2024
VCE
Mathematical Methods
Year 12
Trial Examination 2
Detailed Answers

Kilbaha Education

Quality educational content

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SECTION A

ANSWERS

1		**A**		B		C		D
2		A		B		**C**		D
3		**A**		B		C		D
4		A		**B**		C		D
5		A		B		**C**		D
6		A		**B**		C		D
7		A		B		C		**D**
8		**A**		B		C		D
9		A		B		**C**		D
10		**A**		B		C		D
11		A		B		C		**D**
12		A		**B**		C		D
13		A		**B**		C		D
14		A		**B**		C		D
15		A		B		C		**D**
16		A		B		**C**		D
17		A		B		C		**D**
18		A		B		C		**D**
19		A		B		**C**		D
20		**A**		B		C		D

SECTION A

Question 1 **Answer A**

Both functions have periods $T=\dfrac{2\pi}{\dfrac{\pi}{c}}=2c$, both functions have an amplitude of b,

both functions have a range of $[a-b, a+b]$

Question 2 **Answer C**

$f(x)=\log_e(b-x)$ domain $b-x>0,\quad x<b$

$g(x)=\sqrt{x+b}$ domain $x+b\ge 0,\quad x\ge -b$,

domain of $\dfrac{f}{g}=-b<x<b=(-b,b)$ since $g(x)\ne 0$

Define $f(x)=\ln(b-x)$	*Done*
Define $g(x)=\sqrt{x+b}$	*Done*
$\text{domain}\left(\dfrac{f(x)}{g(x)},x\right)$	$-b<x<b$

Question 3 **Answer A**

$\int_0^{4a} f(x)\,dx$

$=\int_0^{2a} f(x)\,dx+\int_{2a}^{4a} f(x)\,dx$

$=\int_0^{2a} f(x)\,dx-\int_{4a}^{2a} f(x)\,dx$

$=\int_0^{2a} f(x)\,dx-\int_{4a}^{2a} f(u)\,du$ let $u=4a-x$

$=\int_0^{2a} f(x)\,dx+\int_0^{2a} f(4a-x)\,dx=\int_0^{2a}\left(f(x)+f(4a-x)\right)dx$

Define $f(x)=x^3$	*Done*
$\int_0^{4\cdot a} f(x)\,dx$	$64\cdot a^4$
$\int_0^{2\cdot a} (f(x)+f(4\cdot a-x))\,dx$	$64\cdot a^4$

reflect in the y-axis

and translate $4a$ units to the right

Question 4 **Answer B**

x	1	2	3	4
$f(x)=\dfrac{1}{x}$	1	$\dfrac{1}{2}$	$\dfrac{1}{3}$	$\dfrac{1}{4}$

$a=1,\quad b=4,\quad n=3.\quad h=\dfrac{b-a}{n}=1$

$A_T=\dfrac{h}{2}\left(f(1)+2\left(f(2)+f(3)\right)+f(4)\right)=\dfrac{1}{2}\left(1+2\left(\dfrac{1}{2}+\dfrac{1}{3}\right)+\dfrac{1}{4}\right)=\dfrac{35}{24}$

Question 5 **Answer C**

Consider the function $f : R \to R$, $f(x) = x + 2$.

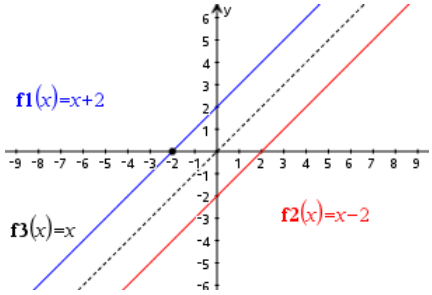
$f :\ y = x + 2$

$f^{-1} : x = y + 2$

$f^{-1}(x) = x - 2$

$f(x) = f^{-1}(x) \Rightarrow x + 2 = x - 2$

This equation is inconsistent, there is no solution, with $y = x$ as all three lines are parallel. Colin is correct.



Consider the function $f : R \to R$, $f(x) = -x^3$.

$f :\quad y = -x^3$

$f^{-1} :\quad x = -y^3\ ,\ y = -x^{\frac{1}{3}} = -\sqrt[3]{x}$

both the function *f* and its inverse have domain and range *R*.

solving $\quad f(x) = f^{-1}(x) \quad \Rightarrow \quad (1) \quad -x^3 = -\sqrt[3]{x}$

$(1) \quad x^9 = x \ \Rightarrow x = 0\ , \pm 1$

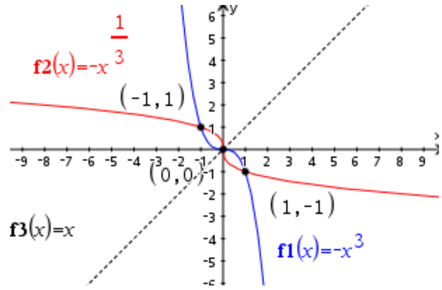
There are three points of intersection between the function and its inverse, with coordinates $(-1,1)$, $(1,-1)$ and the origin $(0,0)$. Only the point at the origin only lies on the line $y = x$. Ben is correct.

Define $f1(x)=-x^3$ Done

solve$(f1(y)=x,y)$ $y=-x^{\frac{1}{3}}$

Define $f2(x)=-x^{\frac{1}{3}}$ Done

solve$(f1(x)=f2(x),x)$ $x=-1$ or $x=0$ or $x=1$



Other functions are possible to show that both Ben and Colin are correct.

Question 6 **Answer B**

BR or RB, 2 ways of drawing one red and one blue, without replacement

Box A: $\dfrac{1}{2}\left[\dfrac{(r-1)(b+1)}{(b+r)(b+r-1)} + \dfrac{(b+1)(r-1)}{(b+r)(b+r-1)}\right] = \dfrac{(r-1)(b+1)}{(b+r)(b+r-1)}$

Box B: $\dfrac{1}{2}\left[\dfrac{(r+1)(b-1)}{(b+r)(b+r-1)} + \dfrac{(b-1)(r+1)}{(b+r)(b+r-1)}\right] = \dfrac{(r+1)(b-1)}{(b+r)(b+r-1)}$

Box A or B $\quad \dfrac{(r-1)(b+1)}{(b+r)(b+r-1)} + \dfrac{(r+1)(b-1)}{(b+r)(b+r-1)} = \dfrac{2(br-1)}{(b+r)(b+r-1)}$

$$\frac{(r-1)\cdot(b+1)+(r+1)\cdot(b-1)}{(r+b)\cdot(r+b-1)} \qquad \frac{2\cdot(b\cdot r-1)}{(r+b)\cdot(r+b-1)}$$

Question 7 **Answer D**

When $n=2$, $y=\sqrt{x-a}$, $\dfrac{dy}{dx}=\dfrac{1}{2\sqrt{x-a}}$

When $n=-3$,

$y=\dfrac{1}{\sqrt[3]{x-a}}=(x-a)^{-\frac{1}{3}}$, $\dfrac{dy}{dx}=-\dfrac{1}{3}(x-a)^{-\frac{4}{3}}$

When $n=-\dfrac{1}{2}$, $y=\dfrac{1}{(x-a)^2}$, $\dfrac{dy}{dx}=-\dfrac{2}{(x-a)^3}$

All of **A. B** and **C.** are not differentiable at $x=a$.

