

The Mathematical Association of Victoria
Trial Examination 2024
MATHEMATICAL METHODS

Trial Written Examination 1 - SOLUTIONS

Question 1

a. $y = \frac{\log_e(x)}{x^2}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{x^2 \times \frac{1}{x} - \log_e(x) \times 2x}{x^4} \\ &= \frac{x - 2x \log_e(x)}{x^4} \\ &= \frac{1 - 2 \log_e(x)}{x^3}\end{aligned}$$

1A

b. $g(x) = x \tan^2(x)$

$g'(x) = x \times 2 \tan(x) \sec^2(x) + \tan^2(x) \times 1$ **1M**

$g'\left(\frac{\pi}{4}\right) = \frac{\pi}{4} \times 2 \tan\left(\frac{\pi}{4}\right) \sec^2\left(\frac{\pi}{4}\right) + \tan^2\left(\frac{\pi}{4}\right)$

$$= \frac{\frac{\pi}{2}}{\cos^2\left(\frac{\pi}{4}\right)} + 1$$

$$\begin{aligned}&= \frac{\frac{\pi}{2}}{\frac{1}{2}} + 1 \\ &= \pi + 1\end{aligned}$$

1A

Question 2

$\log_e(3x^2 - 1) - \log_e(1 - 3x) = \log_e(2)$

$\log_e\left(\frac{3x^2 - 1}{1 - 3x}\right) = \log_e(2)$ **1M**

Equate $\frac{3x^2 - 1}{1 - 3x} = 2$

$3x^2 - 1 = 2 - 6x$

$3x^2 + 6x - 3 = 0$

$3(x^2 + 2x - 1) = 0$ **1M**

$x^2 + 2x - 1 = 0$

$x = \frac{-2 \pm \sqrt{4 + 4}}{2}$

$= \frac{-2 \pm 2\sqrt{2}}{2}$

$$= -1 \pm \sqrt{2}$$

effect $x = -1 + \sqrt{2}$ as $1 - 3x > 0$, $x < \frac{1}{3}$

For real solution $x = -1 - \sqrt{2}$

1A**Question 3**

$$-2x + ky = m \Rightarrow y = \frac{2x + m}{k}$$

$$(1 + k^2)x + y = 2 \Rightarrow y = -(1 + k^2)x + 2$$

Equate gradients $\frac{2}{k} = -(1 + k^2)$

1M

$$1 + k^2 + \frac{2}{k} = 0$$

$$k^3 + k + 2 = 0$$

sing factor theorem

$$(k + 1)(k^2 - k + 2) = 0$$

$$\Delta < 0 \text{ for } k^2 - k + 2 \text{ giving no real solution}$$

$$k = -1$$

1MSubstitute $k = -1$ in equations

$$-2x - y = m$$

$$2x + y = 2$$

values of k and m for which there are no solutions

$$k = -1, m \neq -2$$

Answer $k = -1, m \in \mathbb{R} \setminus \{-2\}$

1A**Question 4**

$$2\sin^2(x) + \sin(x) = 1 \text{ for } x \in [-\pi, \pi]$$

$$2\sin^2(x) + \sin(x) - 1 = 0$$

$$(2\sin(x) - 1)(\sin(x) + 1) = 0$$

$$2\sin(x) = 1, \sin(x) = -1$$

1M

$$\sin(x) = \frac{1}{2} \text{ for } x \in [-\pi, \pi] \text{ gives}$$

$$\sin(x) = -1 \text{ for } x \in [-\pi, \pi] \text{ gives}$$

$$x = \frac{\pi}{6}, \frac{5\pi}{6} \quad \mathbf{1A}$$

$$x = -\frac{\pi}{2} \quad \mathbf{1A}$$

Question 5

$$f: \mathbb{R} \setminus \{2\} \rightarrow \mathbb{R}, f(x) = \frac{1}{(2-x)^2} \text{ and } g: [-2, k) \rightarrow \mathbb{R}, g(x) = 2x + 1$$

