



BILLANOOK COLLEGE

NAME: Answers

Student Number:

MATHEMATICAL METHODS (CAS) UNITS 3 & 4

Practice July Exam

Exam 1 TECHNOLOGY FREE

Friday 21st July, 2017

Reading time: 15 minutes 11:15am- 11:30am

Writing time: 1 hour 11:30am – 12:30pm

QUESTION AND ANSWER BOOKLET

Structure of Booklet

<i>Number of Questions</i>	<i>Number of questions to be answered</i>	<i>Number of marks</i>
10	10	40

- Students are permitted to bring into the test room: pens, pencils, highlighters, erasers, sharpeners, rulers.
- Students are NOT permitted to bring into the test room: notes of any kind, a calculator of any type, blank sheets of paper and/or white out liquid/tape.

Materials supplied

- Question and answer book with a detachable sheet of miscellaneous formulas.

Instructions

- Write your **name** in the space provided above on this page.
- All written responses must be in English.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the test room.

Instructions

Answer **all** questions in the space provided.

In all questions where a numerical answer is required an exact value must be given unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

Question 1 (5 marks)

- a. If $y = x^2 \sin(x)$, find $\frac{dy}{dx}$.

2 marks

$$\frac{dy}{dx} = x^2 \cos(x) + 2x \sin(x)$$

- b. If $f(x) = \sqrt{x^2 + 3}$, find $f'(1)$.

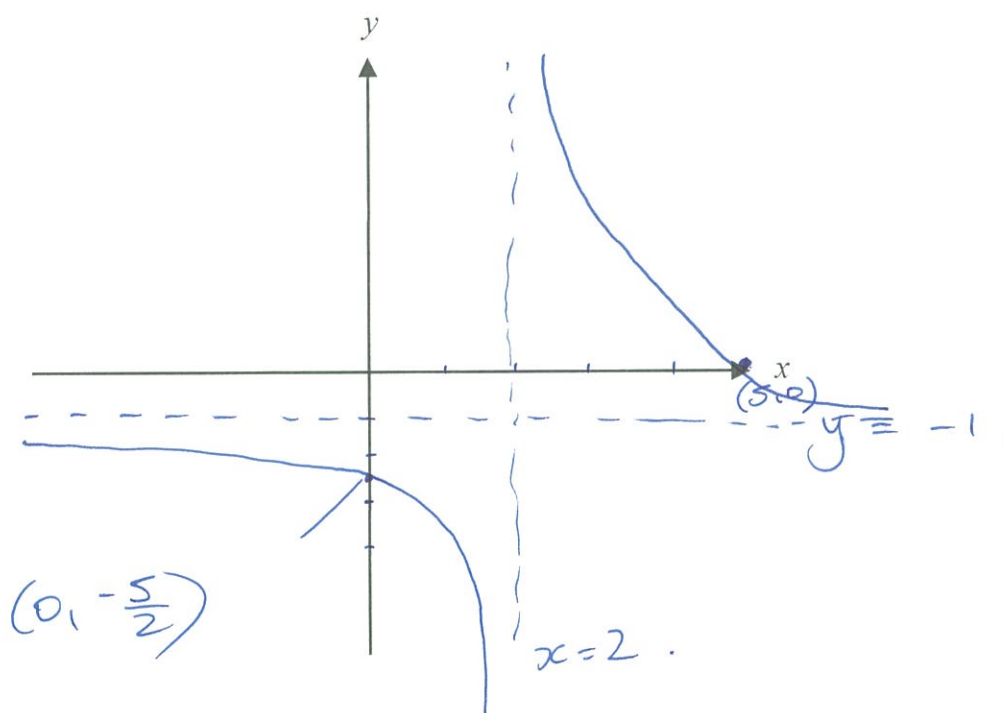
3 marks

$$f'(x) = \frac{1}{2\sqrt{x^2+3}} \times 2x$$

$$f'(x) = \frac{x}{\sqrt{x^2+3}}$$

Question 2 (3 marks)

Sketch the graph of $f: \mathbb{R} \setminus \{2\} \rightarrow \mathbb{R}$, $f(x) = -1 + \frac{3}{x-2}$ on the set of axes below. Label axes intercepts with their coordinates. Label asymptotes with their equations



$$\begin{aligned}
 \times \text{ int } (y=0) \quad & 0 = -1 + \frac{3}{x-2} \\
 & 1 = \frac{3}{x-2} \\
 & x-2 = 3 \\
 & x = 5. \quad (5,0)
 \end{aligned}$$

$$\begin{aligned}
 \times \text{ int } (x)=0 \quad & y = -1 + \frac{3}{-2} \\
 & y = -\frac{5}{2} \quad (0, -\frac{5}{2})
 \end{aligned}$$