When $n=\dfrac{1}{2}$, $y=(x-a)^2$ is differentiable at $x=a$

Define $f(x)=(x-a)^{\frac{1}{n}}\,\vert n=2$	*Done*
$\frac{d}{dx}(f(x))\vert x=a$	undef
Define $f(x)=(x-a)^{\frac{1}{n}}\,\vert n=-3$	*Done*
$\frac{d}{dx}(f(x))\vert x=a$	undef
Define $f(x)=(x-a)^{\frac{1}{n}}\,\vert n=\frac{-1}{2}$	*Done*
$\frac{d}{dx}(f(x))\vert x=a$	undef
Define $f(x)=(x-a)^{\frac{1}{n}}\,\vert n=\frac{1}{2}$	*Done*
$\frac{d}{dx}(f(x))\vert x=a$	0

Question 8 **Answer A**

$$y=4\tan\left(2\left(x-\frac{\pi}{3}\right)\right)=\frac{4\sin\left(2\left(x-\frac{\pi}{3}\right)\right)}{\cos\left(2\left(x-\frac{\pi}{3}\right)\right)}$$

crosses the x-axis when $y=0$ so $\sin\left(2\left(x-\frac{\pi}{3}\right)\right)=0$

has vertical asymptotes when $\cos\left(2\left(x-\frac{\pi}{3}\right)\right)=0$

$\text{solve}\left(\sin\left(2\cdot\left(x-\frac{\pi}{3}\right)\right)=0,x\right)$	$x=\frac{(3\cdot \mathbf{n1}-1)\cdot\pi}{6}$	$x=\frac{(3\cdot(k+1)-1)\cdot\pi}{6}$	$x=\frac{(3\cdot k+2)\cdot\pi}{6}$
$\text{solve}\left(\cos\left(2\cdot\left(x-\frac{\pi}{3}\right)\right)=0,x\right)$	$x=\frac{(6\cdot \mathbf{n2}-5)\cdot\pi}{12}$	$x=\frac{(6\cdot(k-1)-5)\cdot\pi}{12}$	$x=\frac{(6\cdot k-11)\cdot\pi}{12}$

Question 9 **Answer C**

$$\Pr(X \geq 2) = 1 - \left[\Pr(X=1) + \Pr(X=0)\right] = 1 - \left[\binom{10}{1} 0.37 \times 0.63^9 + 0.63^{10}\right]$$

$n = 10,\ q = 0.63,\ p = 0.37$, at least two successes in ten trials each with a probability of 0.37

Question 10 **Answer A**

$$\left(\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\ ,\ \hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$$

z95:=invNorm(0.975,0,1)	1.9600
z99:=invNorm(0.995,0,1)	2.5758

$CI:\quad 99\%,\quad z = 2.5758 \quad \left(\hat{p} - 2.5758\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\ ,\ \hat{p} + 2.5758\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right) = (a, b)$

$(1)\ a = \hat{p} - 2.5758\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \qquad (2)\ b = \hat{p} + 2.5758\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

$\frac{1}{2}((1)+(2)) \quad \hat{p} = \frac{a+b}{2}, \qquad \frac{1}{2}((2)-(1)) \quad \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \frac{b-a}{2 \times 2.5758}$

$CI:\quad 95\%,\quad z = 1.96$

$$\left(\hat{p} - 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\ ,\ \hat{p} + 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$$

$$\left(\frac{a+b}{2} - \frac{1.96(b-a)}{2 \times 2.5758}\ ,\ \frac{a+b}{2} + \frac{1.96(b-a)}{2 \times 2.5758}\right)$$

$$\left(\frac{a}{2}\left(1 + \frac{1.96}{2.5758}\right) + \frac{b}{2}\left(1 - \frac{1.96}{2.5758}\right)\ ,\ \frac{a}{2}\left(1 - \frac{1.96}{2.5758}\right) + \frac{b}{2}\left(1 + \frac{1.96}{2.5758}\right)\right)$$

$(0.88a + 0.12b,\ 0.12a + 0.88b)$

zInterval_1Prop 20,100,0.99: *stat.results*

"Title"	"1−Prop z Interval"
"CLower"	0.097
"CUpper"	0.303
"p̂"	0.200
"ME"	0.103
"n"	100.000

0.88· *stat.CLower*+0.12· *stat.CUpper*	0.122
0.12· *stat.CLower*+0.88· *stat.CUpper*	0.278

zInterval_1Prop 20,100,0.95: *stat.results*

"Title"	"1−Prop z Interval"
"CLower"	0.122
"CUpper"	0.278
"p̂"	0.200
"ME"	0.078
"n"	100.000

Question 11 **Answer D**

$f(x)=g(x)e^{g(x)}$ using the product rule

$$f'(x)=\frac{d}{dx}\left(g(x)\right)\times e^{g(x)}+g(x)\frac{d}{dx}\left(e^{g(x)}\right)$$
$$f'(x)=g'(x)e^{g(x)}+g(x)g'(x)e^{g(x)}$$
$$f'(x)=g'(x)e^{g(x)}\left(1+g(x)\right)$$
$$f'(2)=g'(2)e^{g(2)}\left(1+g(2)\right)=3e^4(1+4)$$
$$f'(2)=15e^4$$

Question 12 **Answer B**

$$f(x)=\int g(x)\,dx \quad \Rightarrow \quad g(x)=\frac{d}{dx}\left(f(x)\right)=f'(x)$$
$$h(x)=\frac{d}{dx}\left(g(x)\right)=\frac{d}{dx}\left(f'(x)\right)$$

Question 13 **Answer B**

$n=18,\quad \hat{P}=\frac{X}{18} \qquad X \stackrel{d}{=} Bi(n=18, p=?)$

$$\Pr\left(\hat{P}=\frac{1}{3}\right)=\Pr(X=6)=\binom{18}{6}p^6(1-p)^{12}=0.1873$$

solving gives $p=0.3$,

$$\Pr\left(\hat{P}<\frac{1}{2}\right)=\Pr(X<9)$$
$$=\Pr(X\le 8)=0.940$$

solve$\left(\text{nCr}(18,6)\cdot p^6\cdot(1-p)^{12}=0.1873,p\right)\mid 0<p<0.35$	
	$p=0.3000$
binomCdf$(18,0.3,0,8)$	0.9404

Question 14 **Answer B**

If $f(x)$ is a non-zero odd function, then $f(-x)=-f(x)$

Let $f(x)=x^3$, $f(-x)=(-x)^3=-x^3=-f(x)$

$f_o f(x)=f(f(x))=f(x^3)=x^9$ which is an odd function **B.** is true.

All of **A. C.** and **D.** are false.