a. For $k = 1$, $g: [-2, 1) \rightarrow \mathbb{R}, g(x) = 2x + 1$

For $f \circ g$ to exist, test range $(g) \subseteq \text{dom}(f)$

$$[-3, 3) \not\subseteq \mathbb{R} \setminus \{2\}$$

 $f \circ g$ does not exist**1A**

b. For $f \circ g$ to exist

$$[-3, 2k+1) \in \mathbb{R} \setminus \{2\}$$

$$2k+1=2 \Rightarrow k=\frac{1}{2} \text{ such that } f \circ g \text{ exists} \quad \mathbf{1A}$$

c. From **part b.** $k=\frac{1}{2}$

$$\text{Dom } f \circ g: \left[-2, \frac{1}{2}\right)$$

$$\text{rule } (f \circ g)(x) = \frac{1}{(1-2x)^2} \quad \mathbf{1A} \text{ both rule and domain}$$

Question 6

$$\mathbf{a.} \int_2^3 \frac{3}{2x-1} dx = \frac{3}{2} \int_2^3 \frac{2}{2x-1} dx$$

$$= \frac{3}{2} [\log_e(2x-1)]_2^3 \quad \mathbf{1M}$$

$$= \frac{3}{2} (\log_e(5) - \log_e(3))$$

in the form $a \log_e(b)$ where $a, b \in \mathbb{Q}$

$$\int_2^3 \frac{3}{2x-1} dx = \frac{3}{2} \log_e\left(\frac{5}{3}\right) \quad \mathbf{1A}$$

b. Two trapeziums of width $\frac{1}{2}$

$$\text{Approximate area} = \frac{1}{2} \left(f(2) + 2f\left(\frac{5}{2}\right) + f(3) \right)$$

$$= \frac{1}{4} \left(\frac{3}{2(2)-1} + 2 \left(\frac{3}{2\left(\frac{5}{2}\right)-1} \right) + \frac{3}{2(3)-1} \right) \quad \mathbf{1M}$$

$$= \frac{1}{4} \left(1 + \frac{3}{2} + \frac{3}{5} \right)$$

$$= \frac{1}{4} \times \frac{31}{10}$$

$$\text{Approximate area} = \frac{31}{40} \quad \mathbf{1A}$$

$$\mathbf{c.} \frac{31}{40} > \frac{3}{2} \log_e\left(\frac{5}{3}\right)$$

Area in **part b.** is an over estimate of the actual area because the upper straight side of each trapezia sits above the concave upward curve. $\mathbf{1A}$

