

Question 1

a. $f(x) = \log_e \left(\sqrt{x^3 + 1} \right) = \log_e \left((x^3 + 1)^{\frac{1}{2}} \right) = \frac{1}{2} \log_e (x^3 + 1)$

$$f'(x) = \frac{1}{2} \frac{\frac{d}{dx}(x^3 + 1)}{(x^3 + 1)} = \frac{3x^2}{2(x^3 + 1)} \quad \text{M1}$$

$$f'(2) = \frac{3 \times 4}{2(8 + 1)}$$

$$f'(2) = \frac{2}{3} \quad \text{A1}$$

b. $\frac{d}{dx} \left(\frac{x}{\sqrt{4x+9}} \right) = \frac{px+r}{(4x+9)^n}$ using the quotient rule

$$u = x, \quad v = \sqrt{4x+9} = (4x+9)^{\frac{1}{2}}$$

$$\frac{du}{dx} = 1, \quad \frac{dv}{dx} = 4 \times \frac{1}{2} \times (4x+9)^{-\frac{1}{2}} = \frac{2}{\sqrt{4x+9}} \quad \text{M1}$$

$$\frac{d}{dx} \left(\frac{x}{\sqrt{4x+9}} \right) = \frac{\sqrt{4x+9} - \frac{2x}{\sqrt{4x+9}}}{4x+9}$$

$$= \frac{4x+9-2x}{\sqrt{4x+9}(4x+9)} = \frac{2x+9}{(4x+9)^{\frac{3}{2}}}, \quad p=2, \quad q=9, \quad n=\frac{3}{2} \quad \text{A1}$$

Question 2

a. $3^{x^2+6x} = \frac{1}{243} = 3^{-5}$

$$x^2 + 6x = -5$$

$$x^2 + 6x + 5 = 0$$

$$(x+5)(x+1) = 0$$

$$x = -5, -1 \quad \text{A1}$$

b. $\log_2(x^2 + 4\sqrt{2}) + \log_2(x^2 - 4\sqrt{2}) = 5$

$$\log_2 \left((x^2 + 4\sqrt{2})(x^2 - 4\sqrt{2}) \right) = 5$$

$$(x^2 + 4\sqrt{2})(x^2 - 4\sqrt{2}) = 2^5 \quad \text{M1}$$

$$x^4 - 32 = 32$$

$$x^4 = 64, \quad x^2 = 8$$

$$x = \pm 2\sqrt{2} \quad \text{A1}$$

Question 3

$$\begin{aligned}(1) \quad 2ax - 2by &= 5 \quad \Rightarrow \quad 2by = 2ax - 5 \\(2) \quad (1-3b)x + 12y &= 2 - 4b \quad \Rightarrow \quad 12y = (3b-1)x + 2 - 4b \\(1) \quad y &= \frac{ax}{b} - \frac{5}{2b} \quad (2) \quad y = \frac{(3b-1)x}{12} + \frac{1-2b}{6}\end{aligned}$$

for an infinite number of solutions the gradients and the y-intercepts are both equal, so

$$(3) \quad \frac{a}{b} = \frac{3b-1}{12} \quad (4) \quad -\frac{5}{2b} = \frac{1-2b}{6} \quad \text{M1}$$

$$\begin{aligned}(3) \quad a &= \frac{b(3b-1)}{12} \quad (4) \quad -30 = 2b(1-2b) = 2b - 4b^2 \\4b^2 - 2b - 30 &= 0 \\2b^2 - b - 15 &= 0 \quad \text{M1} \\(2b+5)(b-3) &= 0\end{aligned}$$

$$b = 3 \quad a = \frac{3(9-1)}{12} = 2, \quad \text{or} \quad b = -\frac{5}{2} \quad a = \frac{-\frac{5}{2}\left(-\frac{15}{2}-1\right)}{12} = -\frac{5}{2} \times \frac{-17}{2} \times \frac{1}{12} = \frac{85}{48}$$

an infinite number of solutions when $a = 2, b = 3$ or $a = \frac{85}{48}, b = -\frac{5}{2}$ A1