Question 3 (2 marks)

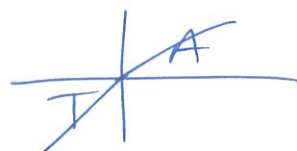
Find $\int_1^3 \left(\frac{2}{x} + 1 \right) dx$.

$$\begin{aligned}
 & \left[2 \log_e(x) + x \right]_1^3 \\
 &= (2 \log_e(3) + 3) - (2 \log_e(1) + 1) \quad \log_e 1 = 0 \\
 &= 2 \log_e(3) + 2.
 \end{aligned}$$

Question 4 (3 marks)

Let $f(x) = \frac{1}{\sqrt{3}} \cos(x)$ and $g(x) = \sin(x)$.

a. Solve the equation $f(x) = g(x)$ for $x \in [0, 2\pi]$.



2 marks

$$\begin{aligned}
 \frac{1}{\sqrt{3}} \cos(x) &= \sin(x) \\
 \frac{1}{\sqrt{3}} &= \tan(x) \quad x \in [0, 2\pi] \\
 x &= \pi/6, \pi/6.
 \end{aligned}$$

b. Evaluate $f(g(0))$.

1 mark

$$\begin{aligned}
 f(g(0)) &= \frac{1}{\sqrt{3}} \cos(\sin(0)) \\
 &= \frac{1}{\sqrt{3}} \cos(0) \\
 &= \frac{1}{\sqrt{3}}.
 \end{aligned}$$

Question 5 (2 marks)

Solve the following equation:

$$\frac{4000}{2 + 7^{3x}} = 5$$

$$4000 = 5(2 + 7^{3x})$$

$$800 = 2 + 7^{3x}$$

$$798 = 7^{3x}$$

$$\log_7(798) = \log_7(7)^{3x}$$

$$\log_7(798) = 3x$$

$$x = \frac{1}{3} \log_7(798)$$

Question 6 (2 marks)The tangent to the curve $y = \frac{3}{x} - 2$ at the point $x = a$, where $a > 0$, has a gradient of -9 .Find the value of a .

$$\frac{dy}{dx} = -\frac{3}{x^2} \quad \text{grad of tangent} = -\frac{3}{a^2}$$

$$-\frac{3}{a^2} = -9 \quad a^2 = \frac{-3}{-9} \quad a = \pm \sqrt{\frac{1}{3}}$$

$$\text{but as } a > 0 \quad a = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}}$$

$$\text{or } \frac{\sqrt{3}}{3}$$

Question 7 (3 marks)

Solve the equation $\log_e(x) + \log_e(3x+2) = 2\log_e(x+1)$ for x , where $x > 0$.

$$\log_e(x(3x+2)) = \log_e(x+1)^2$$

So $3x^2 + 2x = x^2 + 2x + 1$

$$2x^2 = 1$$

$$x^2 = \frac{1}{2}$$

$$x = \pm\sqrt{\frac{1}{2}} \quad \text{but as } x > 0 \text{, reject } x = -\sqrt{\frac{1}{2}}$$

$$\therefore x = \sqrt{\frac{1}{2}}$$

Question 8 (9 marks)

Consider the function with the rule $f(x) = \frac{x-2}{x+2}$

a. Find the rule, f^{-1} , for the inverse of f .

3 marks

$$y = \frac{x-2}{x+2} \quad \text{swap } x \text{ \& } y \text{ to find inverse}$$

$$x = \frac{y-2}{y+2} \Rightarrow yx + 2x = y - 2$$

$$yx - y = -2 - 2x$$

$$y(x-1) = -2-2x$$

$$y = \frac{-2-2x}{x-1}$$

or

$$y = \frac{2x+2}{1-x}$$

- b. Find the domain and range of the inverse of f .