Define $f(x)=x^3$	*Done*
$f(-x)=-f(x)$	true
$f(f(-x))=-f(f(x))$	true
Define $a(x)=f(x^2)\cdot\cos(x)$	*Done*
$a(-x)=-a(x)$	$x^6\cdot\cos(x)=-x^6\cdot\cos(x)$
Define $c(x)=f(x^3)\cdot\sin(x)$	*Done*
$c(-x)=-c(x)$	$x^9\cdot\sin(x)=-x^9\cdot\sin(x)$
Define $d(x)=f(x^2)-f(x)$	*Done*
$d(-x)=-d(x)$	$x^6+x^3=x^3-x^6$

Question 15 **Answer D**

$\sum \Pr(X=x)=1$ for all p, however since they are probabilities $0<1-\frac{5p}{6}<1 \Rightarrow 0<p<\frac{6}{5}$

$$E(X)=\sum x\Pr(X=x)=1\times\frac{p}{2}+2\times\frac{p}{3}+3\left(1-\frac{5p}{6}\right)=\frac{1}{3}(9-4p)$$

$$E(X^2)=\sum x^2\Pr(X=x)=1\times\frac{p}{2}+4\times\frac{p}{3}+9\left(1-\frac{5p}{6}\right)=\frac{1}{3}(27-17p)$$

$$\mathrm{Var}(X)=E(X^2)-(E(X))^2=\frac{p}{9}(21-16p)=\frac{1}{9}(21p-16p^2)$$

for maximum variance $\frac{dV}{dp}=\frac{1}{9}(21-32p)=0,\quad p=\frac{21}{32}$

A. B. C. are all true, **D.** is false

A xv	B pv
1	p/2
2	p/3
3	1−5*p/6

$ex:=\mathrm{sum}(xv\cdot pv)$ $\quad 3-\frac{4\cdot p}{3}$

$ex2:=\mathrm{sum}(xv^2\cdot pv)$ $\quad 9-\frac{17\cdot p}{3}$

$vx:=ex2-ex^2$ $\quad \frac{7\cdot p}{3}-\frac{16\cdot p^2}{9}$

$\frac{d}{dp}(vx)$ $\quad \frac{7}{3}-\frac{32\cdot p}{9}$

$\mathrm{solve}\left(\frac{d}{dp}(vx)=0,p\right)$ $\quad p=\frac{21}{32}$

Question 16 **Answer C**

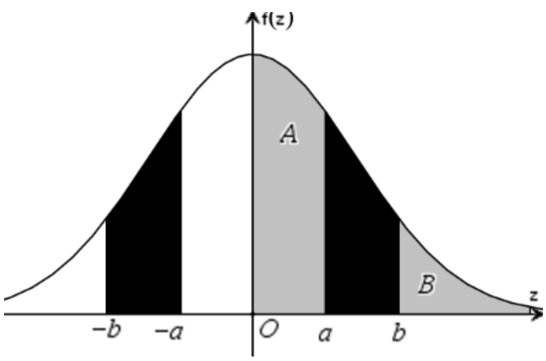
$$\Pr(-b<Z<-a\mid Z<0)$$

$$=\frac{\Pr(-b<Z<-a)}{\Pr(Z<0)}=\frac{\Pr(a<Z<b)}{\Pr(Z>0)} \quad \text{since } 0<a<b<3$$

$$=\frac{0.5-\left[\Pr(0<Z<a)+\Pr(Z>b)\right]}{0.5}$$

$$=\frac{0.5-(A+B)}{0.5}$$

$$=1-2(A+B)$$



Question 17 **Answer D**

$3x-2y+z=1$

$-x+y-z=2$

Allan stated that the general solution could be expressed as
$x=k,\ y=2k-3,\ z=k-5$ for $k\in R$.

Ben stated that the general solution could be expressed as
$x=\dfrac{k+3}{2},\ y=k,\ z=\dfrac{k-7}{2}$ for $k\in R$.

Colin stated that the general solution could be expressed as
$x=k+5,\ y=2k+7,\ z=k$ for $k\in R$.

All of Allan, Ben and Colin are correct.

$eq1:=3\cdot x-2\cdot y+z=1$	$3\cdot x-2\cdot y+z=1$
$eq2:=^-x+y-z=2$	$^-x+y-z=2$
$\text{solve}(eq1 \text{ and } eq2,\{y,z\})\vert x=k$	$y=2\cdot k-3 \text{ and } z=k-5$
$\text{solve}(eq1 \text{ and } eq2,\{x,z\})\vert y=k$	$x=\dfrac{k+3}{2} \text{ and } z=\dfrac{k-7}{2}$
$\text{solve}(eq1 \text{ and } eq2,\{x,y\})\vert z=k$	$x=k+5 \text{ and } y=2\cdot k+7$

Question 18 **Answer D**

Newton's method will succeed if the gradient is defined and non-zero at x_0. Only **D.** has $f'(x_0)\neq 0$

$f(x)=\sqrt{3x^2-7x+2},\quad f'(2)$ is not defined

$f(x)=3x^3-13x^2+16x-4,\quad f'(2)=0$

$f(x)=(3x-1)\log_e(x-2),\quad f'(2)$ is not defined

$f(x)=(3x-1)e^{x-2},\quad f'(2)\neq 0$

Define $fa(x)=\sqrt{3\cdot x^2-7\cdot x+2}$	*Done*
$\dfrac{d}{dx}(fa(x))\vert x=2$	undef
Define $fb(x)=3\cdot x^3-13\cdot x^2+16\cdot x-4$	*Done*
$\dfrac{d}{dx}(fb(x))\vert x=2$	0

Define $fc(x)=(3\cdot x-1)\cdot \ln(x-2)$	*Done*
$\dfrac{d}{dx}(fc(x))\vert x=2$	undef
Define $fd(x)=(3\cdot x-1)\cdot \mathbf{e}^{x-2}$	*Done*
$\dfrac{d}{dx}(fd(x))\vert x=2$	8

Question 19 **Answer C**

iterations	$xleft$	$xmid$	$xright$	$f(xleft).f(xmid)$
0	2	2.5	3	–1.25
1	2	2.25	2.5	–0.0625
2	2	2.125	2.25	0.4844
3	2.125	2.1875	2.25	0.1041

xleft	xmid	xright	f(xleft)*f(xmid)
2.0000	2.5000	3.0000	-1.2500
2.0000	2.2500	2.5000	-0.0625
2.0000	2.1250	2.2500	0.4844
2.1250	2.1875	2.2500	0.1041

```
Define bisection()=
Prgm
maxiter:=4
xleft:=2
xright:=3
Define f(x)=x^2-5
If f(xleft)·f(xright)>0 Then
  Disp "starting values will not converge"
  Return
EndIf
i:=0
Disp " xleft    xmid    xright   f(xleft)· f(xmid)"
While i<maxiter
   xmid:=(xleft+xright)/2
   Disp xleft,xmid,xright,"   ",f(xleft)·f(xmid)
   If f(xleft)·f(xmid)<0 Then
     xright:=xmid
   Else
     xleft:=xmid
   EndIf
 i:=i+1
EndWhile
EndPrgm
```

Question 20 **Answer A**

$\sin^2(x)=\frac{a}{c},\quad \sin(x)=\sqrt{\frac{a}{c}}\ ,\ \cos^2(y)=\frac{b}{c},\quad \cos(y)=\sqrt{\frac{b}{c}}$

since $0<a<b<c<1$, $\sin(x)>0$ and $\cos(y)>0$

$$\log_2(\sin(x)\cos(y))$$

$$=\log_2\left(\sqrt{\frac{a}{c}}\sqrt{\frac{b}{c}}\right)=\log_2\left(\frac{\sqrt{ab}}{c}\right)=\log_2\left(\sqrt{ab}\right)-\log_2(c)$$

$$=\log_e\left((ab)^{\frac{1}{2}}\right)-\log_2(c)=\frac{1}{2}(\log_2(ab))-\log_2(c)$$

$$=\frac{1}{2}(\log_2(a)+\log_2(b))-\log_2(c)$$

END OF SECTION A SUGGESTED ANSWERS

SECTION B

Question 1

a. $f: R \to R, \quad f(x) = x^4 - 4x^3 + 3$

$m(x) = f'(x) = 4x^2(x-3) = 0$

for turning points

$x = 0, \; x = 3, \quad f(3) = -24$

$(3, -24)$ is an absolute minimum turning point A1

b. $m'(x) = 12x^2 - 24x$

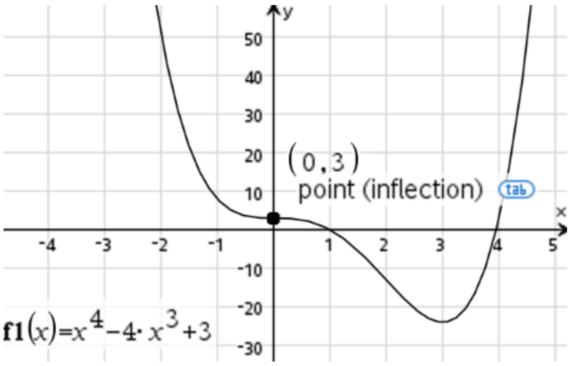
$m'(x) = 12x(x-2) = 0$ for inflection points

