
Review how to do polynomial long division in Unit 3, Week 1 of the workbook.

It is probably easier to do the long division method, as the factor theorem method may not always work, as solutions can also be fractional, which would be impossible to guess when finding a suitable x for f(x) = 0. Alternatively, we can equate coefficients as you will see in Question 13 below.

In summary:
The **remainder theorem** states that when P(x) is divided by (bx + a), then the remainder is P(-a/b).
The **factor theorem** states that if P(-a/b) = 0, then (bx + a) is a factor.
To choose which numbers we put into the factor theorem, we look at the factors of the constant.
Note: When dividing by a nonlinear factor such as $x^2 - 3$, when long dividing, we must write as our divisor, $x^2 + 0x - 3$.

Testing Understanding
1. Sketch the following graphs, showing all relevant features, including turning points:
   a) $f(x) = 3(x - 2)^2 - 4$
   b) $f(x) = -2x^2 + 9x + 11$
   c) $f(x) = -2(3 - x)^2 + 1$

d) $f(x) = 3x^2 - 2x - 1$

2. For which values of m does $mx^2 - 2mx + 3 = 0$ have:
   a) two solutions for x
   b) one solution for x
3. For which values of a does $(a-3)x^2 + 2ax + a + 2 = 0$ have:
   a) no solutions for x
   b) one solution for x
   c) two solutions for x
4. Prove $x^2 + (a+1)x + a - 2 = 0$ always has two distinct solutions.
5. For which values of k does $kx^2 - 2kx = 5$ have:
   a) no solutions for x
   b) one solution for x
   c) two solutions for x
6. Determine the rule for a parabola with x-intercepts -3 and -3/2 and passes the point (1, 20)
7. A parabola with turning point (-2, -3) passes through the point (-3, -5). Find its rule.
8. A parabola passes through (2, -14), (0, 10) and (-4, 10). Find its rule.
9. A parabola with y-intercept 6 also passes through (2, 2) and (4, 6). Find its rule.
10. A parabola with x-intercepts -1 and 5 has a y-intercept of -5. Find its rule.
11. Let $P(x) = x^2 - x^3 - 2x + c$. If P(1) = 6, find the value of c.
12. Let $P(x) = x^5 - 2x^4 + ax^3 + bx^2 + 12x - 36$. If P(1) = P(3) = 0, find a and b.
13. If for all values of x that $x^3 - x^2 - 6x - 4 = (x+1)(x^2 + bx + c)$, find b and c by equating coefficients. Hence, write $x^3 - x^2 - 6x - 4$ as a product of three linear factors.
14. Show that $x^3 - 5x^2 - 2x + 24$ cannot be written in the form $a(x + c)^3 + b$.
15. Find the values for A, B, C given $x^3 - 9x^2 + 27x - 22 = A(x+B)^3 + C$
16. Fully factorise the following expressions, using any method you like:
   a) $x^3 + x^2 - 24x + 36$
   b) $x^3 - 31x - 30$
   c) $x^3 - x^2 + 2x - 8$
   d) $x^3 - 12x + 16$
   e) $8x^3 + 10x^2 - 83x + 20$
   f) $x^3 + 6x^2 + 10x + 4$ (hint: complete the square)
17. The polynomial $f(x) = 2x^3 + ax^2 - bx + 3$ has factor $x + 3$, but when divided by $x - 2$, has a remainder of 15. Calculate a and b, and find the other two linear factors.

Methods Unit 3 – Week 8 – Further Polynomials

Sketching Polynomials

The general cubic function $f(x) = ax^3 + bx^2 + cx + d$ and the general quartic function $f(x) = ax^4 + bx^3 + cx^2 + dx + e$ can be graphed, mainly through factorising and solving, thereby determining intercepts. Note that turning points and points of inflections cannot yet be determined at this point without calculus, instead, use a CAS calculator if necessary.