Question 4

$$n = 6, \quad \hat{P} = \frac{X}{n} = \frac{X}{6} \quad X \stackrel{d}{=} Bi(n=6, p=?)$$

$$\Pr\left(\hat{P} = \frac{1}{3}\right) = \Pr(X = 2) = \Pr\left(\hat{P} = \frac{1}{2}\right) = \Pr(X = 3)$$

$$\Pr(X = 2) = \binom{6}{2} p^2 (1-p)^4 = \Pr(X = 3) = \binom{6}{3} p^3 (1-p)^3 \quad \text{A1}$$

$$\frac{6 \times 5}{2} p^2 (1-p)^4 - \frac{6 \times 5 \times 4}{3 \times 2} p^3 (1-p)^3 = 0 \quad \text{M1}$$

$$5p^2 (1-p)^3 [3(1-p) - 4p] = 5p^2 (1-p)^3 (3-7p) = 0$$

$$\text{since } 0 < p < 1, \quad p = \frac{3}{7}, \quad X \stackrel{d}{=} Bi\left(n=6, p=\frac{3}{7}\right)$$

$$\Pr(\hat{P} = 1) = \Pr(X = 6) = \binom{6}{6} \left(\frac{3}{7}\right)^6 \left(\frac{4}{7}\right)^0 = \left(\frac{3}{7}\right)^6, \quad a = 3, \quad b = 7, \quad n = 6 \quad \text{A1}$$

Question 5

$$f(x) = \begin{cases} \sqrt{5-x^2}, & x \leq 2 \\ ax^2 + bx, & x > 2 \end{cases}, \quad \text{let } g(x) = ax^2 + bx$$

Since the function is continuous at $x = 2$,

$$f(2) = 1 = g(2) \Rightarrow (1) \quad 4a + 2b = 1 \quad \text{A1}$$

Since the join is smooth, the gradients are also equal.

$$f'(x) = \frac{d}{dx}(\sqrt{5-x^2}) = \frac{-x}{\sqrt{5-x^2}}$$

$$g'(x) = 2ax + b \quad \text{M1}$$

$$f'(2) = -2 = g'(2) \Rightarrow (2) \quad 4a + b = -2$$

$$(1) - (2) \quad b = 3, \quad 4a = -5, \quad a = -\frac{5}{4} \quad \text{A1}$$

Question 6

$$y = f'(x) = 5 \sin\left(\frac{x}{2}\right) + me^{-2x} + 4$$

at the origin the gradient is zero, $f'(0) = 0$

$$0 = 5 \sin(0) + me^0 + 4 = m + 4 = 0, \quad \text{A1}$$

$$m = -4$$

$$y = f(x) = \int \left(5 \sin\left(\frac{x}{2}\right) - 4e^{-2x} + 4 \right) dx \quad \text{A1}$$

$$y = f(x) = -10 \cos\left(\frac{x}{2}\right) + 2e^{-2x} + 4x + c$$

$$f(0) = 0, \quad 0 = -10 \cos(0) + 2e^0 + c = -8 + c, \quad \text{A1}$$

$$c = 8$$

$$y = f(x) = -10 \cos\left(\frac{x}{2}\right) + 2e^{-2x} + 4x + 8$$

Question 7

a. $2\sin^2(2x) + \cos(2x) - 1 = 0$

$$2(1 - \cos^2(2x)) + \cos(2x) - 1 = 0$$

$$2 - 2\cos^2(2x) + \cos(2x) - 1 = 0$$

M1

$$-2\cos^2(2x) + \cos(2x) + 1 = 0$$

$$2\cos^2(2x) - \cos(2x) - 1 = 0$$

$$(2\cos(2x) + 1)(\cos(2x) - 1) = 0$$

A1

$$\cos(2x) = -\frac{1}{2}, \quad \cos(2x) = 1$$

$$2x = 2n\pi \pm \cos^{-1}\left(-\frac{1}{2}\right), \quad 2x = 2n\pi \pm \cos^{-1}(1)$$

