

MATHEMATICAL METHODS

Units 3 & 4 – Written examination 2



2024 Trial Examination

SOLUTIONS

SECTION 1: Multiple-choice questions (1 mark each)

Question 1

Answer: D

Explanation:

Period = $\frac{\pi}{3}$, Range = R

Question 2

Answer: C

Explanation:

For a unique solution require $\frac{p}{3} \neq \frac{2}{p-1}$

$$p^2 - p - 6 \neq 0$$

$$(p - 3)(p + 2) \neq 0$$

Hence $p \in R \setminus \{-2, 3\}$

Question 3

Answer: C

Explanation:

$$\int_b^a f(x) dx = - \left(\int_a^c f(x) dx - \int_b^c f(x) dx \right) \\ = -(10 - 2) = -8$$

Question 4

Answer: C

Explanation:

Two successive stationary points are adjacent local maxima and minima. The horizontal distance is half a period $= \frac{\pi}{b}$, the vertical distance is $2a$

$$\text{Hence distance} = \sqrt{(2a)^2 + \left(\frac{\pi}{b}\right)^2}$$

Question 5

Answer: D

Explanation:

$$\begin{aligned} \text{dom}(f + g) &= \text{dom}(f) \cap \text{dom}(g) \\ &= [-5, \infty) \cap \mathbb{R} \setminus \{2\} \\ &= [-5, 2) \cup (2, \infty) \end{aligned}$$

Question 6

Answer: C

Explanation:

The graph is a polynomial of order 4. $\frac{d^2y}{dx^2}$ will be of order 2. This can have up to 2 solutions, hence a maximum of two points of inflection.

Question 7

Answer: D

Explanation:

$$\begin{aligned}
 \Pr(a, a) + \Pr(b, b) &= 1 - \Pr(a, b) - \Pr(b, a) = 1 - 2\Pr(a, b) \\
 &= 1 - 2 \times \frac{a}{a+b} \times \frac{b}{a+b-1} \\
 &= 1 - \frac{2ab}{(a+b)(a+b-1)}
 \end{aligned}$$

Question 8

Answer: B

Explanation:

Look for $f(-x) = f(x)$
 $\sin\left(x + \frac{\pi}{2}\right) = \cos(x)$ which is even.

Question 9

Answer: A

Explanation:

Find when gradient of curve is equal to gradient of tangent passing through (0, 0)

$$\begin{aligned}
 \frac{dy}{dx} &= 2x + 5 \\
 \frac{f(x) - f(0)}{x - 0} &= 2x + 5 \\
 x^2 + 5x + 1 &= 2x^2 + 5x \\
 x^2 - 1 &= 0 \\
 x &= \pm 1
 \end{aligned}$$

Check which passes through y, not just parallel.

$$\begin{aligned}
 \text{At } x = 1, y_T &= 7x \quad (1, 7) \\
 \text{At } x = -1, y_T &= 3x \quad (-1, -3) \\
 y(1) &= 7 \\
 y(-1) &= -3 \\
 \text{Hence at both } x &= \pm 1
 \end{aligned}$$

Question 10

Answer: A

Explanation:

Use CAS

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binomialCdf(60, 80, 80, 0.7) / binomialCdf(50, 80, 80, 0.7)  
0.2101951456
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Question 11

Answer: D

Explanation:

$$\int_0^5 kx^2 + 3 \, dx = 1$$
$$k = -\frac{42}{125}$$

Question 12

Answer: B

Explanation:

$$a + a^2 + 2a + 0.4 - a^2 = 1$$
$$a = 0.2$$

$$E(X) = (0 \times 0.2) + (1 \times 0.04) + (2 \times 0.4) + (3 \times 0.36)$$
$$= 1.92$$

Question 13

Answer: A

Explanation:

$$\sqrt{\frac{0.65 \times 0.35}{n}} = 0.1$$

$$n = \frac{91}{4}, \quad \text{so } n = 23$$

Question 14

Answer: D

Explanation:

Look for graph that is a reflection through the line $y = x$.

Question 15

Answer: A

Explanation:

$$\frac{y(4) - y(0)}{4 - 0}$$

$$= 30$$

Question 16

Answer: D

Explanation:

$$\frac{1}{5-1} \int_1^5 x + \sqrt{x} \, dx = \frac{5\sqrt{5} + 17}{6}$$

Question 17*Answer:* C*Explanation:*

$$(2)^3 - 27 = b(2)^2 - 15(2) - 1$$

$$b = 3$$

Question 18*Answer:* A*Explanation:*

$$\text{As } x \rightarrow 0, f(x) \rightarrow \infty$$

$$\text{As } x \rightarrow n, f(x) = \frac{m}{\sqrt{n^2}} = \frac{m}{n}$$

$$\text{Hence } \left[\frac{m}{n}, \infty \right)$$

Question 19*Answer:* D*Explanation:*

n	x	f(x)	df(x)
0	5	-13	2
1	11.5	42.25	15
2	8.683333	7.933611	9.366667
3	7.836329	0.717417	7.672657
4	7.742826	0.008743	7.485651
5	7.741658	1.36E-06	7.483315