2 marks

$$\text{Domain: } \mathbb{R} \setminus \{1\}$$

$$\text{Range: } \mathbb{R} \setminus \{-2\}$$

- c. Show that $f^{-1}(x)$ can be written in the form of $a + \frac{b}{1-x}$ and hence find $\int_0^{\frac{1}{2}} f^{-1}(x) dx$

4 marks

Use long division.

$$\frac{2+2x}{1-x} = -2 + \frac{4}{1-x}$$

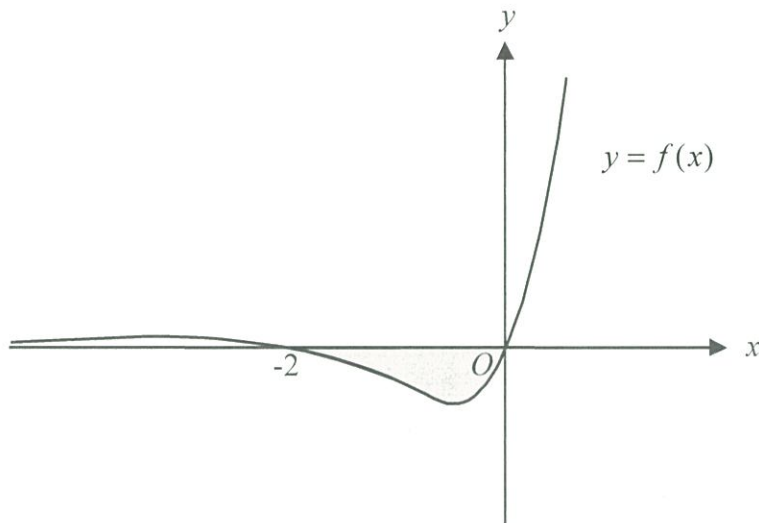
$$\begin{array}{r} -2 \\ -x+1 \overline{) 2x+2} \\ \underline{-(2x-2)} \\ 4 \end{array}$$

$$\int_0^{\frac{1}{2}} \left(-2 + \frac{4}{1-x}\right) dx = \left[-2x - 4 \log_e(1-x)\right]_0^{\frac{1}{2}}$$

$$= -1 - 4 \log_e\left(\frac{1}{2}\right) \Rightarrow \boxed{-1 + 4 \log_e(2)}$$

Question 9 (4 marks)

The graph of $f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = (x^2 + 2x)e^x$ is shown below.



The region enclosed by the graph of f and the x -axis is shaded.

- a. Find the derivative of $(3 - x^2)e^x$. Give your answer in the form $ae^x - f(x)$, where a is a positive constant. 1 mark

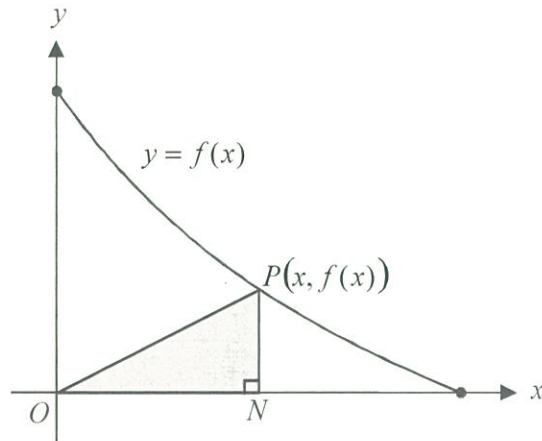
$$\begin{aligned} \frac{d}{dx} (3 - x^2)e^x &\Rightarrow (3 - x^2)e^x + -2xe^x \\ \Rightarrow 3e^x - x^2e^x - 2xe^x &\Rightarrow 3e^x - (x^2 + 2x)e^x \\ &\Rightarrow 3e^x - f(x) \end{aligned}$$

- b. Use your answer to part a. to find the area of the shaded region. 3 marks

$$\begin{aligned} - \int_{-2}^0 f(x) dx & \quad \frac{d}{dx} (3 - x^2)e^x = 3e^x - f(x) \\ f(x) &= 3e^x - \frac{d}{dx} (3 - x^2)e^x \\ \text{So, } \int_{-2}^0 f(x) dx &= \int_{-2}^0 3e^x dx - \int_{-2}^0 \frac{d}{dx} (3 - x^2)e^x dx \\ &= [3e^x]_{-2}^0 - [(3 - x^2)e^x]_{-2}^0 \\ &= (3 - 3e^{-2}) - (3 - e^{-2}) \\ &= -4e^{-2} \\ \therefore - \int_{-2}^0 f(x) dx &= 4e^{-2} \end{aligned}$$

Question 10 (7 marks)

Let $f:[0,1] \rightarrow \mathbb{R}$, $f(x)=1-x^{\frac{2}{3}}$. The graph of f is shown below.



The right-angled triangle NOP has vertex N on the x -axis, and vertex O at the origin. The vertex P lies on the graph of f and has coordinates $(x, f(x))$ as shown.

- a. Find the area A , of triangle NOP in terms of x .

