

Student Name: SOLUTIONS

Home Group: _____

Teacher's name: (please circle): Ms Nation Ms O'Rielly



Mathematical Methods

Unit 2

Wednesday 8th November 2017

Part I

Total 57 marks

- Topics covered:
- Combinatorics
 - Circular Functions
 - Rates of Change
 - Differential Calculus
 - Integral Calculus
 - Exponential Functions and Logarithms

Complete working must be shown and simplified wherever possible in order to gain full marks.

Reading Time: 15 minutes

Writing Time: 60 minutes

Students are NOT permitted to use any calculators or reference books for this section.

No paper or electronic dictionaries may be used.

Useful formulae:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$${}^nC_r = \frac{n!}{(n-r)!r!}$$

$${}_nP_r = \frac{n!}{(n-r)!}$$

Newton's Iterative formula for approximating roots of a polynomial:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Any question worth more than 1 mark must have the appropriate working shown to justify the extra marks.

- 1) A menu offers a choice of five entrees, four mains and three desserts.

Find the number of meal choices possible

- a) if one of each must be chosen for a 3 course meal.

$$5 \times 4 \times 3 = 60$$

- b) if you have of choice of not having the dessert if you prefer.

3 course or 2 course

$$60 + 5 \times 4 = 80$$

(2 marks)

- 2) The digits 0, 1, 2, 3 and 4 are used to make a 3-digit number. No digit is repeated.

- a) How many different 3-digit numbers are possible, if 0 cannot be the first digit?

$$4 \times 4 \times 3 = 48$$

- b) If any of the 3-digit numbers in part a is equally likely to have been made, find the probability that number made is greater than or equal to 230.

$$\begin{aligned} \text{Number} \geq 300 &= 2 \times 4 \times 3 = 24 \\ + \text{No. } 230 \leq x < 300 &= 1 \times 2 \times 3 = 6 \\ \hline \text{Total} &= 30 \end{aligned} \quad \textcircled{1}$$

$$Pr(\geq 230) = \frac{30}{48} = \frac{5}{8} \quad \textcircled{1}$$

(1 + 2 = 3 marks)

- 3) Evaluate $^{100}C_2$

$$\begin{aligned} &= \frac{100!}{98! \cdot 2!} = \frac{100 \times 99}{2} = 2 \overline{)9900} \\ &= \boxed{4950} \end{aligned}$$

(1 mark)

- 4) In how many ways can four girls be selected for a table tennis team, if seven girls try out?

$$\begin{aligned} {}^7C_4 &= \frac{7!}{4! \cdot 3!} = \frac{7 \times 6 \times 5}{3 \times 2} = 35 \quad \textcircled{1} \\ \textcircled{1} \end{aligned}$$

(2 mark)

5) Find the exact values of:

$$\begin{aligned} \text{a) } \sin 120^\circ &= \sin(180^\circ - 60^\circ) \\ &= \sin 60^\circ \\ &= \frac{\sqrt{3}}{2} \end{aligned}$$

$$\begin{aligned} \text{b) } \tan \frac{4\pi}{3} &= \tan\left(\pi + \frac{\pi}{3}\right) \\ &= \tan \frac{\pi}{3} \\ &= \sqrt{3} \end{aligned}$$

$$\begin{aligned} \text{c) } \sin\left(-\frac{\pi}{6}\right) &= -\sin \frac{\pi}{6} \\ &= -\frac{1}{2} \end{aligned}$$

$$\begin{aligned} \text{d) } \cos \frac{9\pi}{4} &= \cos\left(\frac{9\pi}{4} - 2\pi\right) \\ &= \cos \frac{\pi}{4} \\ &= \frac{\sqrt{2}}{2} \end{aligned}$$

(4 marks)