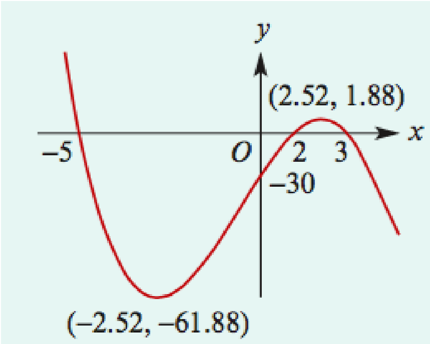
- The maximum amount of x-intercepts is equal to the number of degrees in a polynomial, e.g. a cubic can have 0-3 x-intercepts, a quartic, 0-4 x-intercepts.
- Not all graphs have a stationary point, e.g. $x^3 + x$ has no points of zero gradients
- When there is a single, linear factor, then the graph cuts through the corresponding x-intercept
- When there are repeated factors, e.g. $(x – 3)^2$, this indicates the presence of a turning point at (3, 0). Where there are 3 repeated factors, e.g. $(x – 1)^3$, there's a stationary point of inflection at that point.
- Use the following properties below to assist on sketching graphs. These are derived by looking at the positive and negative graphs of $y = x^2$ and $y = x^3$.

Degree	Positive/Negative	As x moves towards $-\infty$	As x moves towards ∞
Even (similar to x^2)	a > 0 (positive)	$y \to \infty$	$y \to \infty$
	a < 0 (negative)	$y \to -\infty$	$y \to -\infty$
Odd (similar to x^3)	a > 0 (positive)	$y \to -\infty$	$y \to \infty$
	a < 0 (negative)	$y \to \infty$	$y \to -\infty$

Example

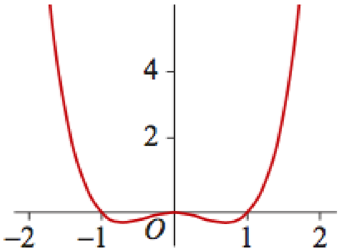
Sketch f(x) = $-x^3 + 19x - 30$

Factorising using any method yields f(x) = -(x + 5)(x – 2)(x – 3)
This yields x-intercepts of -5, 2 and 3.
Find the y-intercept, when x = 0: f(0) = -30.
Because it is a negative cubic, we look at the fourth row above, as x approaches negative infinity, y approaches infinity, and as x approaches infinity, y approaches negative infinity.
We use this information to sketch (turning points calculated using CAS):



Sketch f(x) = $x^4 – x^2$

Factorising yields: f(x) = $x^2(x^2 – 1) = x^2(x + 1)(x – 1)$. This yields x-intercepts of 0, -1 and 1. Also, there is a turning point at 0, since it is a repeated factor. The y-intercept is also 0.
Because it is a positive quartic, we look at the first row above (it functions similarly to a positive quadratic), where as x moves towards negative or positive infinity, y approaches infinity.
We use this information to sketch:



Determining Rules of Polynomials

As with the quadratics, it depends on the information you have:

- For a polynomial of degree n, then you will need n + 1 points to solve
- If you are given a stationary point and another point, you can substitute into $y = a(x-h)^n + k$
- If you are given x-intercepts and a point: $y = a(x-e)(x-f)$ ... as required.

Solving Literal Polynomials

Literal polynomials can also be solved. There are a range of techniques that can be used to solve:

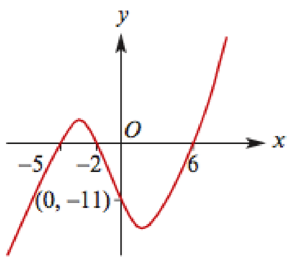
- If it can be factorised easily, do so, and use the null factor law.

- If it is a quadratic, use the quadratic formula. This is usually used for 'find the values of __' so that graphs intersect once, twice or no times at all.
- To find the points of intersection between two graphs, equal them to each other.
- Sometimes, you can substitute one equation into another to solve simultaneously

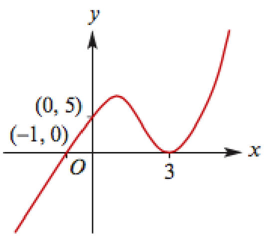
Note: If n is an odd number and a = b^n, then b = $a^{1/n}$. If n is even and a = b^n, then b = $\pm a^{1/n}$