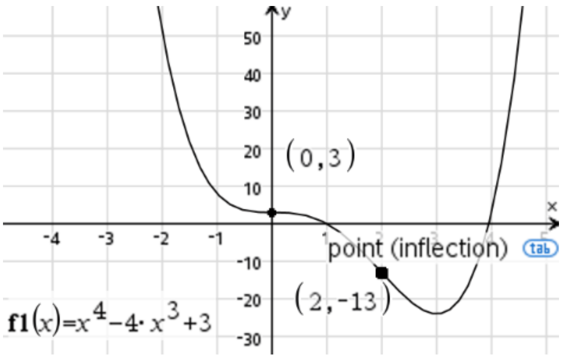
$x = 0, 2$

$f(0) = 3, \quad f(2) = -13$

$(0, 3)$ is a stationary point of inflection

$(2, -13)$ is a point of inflexion A1





c. at $x = 2 \quad f(2) = -13, \; f'(2) = -16$

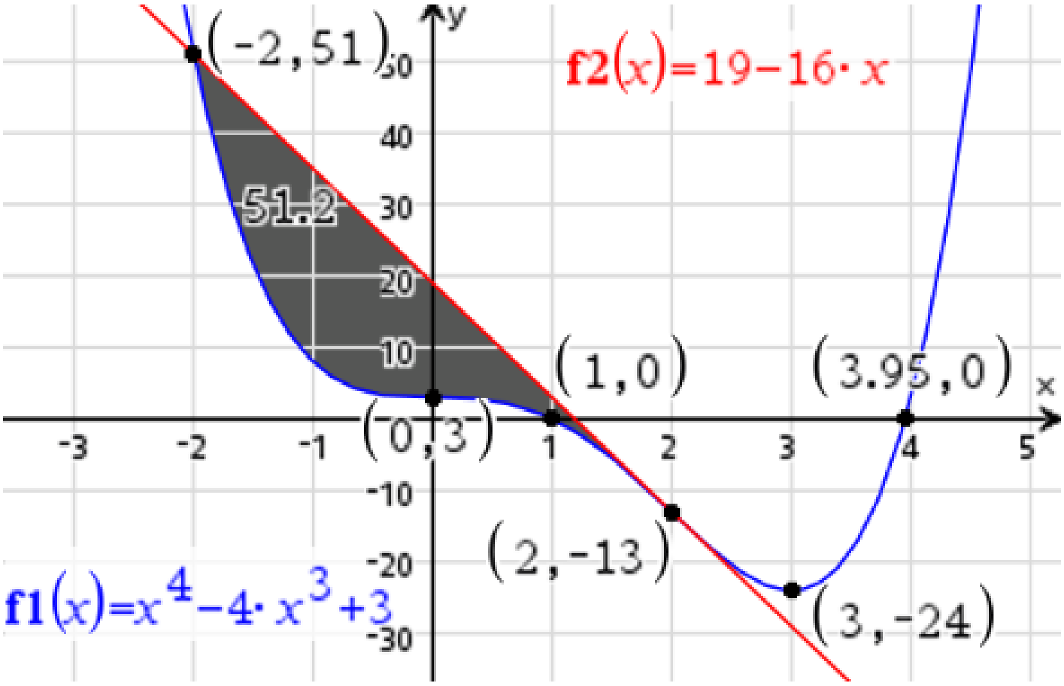
the tangent line at $x = 2$ is

$y + 13 = -16(x-2) = -16x + 32$

$y = g(x) = -16x + 19$ A1

note the curve crosses the tangent at the point of inflection.

d. G2



e. $g(x)-f(x)$

$=(-16x+19)-(x^4-4x^3+3)$

$=-x^4+4x^3-16x+16$ M1

$=-(x+2)(x-2)^3=(x+2)(2-x)^3$

$g(x)=f(x), \ \Rightarrow x=\pm 2$

i. the area is $A=\int_{-b}^{b}\left(g(x)-f(x)\right)dx$

$A=\int_{-2}^{2}(x+2)(2-x)^3\,dx$ A1

$b=2, \quad n=3$

ii. $A=\dfrac{256}{5}=51\dfrac{1}{5}=51.2$ A1

f. solving $\dfrac{f(c)-f(1)}{c-1}=-12, \ c>1$ gives $c=\sqrt{3}$ or $c=3$ A1

g. $h:R\to R, \ \ h(x)=f(x)+k$, need to translate the graph up by 24 or more units,

so that for the graph to not cross the x-axis, require $k>24$ or $k\in(24,\infty)$ A1

Define $f1(x)=x^4-4\cdot x^3+3$	*Done*
$\text{factor}(f1(x))$	$(x-1)\cdot(x^3-3\cdot x^2-3\cdot x-3)$
$\text{solve}(f1(x)=0,x)$	$x=1.00000$ or $x=3.95137$
$\text{solve}\left(\frac{d}{dx}(f1(x))=0,x\right)$	$x=0$ or $x=3$
$f1(3)$	-24
$f1(2)$	-13
$\text{tangentLine}(f1(x),x,2)$	$19-16\cdot x$
Define $f2(x)=19-16\cdot x$	*Done*
$\text{solve}(f1(x)=f2(x),x)$	$x=-2$ or $x=2$
$f2(x)-f1(x)$	$-x^4+4\cdot x^3-16\cdot x+16$
$\text{factor}(f2(x)-f1(x))$	$-(x-2)^3\cdot(x+2)$
$\int_{-2}^{2}\left(-(x-2)^3\cdot(x+2)\right)dx$	51.20000
$\text{solve}\left(\frac{f1(c)-f1(1)}{c-1}=-12,c\right)\vert c>1$	$c=\sqrt{3}$ or $c=3$

Question 2 $f: R \to R,\quad f(x) = e^{-x^2}$

a.i. $A(a) = 2af(a) = 2ae^{-a^2}$

$\dfrac{dA}{da} = \left(2 - 4a^2\right)e^{-a^2} = 0$ for maximum area $a = \dfrac{\sqrt{2}}{2}$ M1

ii. $A_{\max} = A\left(\dfrac{\sqrt{2}}{2}\right) = \sqrt{\dfrac{2}{e}}$ A1

iii. the inflexion points are $(\pm 0.7071, 0.6065)$ $a = \dfrac{\sqrt{2}}{2} \approx 0.7071$

yes Jenny's assertion is correct. A1

b.i. $s(a) = \sqrt{a^2 + \left(f(a)\right)^2} = \sqrt{a^2 + e^{-2a^2}}$

$\dfrac{ds}{da} = 0$ for minimum area $a = \dfrac{1}{2}\sqrt{\log_e(4)}$ A1

ii. $s_{\min} = s\left(\dfrac{1}{2}\sqrt{\log_e(4)}\right) = \dfrac{1}{2}\sqrt{\log_e(4) + 2}$ A1

c. $f(x) = e^{-x^2} \quad \to \quad f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)^2}$ A1