6) a) What is the period and the amplitude of the graph of

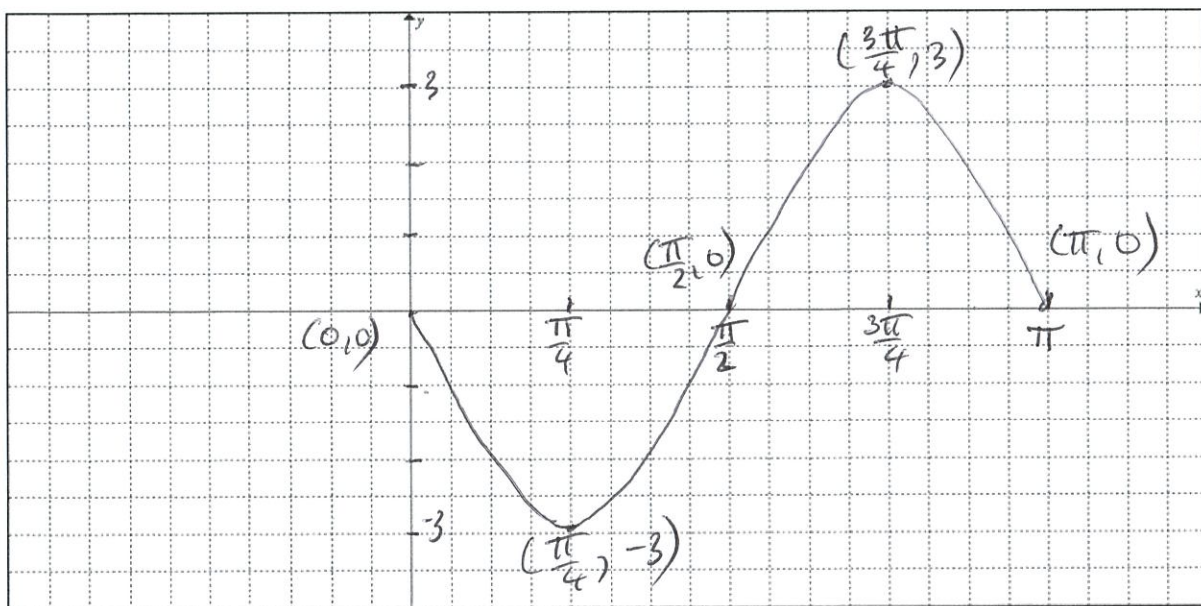
$$y = -3 \sin 2x ?$$

$$\text{period} = \frac{2\pi}{2} = \pi$$

$$\text{amp} : 3$$

* reflected

b) Sketch the graph, showing one complete cycle. Clearly label key points.



shape ①
key points
labelled ①
1 cycle ①

(2 + 3 = 5 marks)

7) Solve the following equation, giving your answer(s) as exact values:

$$\sqrt{2} \sin x + 1 = 0, \quad 0 \leq x \leq 4\pi$$

$$\sqrt{2} \sin x = -1$$

$$\sin x = \frac{-1}{\sqrt{2}}$$

$$= -\frac{\sqrt{2}}{2}$$

$$\frac{\pi}{4} \quad (1)$$

reference angle (Q1) $\theta = \frac{\pi}{4}$

In Q3 : $x = \pi + \theta$

$$= \frac{5\pi}{4} \quad (1)$$

In Q4 : $x = 2\pi - \theta$

$$= 2\pi - \frac{\pi}{4}$$

$$= \frac{7\pi}{4}$$

$$x = \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{5\pi}{4} + 2\pi, \frac{7\pi}{4} + 2\pi$$

$$x = \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{13\pi}{4}, \frac{15\pi}{4} \quad (1)$$

(3 marks)

8) If $\sin \theta = 0.66$, $\cos \theta = 0.75$ and $\tan \theta = 0.87$, write down the value of:

$\sin(2\pi - \theta) = -\sin \theta$ $= -0.66$	$\tan(\pi + \theta) = \tan \theta$ $= 0.87$
$\cos(-\theta) = \cos \theta$ $= 0.75$	$\cos(\pi - \theta) = -\cos \theta$ $= -0.75$

(4 marks)

9) If Evie drives at 60 km/h for 2 hours and 110 km/h for 3 hours, what is her average speed for the entire journey?

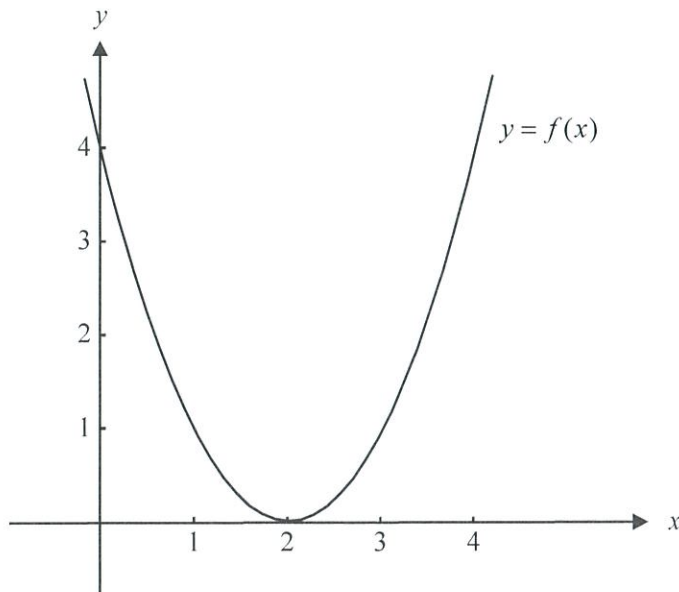
$$\text{Total distance} = 60 \times 2 + 110 \times 3$$

$$= 120 + 330$$

$$= 450 \text{ km}$$

$$\text{Av. speed} = \frac{450}{5} = 90 \text{ km/hr} \quad (1 \text{ mark})$$

10) The graph of the function $f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = (x-2)^2$ is shown below.



a) Find the average rate of change of $y = f(x)$ with respect to x , between $x = 1$ and $x = 4$.