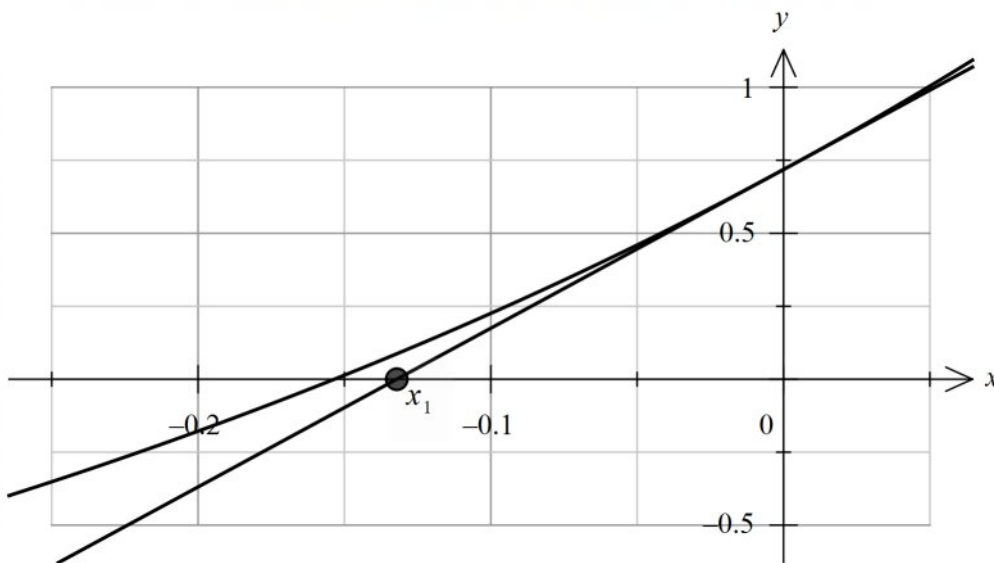
Question 7

$$f(x) = e^{2x+1} - 2$$

$$\text{a. } x_1 = x_0 - \frac{f(0)}{f'(0)}$$

$$x_1 = 0 - \frac{e^{2 \times 0 + 1} - 2}{2e^{2 \times 0 + 1}}$$

$$x_1 = -\frac{e-2}{2e} = \frac{2-e}{2e} = \frac{1}{e} - \frac{1}{2}$$

1A or equivalent form**b.** Tangent line with x_1 labelled.**1A**

$$\text{c. } e^{2x+1} - 2 = 0$$

$$e^{2x+1} = 2$$

$$2x+1 = \log_e(2)$$

$$x = \frac{\log_e(2) - 1}{2}$$

1A

$$\text{Distance} = \frac{2-e}{2e} - \frac{\log_e(2)-1}{2}$$

$$= \frac{2 - e \log_e(2)}{2e}$$

$$= \frac{1}{e} - \frac{\log_e(2)}{2}$$

$$= \frac{1}{e} + \log_e\left(\frac{1}{\sqrt{2}}\right) \quad \text{1H}$$

Question 8

$$f(x) = 1 + \sqrt{2x-3}$$

$$\text{a. } f'(x) = \frac{1}{\sqrt{2x-3}} = 1$$

$$\frac{1}{\sqrt{2x-3}} = 1$$

$$1 = \sqrt{2x-3}$$

$$2x-3=1$$

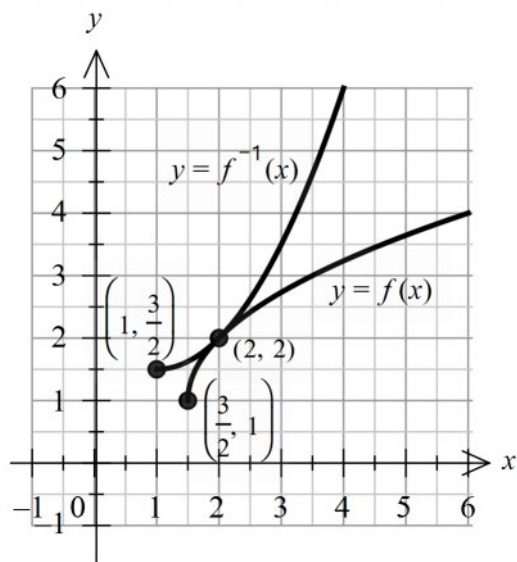
$$x=2$$

(2,2) coordinates **1A**

b. Shape $y = f(x)$ **1A**

Shape $y = f^{-1}(x)$ **1A**

Coordinates **1A**



$$\text{c. } g(x) = 1 + \sqrt{2x-a} = 1 + \sqrt{2\left(x - \frac{a}{2}\right)}$$

As the graph of f touches the line $y = x$ then $a < 3$.

The graphs of g will intersect the line $y = x$ twice until the coordinates of the endpoint are (1,1).

If the graph of f is translated $\frac{1}{2}$ a unit left it will intersect the line $y = x$ at (1,1).

$$g(x) = 1 + \sqrt{2\left(x - \frac{3}{2} + \frac{1}{2}\right)} = 1 + \sqrt{2(x-1)} = 1 + \sqrt{2x-2}$$

Hence $2 \leq a < 3$. **1A**

OR

$$\text{Solve } 1 + \sqrt{2x-a} = x$$

$$\sqrt{2x-a} = x-1$$

$$2x-a = (x-1)^2$$

$$2x-a = x^2 - 2x + 1$$

$$x^2 - 4x + 1 + a = 0$$

$$\Delta = 16 - 4(1+a) > 0$$

$$12 - 4a > 0$$

$$a < 3$$

Substitute $x=1$ into $x^2 - 4x + 1 + a = 0$ to find the minimum value of a .

$$a = 2$$

Hence $2 \leq a < 3$. **1A**

OR

Substitute (1,1) into $g(x) = 1 + \sqrt{2x - a}$

$$1 + \sqrt{2 - a} = 1$$

$$\sqrt{2 - a} = 0$$

$$a = 2$$

Hence $2 \leq a < 3$. **1A**

d. If the graph of f is translated $\frac{1}{2}$ a unit left then the points of intersection are at $x = 1$ and $x = 3$.