$x = \dfrac{x' - \mu}{\sqrt{2}\sigma},\quad x' = \sqrt{2}\sigma x + \mu,\quad \mu > 0$ A2

- dilate by a factor of $\dfrac{1}{\sigma\sqrt{2\pi}}$ parallel to the *y*-axis (or away from the *x*-axis)
- dilate by a factor of $\sqrt{2}\sigma$ parallel to the *x*-axis (or away from the *y*-axis)
- translate by a factor of μ to the right parallel to the *x*-axis (or away from the *y*-axis)

Define $f1(x)=e^{-x^2}$	Done
Define $r(a)=2\cdot a\cdot f1(a)$	Done
$\frac{d}{da}(r(a))$	$(2-4\cdot a^2)\cdot e^{-a^2}$
$\text{solve}\left(\frac{d}{da}(r(a))=0,a\right)\mid a>0$	$a=\frac{\sqrt{2}}{2}$
$r\left(\frac{\sqrt{2}}{2}\right)$	$e^{\frac{-1}{2}}\cdot\sqrt{2}$
$\frac{d^2}{dx^2}(f1(x))$	$(4\cdot x^2-2)\cdot e^{-x^2}$
$\text{solve}\left(\frac{d^2}{dx^2}(f1(x))=0,x\right)$	$x=\frac{-\sqrt{2}}{2}$ or $x=\frac{\sqrt{2}}{2}$

Define $s(a)=\sqrt{a^2+(f1(a))^2}$	Done
$\text{solve}\left(\frac{d}{da}(s(a))=0,a\right)\mid a>0$	$a=\frac{\sqrt{2\cdot\ln(2)}}{2}$
$s\left(\frac{\sqrt{2\cdot\ln(2)}}{2}\right)$	$\frac{\sqrt{2\cdot(\ln(2)+1)}}{2}$

Question 3

a.i. $P \overset{d}{=} Bi\left(n=50,\ p=0.46\right)$

$\text{binomCdf}(50,0.46,26,50) \quad 0.238643$

$\Pr\left(P>25\right)$

$=\Pr\left(26 \le P \le 50\right)=0.2386$ A1

ii. $E \overset{d}{=} Bi\left(n=?,\ p=0.07\right)$

$\Pr\left(E \ge 2\right) \ge 0.3$

$1-\Pr\left(E \le 1\right) \le 0.7$

$1-\left(\Pr\left(E=0\right)+\Pr\left(E=1\right)\right) \le 0.7$

$\Pr\left(E=0\right)+\Pr\left(E=1\right) \ge 0.3$

$0.93^n + n \times 0.93^{n-1} \times 0.07 \ge 0.3$

$n=16$ A1

$\text{solve}\left((0.93)^n+n\cdot(0.03)^{n-1}\cdot 0.07=0.31,n\right) \quad n=16.1385$

$\text{binomCdf}(n,0.07,2,n)|n=15 \quad 0.2832$

$\text{binomCdf}(n,0.07,2,n)|n=16 \quad 0.3098$

$\text{invBinomN}(0.7,0.07,1,1) \quad \begin{bmatrix} 15 & 0.7168 \\ 16 & 0.6902 \end{bmatrix}$

b. $\Pr\left(EH \mid NB\right)=\dfrac{\Pr\left(EH \cap NB\right)}{\Pr\left(NB\right)}=\dfrac{23}{177}$

$=\dfrac{0.23(1-b)}{0.23(1-b)+0.77 \times 0.6}=\dfrac{23}{177}$ M1

$b=0.7, \quad 70\%$ A1

c.i. $B \overset{d}{=} N\left(96,8^2\right)$ time in months

$\Pr\left(B>120 \mid B \ge 108\right)$ M1

$=\dfrac{\Pr\left(B>120\right)}{\Pr\left(B \ge 108\right)}=\dfrac{0.00135}{0.0668}$

$\dfrac{\text{normCdf}(120,\infty,96,8)}{\text{normCdf}(108,\infty,96,8)} \quad 0.0202$

$=0.0202$ A1

ii. $\Pr\left(B>t\right)=0.8$

$\dfrac{t-96}{8}=0.8416$

$t=102.73$ months

8.56 years A1

$\dfrac{\text{invNorm}(0.8,96,8)}{12} \quad 8.5611$

iii. $S \overset{d}{=} Bi\left(n=10,\ p=0.0668\right)$ A1

$\Pr\left(S>2\right)$

$=\Pr\left(3 \le S \le 10\right)=0.0251$ A1

$p:=\text{normCdf}(108,\infty,96,8) \quad 0.0668$

$\text{binomCdf}(10,p,3,10) \quad 0.0251$

d. White: $\hat{p}_W = 0.34$, 95%, $z = 1.96$

$(1)\ CL_W = 0.34 - 1.96\sqrt{\dfrac{0.34(1-0.34)}{n_W}}$

Black: $\hat{p}_B = 0.21$, 99%, $z = 2.576$ M1

$(2)\ CU_B = 0.21 + 2.576\sqrt{\dfrac{0.21(1-0.21)}{n_B}}$ A1

solving $(1)=(2)$ $CL_W = CU_B$ with $n_B = 3n_W$

gives $n_W = 139$ A1

$z95 := \text{invNorm}(0.975,0,1)$ 1.9600

$z99 := \text{invNorm}(0.995,0,1)$ 2.5758

$clw := 0.34 - z95\cdot\sqrt{\dfrac{0.34\cdot(1-0.34)}{nw}}$ $0.3400 - 0.9285\cdot\sqrt{\dfrac{1}{nw}}$

$clu := 0.21 + z99\cdot\sqrt{\dfrac{0.21\cdot(1-0.21)}{nb}}$ $1.0492\cdot\sqrt{\dfrac{1}{nb}} + 0.2100$

$\text{solve}(clw = clu, nw)|nb = 3\cdot nw$ $nw = 139.2732$

e. $T \overset{d}{=} N(\mu = ?, \sigma^2 = ?)$ time in years

$\Pr(T > 7) = 0.86$

$\Pr(T < 7) = 0.14$

$(1)\ \dfrac{7-\mu}{\sigma} = -1.0803$ M1

$\text{invNorm}(0.14,0,1)$ -1.0803

$\text{invNorm}(0.05,0,1)$ -1.6449

$\Pr(T < 5) = 0.05$

$(2)\ \dfrac{5-\mu}{\sigma} = -1.6449$

$\text{solve}\left(\dfrac{7-m}{s} = -1.0803 \text{ and } \dfrac{5-m}{s} = -1.6449, \{m,\blacktriangleright\right.$

$s = 3.5423$ and $m = 10.8268$

solving $(1),(2)$ $\mu = 10.8$, $\sigma = 3.5$ A1

f.
$$f(t) = \begin{cases} b(2t-1) & \text{for } \dfrac{1}{2} \le t \le 1 \\ \dfrac{b}{\left(t-\dfrac{1}{2}\right)^2} & \text{for } 1 < t \le 4 \\ 0 & \text{elsewhere} \end{cases}$$

since it is a probability density function $\int_{0.5}^{4} f(t)\,dt = 1$ solving gives $b = \dfrac{28}{55}$