Using rule $f(1) = (1-2)^2 = 1$
 $f(4) = (4-2)^2 = 4$

$$\text{av. rate of change} = \frac{f(4) - f(1)}{4 - 1} = \frac{4 - 1}{4 - 1} = 1$$

b) Find the instantaneous rate of change of $y = f(x)$ with respect to x at the point where

i) $x = 2$

$$f'(x) = 0$$

(tp of graph)

ii) $x = 4$

$$f'(x) = 2x - 4 \quad (1)$$

$$f'(4) = 2(4) - 4 = 4 \quad (1)$$

(1 + 1 + 1 = 3 marks)

11) If $f(x) = (x+2)(x+3)$, find $f'(3)$.

$$f(x) = x^2 + 3x + 2x + 6 = x^2 + 5x + 6 \quad (1)$$

$$\Rightarrow f'(x) = 2x + 5$$

$$f'(3) = 2(3) + 5 = 6 + 5$$

$$= 11 \quad (1)$$

(2 marks)

12) Find, using first principles, the derivative of

$$f(x) = 3x^2 + x - 2$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{3(x+h)^2 + (x+h) - 2 - (3x^2 + x - 2)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{3x^2 + 6xh + h^2 + x + h - 2 - 3x^2 - x + 2}{h}$$

$$= \lim_{h \rightarrow 0} \frac{6xh + h^2 + h}{h} \quad (1)$$

$$= \lim_{h \rightarrow 0} \frac{h(6x + h + 1)}{h}$$

$$= \lim_{h \rightarrow 0} 6x + h + 1$$

$$= \underline{6x + 1} \quad (1)$$

(1) correct, notation throughout.

(3 marks)

13) Find $\lim_{x \rightarrow 3} \frac{x^2 - x - 12}{x^2 - 16} = \lim_{x \rightarrow 3} \frac{(x-4)(x+3)}{(x+4)(x-4)}$

$$= \lim_{x \rightarrow 3} \frac{x+3}{x+4} \quad (1)$$

$$= \frac{3+3}{3+4} \quad \boxed{= \frac{6}{7}} \quad (1)$$

to

(2 marks)

14) Evaluate:

a) $\int (4x^3 - x^2 + 9) dx$

$$= x^4 - \frac{1}{3}x^3 + 9x + C$$

b) $\int \frac{3x^4 + 5x^3}{2x} dx$

$$= \int \frac{3x^4}{2x} dx + \int \frac{5x^3}{2x} dx \quad (1)$$

$$= \int \frac{3}{2}x^3 dx + \int \frac{5}{2}x^2 dx$$

$$= \frac{3}{8}x^4 + \frac{5}{6}x^3 + C \quad (1)$$

(1 + 2 = 3 marks)

15) A particle moves in a straight line with velocity of $v(t) = 6t^2 - 4t$ (m/s) at time t seconds ($t \geq 0$). The particle has an initial position $x(t)$ of 3m left of the origin, O.

a) Find the equation of the position of the particle, $x(t)$

$$\begin{aligned} x(t) &= \int v(t) dt \\ &= \int 6t^2 - 4t dt \\ &= 2t^3 - 2t^2 + C \quad \textcircled{1} \end{aligned}$$

Initial posⁿ = -3 m (0, -3)

$$\Rightarrow -3 = 2(0)^3 - 2(0)^2 + C$$

$$C = -3$$

$$\underline{x(t) = 2t^3 - 2t^2 - 3} \quad \textcircled{1}$$

b) Find the acceleration of the particle at $t = 2$ seconds

$$\begin{aligned} a(t) &= \frac{dv}{dt} \\ &= 12t - 4 \quad \textcircled{1} \end{aligned}$$

$$\begin{aligned} \text{at } t = 2 \quad a(2) &= 12(2) - 4 \\ &= 20 \text{ m s}^{-2} \quad \textcircled{1} \end{aligned}$$

(2 + 2 = 4 marks)