Question 20*Answer:* B*Explanation:*

$$f(x) = (x + 1)(x - 1)(x - 2)$$

So area between $x = -1$ and $x = 1$ and area between $x = 1$ and $x = 2$

$$\frac{1 - (-1)}{2 \times 4} \left(f(-1) + 2f\left(-\frac{1}{2}\right) + 2f(0) + 2f\left(\frac{1}{2}\right) + f(1) \right) = \frac{5}{2}$$

$$- \left(\frac{2 - 1}{2 \times 2} \left(f(1) + 2f\left(\frac{3}{2}\right) + f(2) \right) \right) = \frac{5}{16}$$

$$\text{Area} = \frac{5}{2} + \frac{5}{16} = \frac{45}{16}$$

$$\int_{-1}^1 f(x) dx - \int_1^2 f(x) dx = \frac{37}{12}$$

$$\frac{\frac{45}{16}}{\frac{37}{12}} \times 100 = 91.22\%$$

SECTION 2: Extended response questions**Question 1** (13 marks)

a. $f'(x) = x^3 e^{-x}(4 - x)$ or equivalent

1 mark

b. $f'(x) = 0$ at $x = 0, 4$

$$f(0) = 0, f(4) = 256e^{-4}$$

(0,0) is a local minimum

(4, $256e^{-4}$) is a local maximum

2 marks

c. $f''(x) = x^2(x - 2)(x - 6)e^{-x}$ or equivalent

Hence points of inflection at $x = 2$ and $x = 6$
 $(2, 16e^{-2})$ and $(6, 1296e^{-6})$

2 marks

d. $f'(x) < 0$ for $x \in (-\infty, 0) \cup (4, \infty)$

1 mark

e. Maximum at $(4, 256e^{-4})$

Hence $c < -256e^{-4}$ or $c = 0$

1 mark

f. $y_T = 27e^{-3}x$

1 mark

g. $\int_0^3 27e^{-3}x - f(x) dx$
 $= 1.615$

2 marks

h. Distance $D(x) = 27e^{-3}x - f(x)$

$\frac{dD}{dx} = 0$ at $x = 1.13374$ and $x = 3$
 $y_T(1.13374) = 1.524$
 So at $(1.13, 1.52)$

3 marks

Question 2 (10 marks)

a. $V = 2x(h - 2x)(2h - x)$ or equivalent

2 marks

b. $x \in \left(0, \frac{h}{2}\right)$

2 marks

c. $V(x) = 4h^2x - 10hx^2 + 4x^3$
 $\frac{dV}{dx} = 12x^2 - 20hx + 4h^2$

1 mark

d. $12x^2 - 20hx + 4h^2 = 0$
 $3x^2 - 5hx + h^2 = 0$
 $x = \frac{5h \pm \sqrt{25h^2 - 12h^2}}{6} = \frac{h(5 \pm \sqrt{13})}{6}$

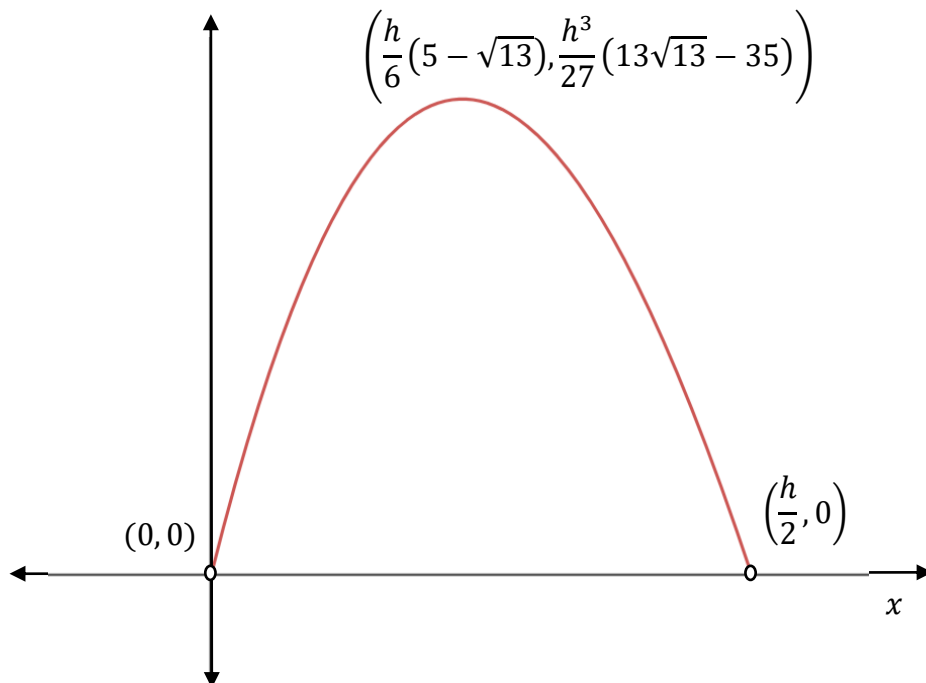
Hence $x = \frac{h}{6}(5 - \sqrt{13})$

$V\left(\frac{h}{6}(5 - \sqrt{13})\right) = \frac{h^3}{27}(13\sqrt{13} - 35) \text{ cm}^3$