$$2x = 2n\pi \pm \frac{2\pi}{3}, \quad 2x = 2n\pi \pm 0$$

$$x = \frac{\pi}{3}(3n \pm 1), \quad x = n\pi \quad n \in \mathbb{Z}.$$

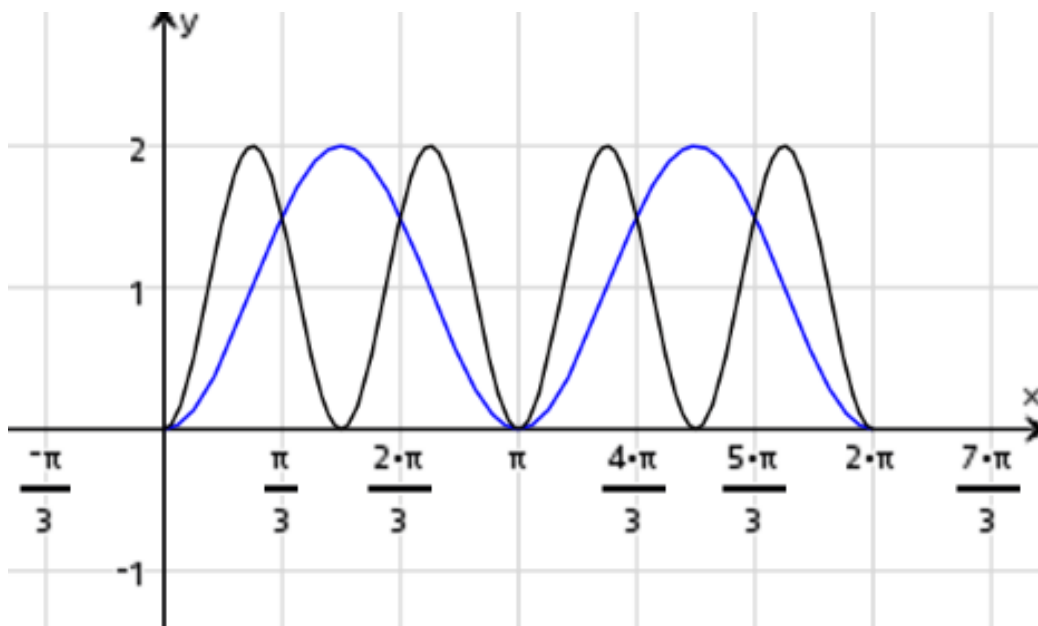
A1

b. The two graphs intersect at $x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi$ from a.

$$y_1 = 2\sin^2(2x)$$

$$y_2 = 1 - \cos(2x) \text{ has a period of } \pi$$

G2



$$2\sin^2(2x) < 1 - \cos(2x), \quad \frac{\pi}{3} < x < \frac{2\pi}{3}, \quad \frac{4\pi}{3} < x < \frac{5\pi}{3}$$

A1

$$\left(\frac{\pi}{3}, \frac{2\pi}{3}\right) \cup \left(\frac{4\pi}{3}, \frac{5\pi}{3}\right)$$

Question 8

a. completing the square

$$h(x) = 6 + 3x - x^2 = -\left(x^2 - 3x + \frac{9}{4}\right) + 6 + \frac{9}{4} = \frac{33}{4} - \left(x - \frac{3}{2}\right)^2$$

$$\text{range } h = \left(-\infty, \frac{33}{4}\right]$$

$$f(x) = \log_e(x-2) \text{ domain } x-2 > 0 \Rightarrow x > 2, \text{ range } R$$

	$f(x)$	$h(x)$
domain	$(2, \infty)$	R
range	R	$\left(-\infty, \frac{33}{4}\right]$

Since range $h \not\subseteq$ domain f , so $f \circ h(x)$ does not exist.