16) Simplify these expressions using appropriate index or logarithm laws:

$$\begin{aligned} \text{a) } \frac{25^{x+3} \times 5^{6x}}{125^{2x-1}} &= \frac{(5^2)^{x+3} \times 5^{6x}}{(5^3)^{2x-1}} \quad \textcircled{1} \\ &= \frac{5^{2x+6} \times 5^{6x}}{5^{6x-3}} \end{aligned}$$

$$= 5^{2x+6+6x-(6x-3)}$$

$$= 5^{2x+9} \quad \textcircled{1}$$

$$\begin{aligned} \text{b) } \frac{(2x^4y^{-3})^3}{2(x^{-3}y^2)^2} &= \frac{2^3 x^{12} y^{-9}}{2 x^{-6} y^4} \quad \textcircled{1} \\ &= 4 x^{12-(-6)} y^{-9-4} \end{aligned}$$

$$= 4 x^{18} y^{-13}$$

$$= \frac{4x^{18}}{y^{13}} \quad \textcircled{1}$$

$$\text{c) } 2 \log_{10} 5 + \log_{10} 4$$

$$= \log_{10} 5^2 + \log_{10} 4$$

$$= \log_{10} 25 \times 4 \quad \textcircled{1}$$

$$= \log_{10} 100$$

$$= \log_{10} 10^2$$

$$= 2 \log_{10} 10$$

$$= 2 \quad \textcircled{1}$$

(3 x 2 = 6 marks)

17) Solve the following equations for x :

a) $9^{2x} = 27^{2x-4}$

$$(3^2)^{2x} = (3^3)^{2x-4}$$

$$3^{4x} = 3^{6x-12} \quad (1)$$

$$4x = 6x - 12$$

$$-2x = -12$$

$$\underline{x = 6} \quad (1)$$

b) $\log_2(3x - 5) = 4$

$$2^4 = 3x - 5 \quad (1)$$

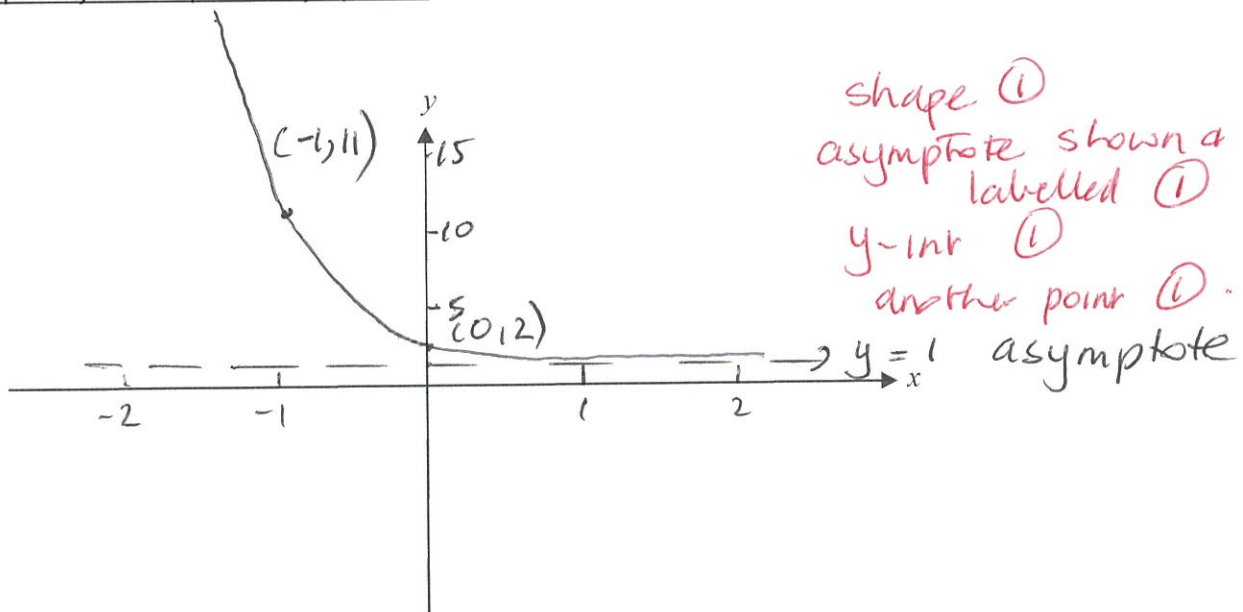
$$16 = 3x - 5$$

$$21 = 3x$$

$$\underline{7 = x} \quad (1)$$

(2 x 2 = 4 marks)

18) a) Sketch the graph of the function $y = 10^{-x} + 1$, $x \in \mathbb{R}$ on the set of axes below. Indicate clearly on the graph any intercepts or asymptotes.



b) What is the range of this function?

$$(1, \infty)$$

(4 + 1 = 5 marks)

END OF EXAM