$E(T) = \int_{0.5}^{4} t\, f(t)\,dt = 1.5331$

$E(T^2) = \int_{0.5}^{4} t^2\, f(t)\,dt = 2.8263$

$sd(T) = \sqrt{E(T^2) - (E(T))^2} = 0.6899$ M1

$E(T) + 2sd(T) = 2.9129$

$E(T) - 2sd(T) = 0.1533 < 0.5$

$\Pr(0.5 \le T \le E(T) + 2sd(T))$

$= \Pr(0.5 \le T \le 2.9129)$

$= \int_{0.5}^{2.9129} f(t)\,dt = 0.9345$ A1

$$\text{Define } f(t) = \begin{cases} b\cdot(2\cdot t-1), & \frac{1}{2} \le t < 1 \\ \dfrac{b}{\left(t-\frac{1}{2}\right)^2}, & 1 \le t \le 4 \end{cases} \qquad \textit{Done}$$

$$\text{solve}\left(\int_{\frac{1}{2}}^{4} f(t)\,dt = 1, b\right) \qquad b = \frac{28}{55}$$

$$\text{Define } f(t) = \begin{cases} b\cdot(2\cdot t-1), & \frac{1}{2} \le t < 1 \\ \dfrac{b}{\left(t-\frac{1}{2}\right)^2}, & 1 \le t \le 4 \end{cases} \Big|\, b = \frac{28}{55} \qquad \textit{Done}$$

$$ex := \int_{\frac{1}{2}}^{4} (t\cdot f(t))\,dt \qquad 1.5331$$

$$ex2 := \int_{\frac{1}{2}}^{4} (t^2\cdot f(t))\,dt \qquad 2.8263$$

$$sdx := \sqrt{ex2 - ex^2} \qquad 0.6899$$

$$ex + 2\cdot sdx \qquad 2.9129$$

$$ex - 2\cdot sdx \qquad 0.1533$$

$$\int_{\frac{1}{2}}^{2.91285} f(t)\,dt \qquad 0.9345$$

Question 4

a. $f:[-\pi,\pi]\to R,\quad f(x)=x^2\cos(x)$

$f(-x)=(-x)^2\cos(-x)$

$\quad =x^2\cos(x)=f(x)$ A1

so f is an even function and the graph of $y=f(x)$

is symmetrical about the y-axis. A1

b. $f'(x)=2x\cos(x)-x^2\sin(x)=x\left(2\cos(x)-x\sin(x)\right)$

for non-zero turning points $g(x)=2\cos(x)-x\sin(x)=0$

$g'(x)=-3\sin(x)-x\cos(x),\quad x_0=0.75$

$x_1=x_0-\dfrac{g(x_0)}{g'(x_0)}=1.117$ M1

$x_2=x_1-\dfrac{g(x_1)}{g'(x_1)}=1.077$

A1

x_0	0.75
x_1	1.117
x_2	1.077

Define $f1(x)=x^2\cdot\cos(x)$ *Done*

Define $m(x)=\dfrac{d}{dx}(f1(x))$ *Done*

$m(x)$ $\quad 2\cdot x\cdot\cos(x)-x^2\cdot\sin(x)$

Define $g(x)=\dfrac{m(x)}{x}$ *Done*

$g(x)$ $\quad 2\cdot\cos(x)-x\cdot\sin(x)$

Define $dg(x)=\dfrac{d}{dx}(g(x))$ *Done*

$dg(x)$ $\quad -x\cdot\cos(x)-3\cdot\sin(x)$

$0.75-\dfrac{g(0.75)}{dg(0.75)}$ $\quad 1.117$

$1.117-\dfrac{g(1.117)}{dg(1.117)}$ $\quad 1.077$

c.i. $y-a^2\cos(a)=\left(2a\cos(a)-a^2\sin(a)\right)(x-a)$

$y=\left(2a\cos(a)-a^2\sin(a)\right)x-a^2\left(\cos(a)-a\sin(a)\right)$ A1

ii. solving $y=0$ when $x=\pi$ and $0<a<\pi$ gives $a=1.151$, and A1

$y=0.852-0.271x$ A1

$\text{tangentLine}\left(f1(x),x,a\right)$

$$2.000\cdot a\cdot\left(\cos(a)-0.500\cdot a\cdot\sin(a)\right)\cdot x-a^2\cdot\left(\cos(a)-a\cdot\sin(a)\right)$$

$\text{solve}\left(a\cdot\left(2\cdot\cos(a)-a\cdot\sin(a)\right)\cdot x-a^2\cdot\left(\cos(a)-a\cdot\sin(a)\right)=0,a\right)|x=$

$$(a-2\cdot\pi)\cdot\cos(a)-a\cdot(a-\pi)\cdot\sin(a)=0 \text{ and } 0<a<\pi$$

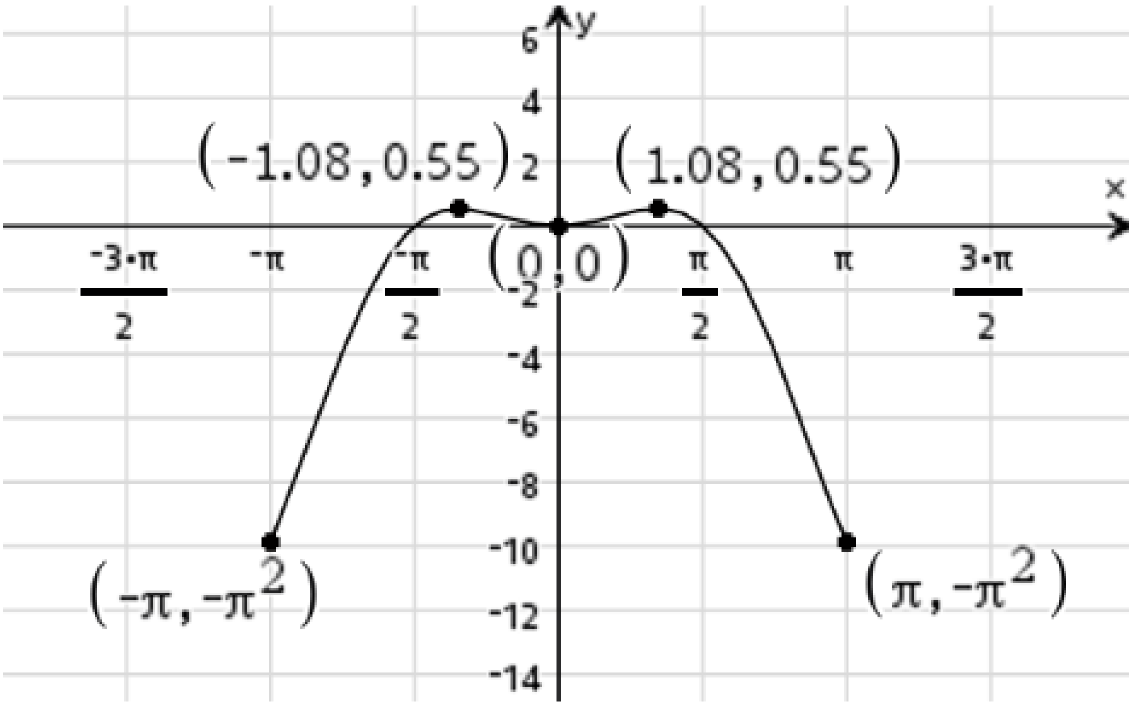
$a\cdot\left(2\cdot\cos(a)-a\cdot\sin(a)\right)\cdot x-a^2\cdot\left(\cos(a)-a\cdot\sin(a)\right)|a=1.1509284$

$$0.852-0.271\cdot x$$

d. endpoints $\left(\pm\pi,-\pi^2\right)$

local minimum turning point $(0,0)$ A1

absolute maximum turning points $(\pm1.08,0.55)$ G2



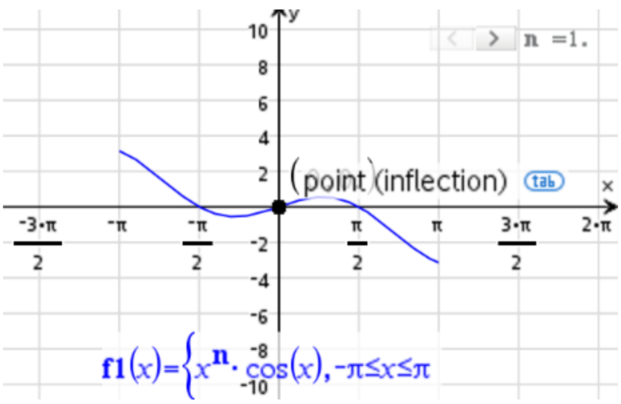
e. f is strictly increasing for $x\in[-3.14,-1.08]$ or $x\in[0,1.08]$ A1