OR

Solve $1 + \sqrt{2x - 2} = x$

$$2x - 2 = (x - 1)^2$$

$$x^2 - 4x + 3 = 0$$

$$(x - 1)(x - 3) = 0$$

$$x = 1 \text{ or } x = 3$$

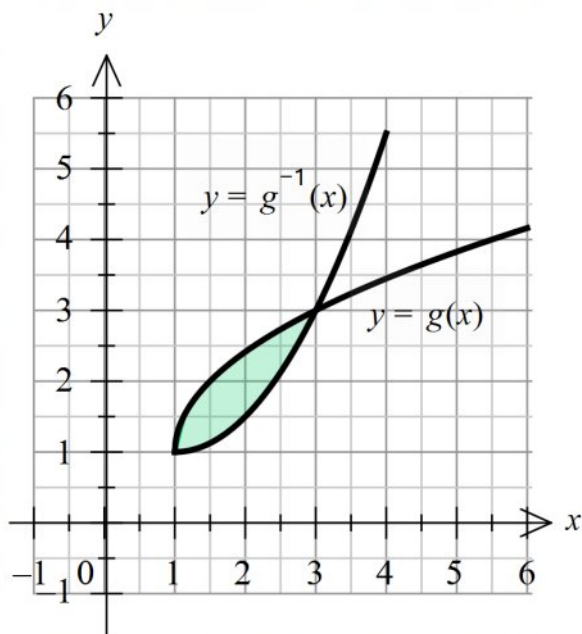
$$2 \int_1^3 (1 + \sqrt{2x - 2} - x) dx \quad \mathbf{1M}$$

$$= 2 \left[x + \frac{1}{3} (2x - 2)^{\frac{3}{2}} - \frac{1}{2} x^2 \right]_1^3 \quad \mathbf{1M}$$

$$= 2 \left(\left(3 + \frac{8}{3} - \frac{9}{2} \right) - \left(1 - \frac{1}{2} \right) \right)$$

$$= 2 \left(-2 + \frac{8}{3} \right)$$

$$= \frac{4}{3} \quad \mathbf{1A \text{ or equivalent form}}$$



Question 9

a. 0.02^3

$$= 0.000\ 008 \text{ or } \frac{1}{125\ 000} \quad \mathbf{1A}$$

b. $X \sim \text{Bi}(4, 0.3)$ **1M**

$$\Pr(X = 4) = \binom{4}{3} 0.3^3 \times 0.7$$

$$= 4 \times 0.027 \times 0.7$$

$$= 0.0756 \text{ or } \frac{189}{2500} \quad \mathbf{1A}$$

c. $2 \times 0.02 \times 0.98$

$$= 0.0392 \text{ or } \frac{49}{1250} \quad \mathbf{1A}$$

d. $0.02n_t = 0.3n_s$

$$n_t = 15n_s, \ n_s \in \mathbb{Z}^+ \quad \mathbf{1A}$$

$$\mathbf{e.} \ d(t) = \begin{cases} t-1 & 1 \leq t \leq 2 \\ a(t-3)^2 + b & 2 < t \leq 3 \\ 0 & \text{elsewhere} \end{cases}$$

$$a(t-3)^2 + b = t-1 \text{ when } t=2$$

$$a+b=1 \quad (1) \quad \mathbf{1A}$$

$$\int_1^2 (t-1) dt + \int_2^3 (a(t-3)^2 + b) dt = 1$$

$$\left[\frac{t^2}{2} - t \right]_1^2 + \left[\frac{a}{3}(t-3)^3 + bt \right]_2^3 = 1$$

$$\frac{1}{2} + \left[\frac{a}{3}(t-3)^3 + bt \right]_2^3 = 1$$

$$\left[\frac{a}{3}(t-3)^3 + bt \right]_2^3 = \frac{1}{2}$$

$$b + \frac{a}{3} = \frac{1}{2} \quad (2) \quad \mathbf{1M} \text{ attempt to set up the second equation}$$

$$a+b=1 \quad (1)$$

Subtract (2) from (1)

$$\frac{2}{3}a = \frac{1}{2}$$

$$a = \frac{3}{4}, \ b = \frac{1}{4} \quad \mathbf{1A}$$

END OF SOLUTIONS