Testing Understanding

1. Sketch the following graphs, showing all relevant features, excluding turning points:
   a) $f(x) = (3 - x)(x - 1)(x - 6)$
   b) $f(x) = (x - 5)(x + 1)(2x - 6)$
   c) $f(x) = (x - 5)^2(x - 4)$
   d) $f(x) = x^3 - 4x^2 + x + 6$
2. Use a calculator to define f(x) = $x^3 - 2x^2 + 1$. Using your CAS, sketch the following graphs on the same axes:
   a) $f(x)$ b) $f(x - 2)$ c) $f(x + 2)$ d) $3f(x)$
3. Sketch the following graphs, showing all relevant features, excluding turning points:
   a) $f(x) = (3 - x)(x + 4)(x - 5)(x - 1)$
   b) $f(x) = x^4 - 2x^3 + 4x + 4$
   c) $f(x) = 81x^4 - 72x^2 + 16$
   d) $f(x) = 3x^3 - 81$
4. Sketch the following graphs, showing all relevant features, excluding turning points:
   a) $f(x) = x^6 - x^2$
   b) $f(x) = x^5 - x^3$
5. A cubic function with point of inflection (5, -2) passes through (4, 0). Find the function.
6. A cubic function with x-intercepts 1, -1 and 2 passes through (3, 120). Find the function.
7. A cubic function of form f(x) = $ax^3 + bx$ passes through (2, -20) and (-1, 20). Find a and b.
8. Find the rule for the following cubic functions:
   a)



   b)



9. Find the rule for the cubic function that passes through (0, 1), (1, 3), (-1, -1), and (2, 11), using a CAS.
10. Find the rule for the quartic function that passes through (-3, 119), (-2, 32), (-1, 9), (0, 8) and (1, 11), using a CAS.
11. Solve the following equations for x
    a) $kx^2 + x + k = 0$
    b) $x^3 - 7ax^2 + 12a^2x = 0$
    c) $x^3 - ax = 0$
    d) $x^4 - a^4 = 0$
    e) $(a - x)^4(a - x^3)(x^2 - a) = 0$
    f) $ax^2 - b = c \text{ where } a, b, c > 0$
    g) $x^3 - c = d$

12. Find the points of intersection between each of the following pairs of equations
    a) $y = \frac{1}{x-2} + 3$ and $y = x$
    b) $x^2 + y^2 = 178$ and $x + y = 16$

c) $x + y = 28$ and $xy = 187$

d) $\frac{y}{4} - \frac{x}{5} = 1$ and $x^2 + 4x + y^2 = 12$

13. What is the condition so that $x^2 + ax + b$ is divisible by $(x + c)$?
14. Find the values of m for which $y = mx – 8$ intersects $y = x^2 – 5x + m$ twice. Hint: This requires use of the quadratic equation and the discriminant.
15. Find the equations of the straight lines that pass through (1, 7) and just touch $y = -3x^2 + 5x + 2$.
16. Show when $y = kx + b$ just touches the curve $y = x^2 + x + 4$, then $k^2 – 2k + 4b – 15 = 0$. Hence find the equation of such lines that passes through the point (0, 3).

Year 12 – Correspondences between old and new workbook, extra questions

Some questions are the same as in the old workbook, so students can do some questions inside the workbook instead of their own exercise book. It also contains other questions which are relevant to the course, that students can do if they have free time:

<table>
<tr><td>Week</td><td>Correspondences in old workbook</td><td>Additional questions in old workbook</td></tr>
<tr><td>1</td><td>Q2 = Q3 in Week 1</td><td>Q1-3 in Week 2</td></tr>
<tr><td>2</td><td>None</td><td>None</td></tr>
<tr><td>3</td><td>Q13 = Q2 in Week 1</td><td>Q1, Q4 in Week 1</td></tr>
<tr><td>4</td><td>Q2e-2h = Q5 in Week 1</td><td>None</td></tr>
<tr><td rowspan="2">5
6</td><td>None</td><td rowspan="2">Q4, Q5cd in Week 2
Q1-3 in Week 4</td></tr>
<tr><td>Q2hi = Q5ab in Week 2
Q11-12 = Q6-7 in Week 2</td></tr>
<tr><td>7</td><td>None</td><td>None</td></tr>
<tr><td>8</td><td>None</td><td>Q1 in Week 3</td></tr>
</table>