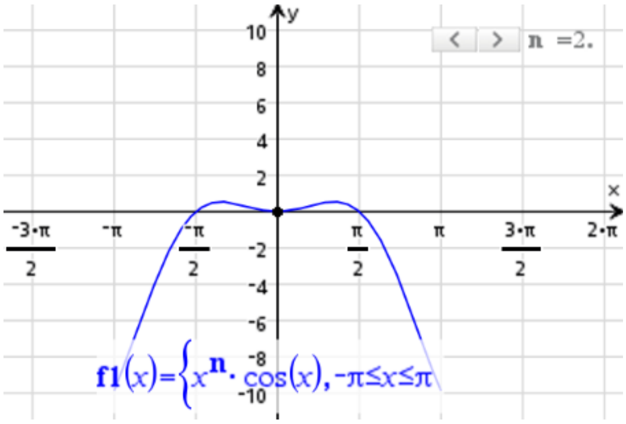
f. $\bar{f}=\dfrac{1}{\pi-(-\pi)}\displaystyle\int_{-\pi}^{\pi}x^2\cos(x)\,dx=-2$ A1

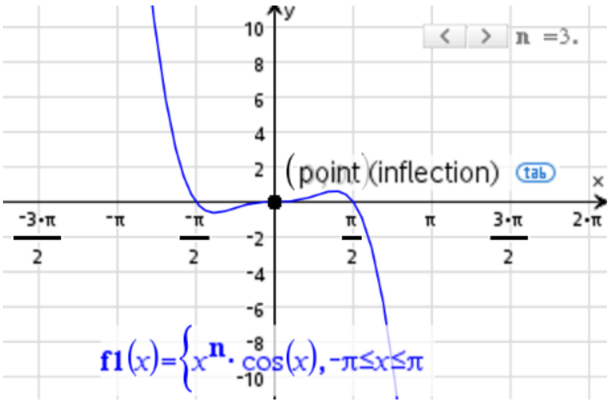
$$\frac{1}{\pi--\pi}\cdot\int_{-\pi}^{\pi}f1(x)\,dx \qquad -2$$

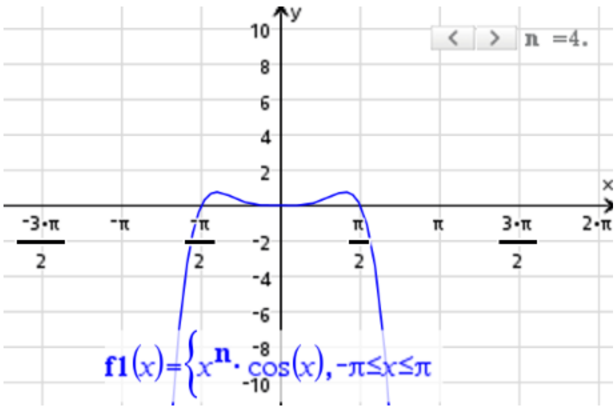
g. $f_n : [-\pi, \pi] \to R, \quad f_n(x) = x^n \cos(x), \quad n \in N.$ A2

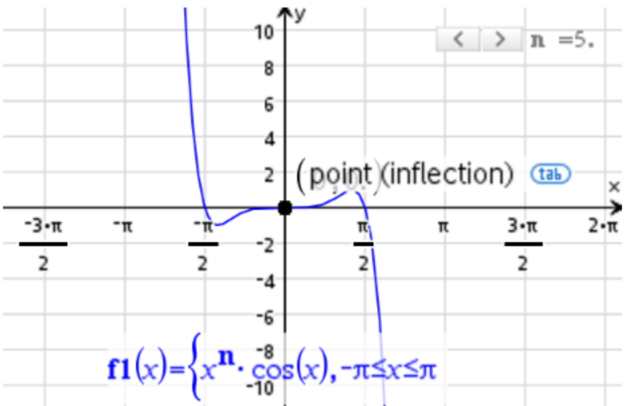
values of n	The graph of $y = f_n(x)$ at the origin has
$n = 1$	a point of inflection
n even, 2,4,6,…. $n = 2k, k \in Z^+$	a local minimum turning point
n odd, 3,5,7,…. $n = 2k+1, k \in Z^+$	a stationary point of inflection