With dimensions $\frac{h}{3}(\sqrt{13} - 2) \text{ cm}$, $\frac{h}{3}(\sqrt{13} + 7) \text{ cm}$ and $\frac{h}{6}(5 - \sqrt{13}) \text{ cm}$

3 marks

e.



2 marks

Question 3 (16 marks)

a. $a = \frac{1}{3}, b = 5$

2 marks

b. $g'(x) = \frac{10}{3}x - x^2$

1 mark

c. $g'(x) = 0$ at $x = 0$ and $x = \frac{10}{3}$

So $x \in (-\infty, 0] \cup \left[\frac{10}{3}, \infty\right)$

2 marks

d. $Area = \frac{5-0}{2 \times 5} (g(0) + 2g(1) + 2g(2) + 2g(3) + 2g(4) + g(5))$
 $= \frac{50}{3}$

2 marks

e. $\int_0^5 g(x) dx = \frac{625}{36}$

2 marks

f. Maximum occurs at $\left(\frac{10}{3}, \frac{500}{81}\right)$

$4m(n-4) = \frac{50}{9}$ and $\frac{10}{3}m\left(n - \frac{10}{3}\right) = \frac{500}{81}$

$m = \frac{25}{36}, n = 6$

2 marks

g. $\int_0^{\frac{10}{3}} \frac{1}{3}x^2(5-x) dx + \int_{\frac{10}{3}}^6 \frac{25}{36}x(6-x) dx$

$= \frac{15100}{729} \approx 20.7$

2 marks

h. $h'(x) = \frac{h(6) - h(\frac{10}{3})}{6 - \frac{10}{3}}$

$$x = \frac{14}{3}, \quad h\left(\frac{14}{3}\right) = \frac{350}{81}$$

$$\left(\frac{14}{3}, \frac{350}{81}\right)$$

3 marks

Question 4 (21 marks)

a. $\Pr(X < 2.9) = 0.015 \quad \Pr(X > 5) = 0.091$

$$\Pr\left(Z < \frac{2.9 - \mu}{\sigma}\right) = 0.015 \quad \Pr\left(Z > \frac{5 - \mu}{\sigma}\right) = 0.091$$

$$\frac{2.9 - \mu}{\sigma} = -2.1701 \quad \frac{5 - \mu}{\sigma} = 1.3346$$

$$\mu = \$4.20 \text{ and } \sigma = \$0.60$$

3 marks

b. $\Pr(X > 3.5) = 0.8783$

1 mark

c. $\Pr(X > 3.5 | X < 4.2) = \frac{\Pr(3.5 < X < 4.2)}{0.5}$

$$= 0.7567$$

2 marks

d. Assuming n_1 and n_2 are symmetric about μ :

$$\Pr(X < n_1) = 0.3 \quad \text{and} \quad \Pr(X > n_2) = 0.3$$

$$n_1 = 3.89 \text{ and } n_2 = 4.51$$

2 marks

e. $\Pr(X > 6|X > 5.5) = \frac{\Pr(X>6)}{\Pr(X>5.5)}$

$$\frac{0.0013}{0.0151} = 0.0892 = 9\%$$

2 marks

f. $\Pr(Y > 5.5) = 0.0151$

$$E(Y) = np = 2800 \times 0.0151 = 42.36$$

1 mark

g. Can be done by trial and error.
 $p = 0.01513$

`binomialCDF(100, 8231, 8231, 0.01513)`

0.9900206068

`binomialCDF(100, 8230, 8230, 0.01513)`

0.9899828285

So minimum number of cards is 8231.

2 marks

h. (0.3965, 0.4835)

There is a 95% confidence that the population proportion of coins that are gold lies between 0.3965 and 0.4835.

2 marks

i. For 95% $k = 1.95996$, for 99% $k = 2.5758$

$$\frac{2.5758}{1.95996} = 1.31$$

The interval would increase by a factor of 1.31

2 marks

j. $\int_0^{\infty} kv^3 e^{-v} dv = 1$

$$k = \frac{1}{6}$$

2 marks

k. $\int_0^{\infty} kv^4 e^{-v} dv$
 $= 4$

2 marks