1 mark

$$A = \frac{1}{2}bh$$

$$= \frac{1}{2} \times x \times f(x)$$

$$= \frac{1}{2} \times x \times (1 - x^{2/3})$$

$$= \frac{x}{2} (1 - x^{2/3})$$

b. Find

i. the value of x for which A is a maximum.

2 marks

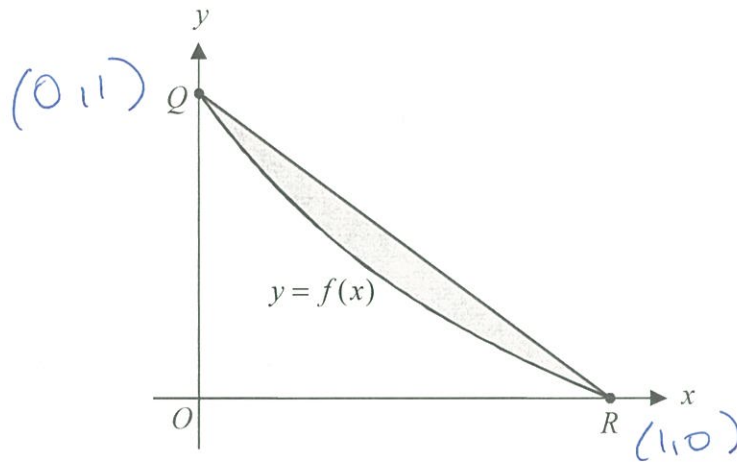
$$\begin{aligned}
 \frac{dA}{dx} &= \left(\frac{x}{2} \times -\frac{2}{3} x^{-1/2} \right) + \left(\frac{1}{2} (1 - x^{2/3}) \right) \\
 &= \frac{-x^{2/3}}{3} + \frac{1}{2} - \frac{x^{2/3}}{2} \\
 &= \frac{-2x^{2/3} - 3x^{2/3}}{6} + \frac{1}{2} \quad \frac{dA}{dx} = 0 \\
 0 &= -\frac{5x^{2/3}}{6} + \frac{1}{2} \\
 -\frac{1}{2} &= -\frac{5x^{2/3}}{6} \Rightarrow 3 = 5x^{2/3} \Rightarrow x^{2/3} = \frac{3}{5} \\
 x &= \left(\frac{3}{5} \right)^{3/2}
 \end{aligned}$$

ii. the maximum area of triangle NOP . Give your answer in the form $\frac{a\sqrt{b}}{c}$ where a , b and c are positive integers.

1 mark

$$\begin{aligned}
 A &= \frac{1}{2} \left(\frac{3}{5} \right)^{3/2} \left(1 - \left(\left(\frac{3}{5} \right)^{3/2} \right)^{2/3} \right) \\
 &= \frac{1}{2} \left(\frac{3}{5} \right)^{3/2} \left(1 - \frac{3}{5} \right) \\
 &= \frac{1}{2} \left(\frac{3}{5} \right)^{3/2} \left(\frac{2}{5} \right) \\
 &= \frac{1}{5} \left(\frac{3}{5} \right)^{3/2} \\
 &= \frac{1}{5} \times \frac{\sqrt{27}}{\sqrt{125}} \\
 &= \frac{3\sqrt{3}}{125\sqrt{5}} \quad \text{OR} \quad \frac{3\sqrt{15}}{125}
 \end{aligned}$$

- c. The point Q lies on the graph of f and on the y -axis. The point R lies on the graph of f and on the x -axis.



Find the area enclosed by the line segment QR and the graph of f .

3 marks

Find Q (y int of $f(x)$).

$$f(x) = 1 - x^{2/3} \quad \text{y int (x=0)} \quad \therefore \text{y int (0,1)}$$

Find R (x int of $f(x)$).

$$f(x) = 1 - x^{2/3}$$

$$x^{2/3} = 1 \quad \text{so } x = 1 \quad (1,0)$$

$$y - 1 = -1(x - 0)$$

$$y = -x + 1 \Rightarrow y = 1 - x$$

$$\text{Area} = \int_0^1 (1 - x) dx - \int_0^1 (1 - x^{2/3}) dx$$

$$= \int_0^1 (1 - x - 1 + x^{2/3}) dx \Rightarrow \int_0^1 (x^{2/3} - x) dx$$

$$= \left[\frac{3}{5} x^{5/3} - \frac{x^2}{2} \right]_0^1 \Rightarrow \left(\frac{3}{5} - \frac{1}{2} \right) - 0$$

END OF QUESTION BOOKLET

$$= \frac{6-5}{10}$$

$$= \frac{1}{10} \text{ sq units.}$$