Question 5

a. $f:\left[\frac{3}{2},\infty\right)\to R$, $f(x)=\sqrt{4x^2-9}$

$f:\quad y=\sqrt{4x^2-9}$

$f^{-1}:\ x=\sqrt{4y^2-9}$, $x^2=4y^2-9$, $y^2=\dfrac{x^2+9}{4}$

domain f = range of $f^{-1}=\left[\frac{3}{2},\infty\right)$

domain f^{-1} = range of $f=[0,\infty)$

$f^{-1}:[0,\infty)\to R$, $f^{-1}(x)=\dfrac{\sqrt{x^2+9}}{2}$ A1

b. solving $f(x)=f^{-1}(x)\quad\Rightarrow x=\sqrt{3}$

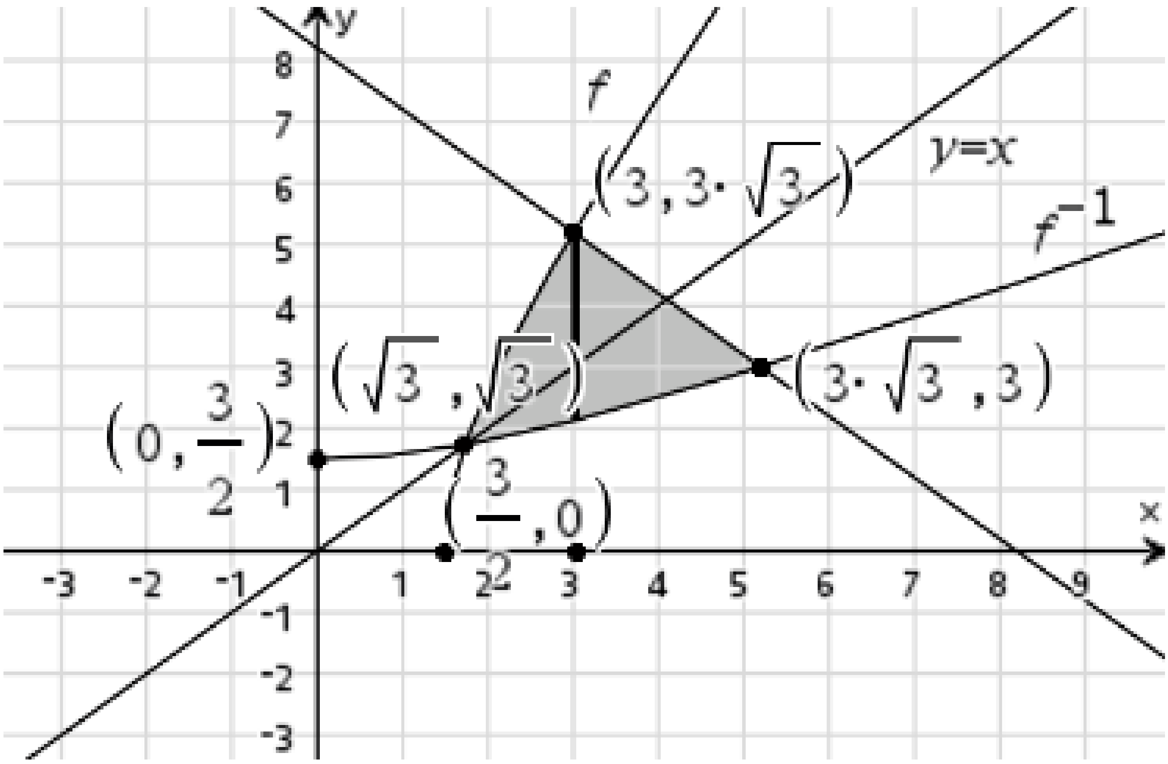
$P\left(\sqrt{3},\sqrt{3}\right)$ A1

Define $f1(x)=\sqrt{4\cdot x^2-9}$ *Done*

Define $f2(x)=\dfrac{\sqrt{x^2+9}}{2}$ *Done*

solve$(f1(x)=f2(x),x)|x>0$ $x=\sqrt{3}$

c. G2



d.i. The line $y = -x + 3\left(\sqrt{3}+1\right)$ intersects the graph of $f(x) = \sqrt{4x^2 - 9}$ at the point $\left(3, 3\sqrt{3}\right)$

and the line $y = -x + 3\left(\sqrt{3}+1\right)$ intersects the graph of $f^{-1}(x) = \dfrac{\sqrt{x^2+9}}{2}$ at the point $\left(3\sqrt{3}, 3\right)$.

The area is $A = \int_{\sqrt{3}}^{3}\left(f(x) - f^{-1}(x)\right)dx + \int_{3}^{3\sqrt{3}}\left(-x + 3\left(\sqrt{3}+1\right) - f^{-1}(x)\right)dx$ M1

$$A = \int_{\sqrt{3}}^{3}\left(\sqrt{4x^2-9} - \frac{\sqrt{x^2+9}}{2}\right)dx + \int_{3}^{3\sqrt{3}}\left(-x + 3\left(\sqrt{3}+1\right) - \frac{\sqrt{x^2+9}}{2}\right)dx \qquad \text{A1}$$

$$\int_{\sqrt{3}}^{3}\left(f1(x) - f2(x)\right)dx + \int_{3}^{3\cdot\sqrt{3}}\left(f4(x) - f2(x)\right)dx \qquad 5.5456$$

ii. $A = 5.546$ A1

e.

$f(x) = \sqrt{4x^2 - 9}$

$f'(x) = \dfrac{4x}{\sqrt{4x^2 - 9}}$

$f'\left(\sqrt{3}\right) = 4 = \tan(\theta_1)$

$f^{-1}(x) = \dfrac{\sqrt{x^2+9}}{2}$

$\dfrac{d}{dx}\left(f^{-1}(x)\right) = \dfrac{x}{\sqrt{x^2+9}}$ A1

$\dfrac{d}{dx}\left(f^{-1}(x)\right)\Big|_{x=\sqrt{3}} = \dfrac{1}{4} = \tan(\theta_2)$

$\theta_1 - \theta_2 = \tan^{-1}(4) - \tan^{-1}\left(\dfrac{1}{4}\right) = 61.9^0$ A1

$$\frac{d}{dx}\left(f1(x)\right)\Big|_{x=\sqrt{3}} \qquad 4$$

$$\frac{d}{dx}\left(f2(x)\right)\Big|_{x=\sqrt{3}} \qquad \frac{1}{4}$$

$$\tan^{-1}(4) - \tan^{-1}\left(\frac{1}{4}\right) \qquad 61.9275$$

f. $f(x)=\sqrt{kx^2-9} \quad x \in \left[\dfrac{3}{\sqrt{k}},\infty\right)$

domain $f=\left[\dfrac{3}{\sqrt{k}},\infty\right)$ since it is a one-one increasing function, so $k>0$.

$f^{-1}(x)=\sqrt{\dfrac{x^2+9}{k}}$

Define $f(x)=\sqrt{k\cdot x^2-9}$ *Done*

Define $g(x)=\dfrac{\sqrt{x^2+9}}{\sqrt{k}}$ *Done*

solve$(f(x)=g(x),x)$

$x=\dfrac{3}{\sqrt{k-1}}$ and $\dfrac{1}{k-1}\geq 0$ or $x=\dfrac{-3}{\sqrt{k-1}}$ and $\dfrac{1}{k-}$ ▸

The graphs do not intersect when $0<k\leq 1$ or $k\in(0,1]$ A1

g. $f(x)=\sqrt{kx^2-9}$, $\quad f'(x)=\dfrac{kx}{\sqrt{kx^2-9}}$, $\quad f'(c)=\dfrac{kc}{\sqrt{kc^2-9}}=\tan(\theta_1)$

$f^{-1}(x)=\sqrt{\dfrac{x^2+9}{k}}$, $\quad \dfrac{d}{dx}\left(f^{-1}(x)\right)=\dfrac{x}{\sqrt{k\left(x^2+9\right)}}$, $\quad \dfrac{d}{dx}\left(f^{-1}(x)\right)\Big|_{x=c}=\dfrac{c}{\sqrt{k\left(c^2+9\right)}}=\tan(\theta_2)$

$f(c)=f^{-1}(c)$ A1

$\Rightarrow \; k=\dfrac{c^2+9}{c^2}, \quad c=\dfrac{3}{\sqrt{k-1}}$

$\theta_1-\theta_2=\tan^{-1}\left(\dfrac{12}{5}\right)=\tan^{-1}(5)-\tan^{-1}\left(\dfrac{1}{5}\right)$

$\tan^{-1}(5)-\tan^{-1}\left(\dfrac{1}{5}\right)$ 67.3801

$\tan^{-1}\left(\dfrac{12}{5}\right)$ 67.3801

solving $\dfrac{kc}{\sqrt{kc^2-9}}=5$ and $\dfrac{c}{\sqrt{k\left(c^2+9\right)}}=\dfrac{1}{5}$

$eq1:=\dfrac{c\cdot k}{\sqrt{c^2\cdot k-9}}=5 \qquad \dfrac{c\cdot k}{\sqrt{c^2\cdot k-9}}=5$

$eq2:=\dfrac{c}{\sqrt{(c^2+9)\cdot k}}=\dfrac{1}{5} \qquad \dfrac{c}{\sqrt{(c^2+9)\cdot k}}=\dfrac{1}{5}$

with $c=\dfrac{3}{\sqrt{k-1}}$

gives $c=\dfrac{3}{2}$

$k=5$

solve$(eq1$ and $eq2,k)|c=\dfrac{3}{\sqrt{k-1}}$ $\quad k=5$ A1

END OF SECTION B SUGGESTED ANSWERS

End of detailed answers for the
2024 Kilbaha VCE Mathematical Methods Trial Examination 2

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