A1

b. solving $g(x) = 6 + 3x - x^2 = 2$

$$\Rightarrow x^2 - 3x - 4 = 0$$

$$(x-4)(x+1) = 0$$

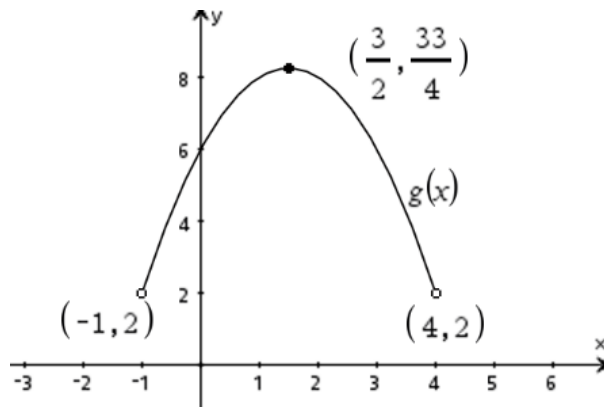
$$\Rightarrow x = -1, 4$$

M1

now $g(-1) = g(4) = 2$, $g\left(\frac{3}{2}\right) = \frac{33}{4}$, so if we now restrict the domain of g ,

as $D = (-1, 4) = \text{domain } f \circ g(x)$ and the range of $g = \left(2, \frac{33}{4}\right]$

A1



so range $g \subseteq$ domain f , so now $f \circ g(x)$ exist.

$$f \circ g(x) = f(g(x)) = f(6 + 3x - x^2) = \log_e(6 + 3x - x^2 - 2)$$

$$f \circ g(x): (-1, 4) \rightarrow R, \quad f \circ g(x) = \log_e(4 + 3x - x^2) = \log_e((4-x)(x+1))$$

A1

Question 9

The two y values are equal at $x = b$ so that (1) $\sin(b) = \sin(b - \alpha) + c$

$$y = \sin(x), \quad \frac{dy}{dx} = \cos(x) \quad \text{and} \quad y = \sin(x - \alpha) + c, \quad \frac{dy}{dx} = \cos(x - \alpha)$$

The gradients are also equal at $x = b$ so that (2) $\cos(b) = \cos(b - \alpha)$ A1

square (1) $\sin^2(b) = \sin^2(b - \alpha) + 2c \sin(b - \alpha) + c^2$

square (2) $\cos^2(b) = \cos^2(b - \alpha)$ now add M1

$$\sin^2(b) + \cos^2(b) = \sin^2(b - \alpha) + \cos^2(b - \alpha) + 2c \sin(b - \alpha) + c^2$$

$$1 = 1 + 2c \sin(b - \alpha) + c^2$$

$$2c \sin(b - \alpha) + c^2 = c(2 \sin(b - \alpha) + c) = 0$$

$$c = -2 \sin(b - \alpha) = 2 \sin(\alpha - b) \quad \text{since } c > 0 \text{ and } 0 < b < \alpha < \frac{\pi}{2} \quad \text{M1}$$

from (1) $\sin(b) = \sin(b - \alpha) + c = \sin(b - \alpha) - 2 \sin(b - \alpha)$

$$\sin(b) = -\sin(b - \alpha) = \sin(\alpha - b)$$

$$b = \alpha - b$$

$$2b = \alpha$$

$$b = \frac{\alpha}{2}$$

$$c = 2 \sin\left(\alpha - \frac{\alpha}{2}\right)$$

$$c = 2 \sin\left(\frac{\alpha}{2}\right) \quad \text{A1}$$

Question 10

a.i.
$$f(x) = \begin{cases} \frac{a}{(2x+1)^2} & 1 \leq x \leq 4 \\ 0 & \text{elsewhere} \end{cases}$$

since it is a probability density function the total area under the curve is 1.

$$\int_1^4 \frac{a}{(2x+1)^2} dx = 1$$

$$a \left[-\frac{1}{2(2x+1)} \right]_1^4 = 1$$

$$a \left(-\frac{1}{18} + \frac{1}{6} \right) = \frac{a(3-1)}{18} = 1$$

$$a = 9$$

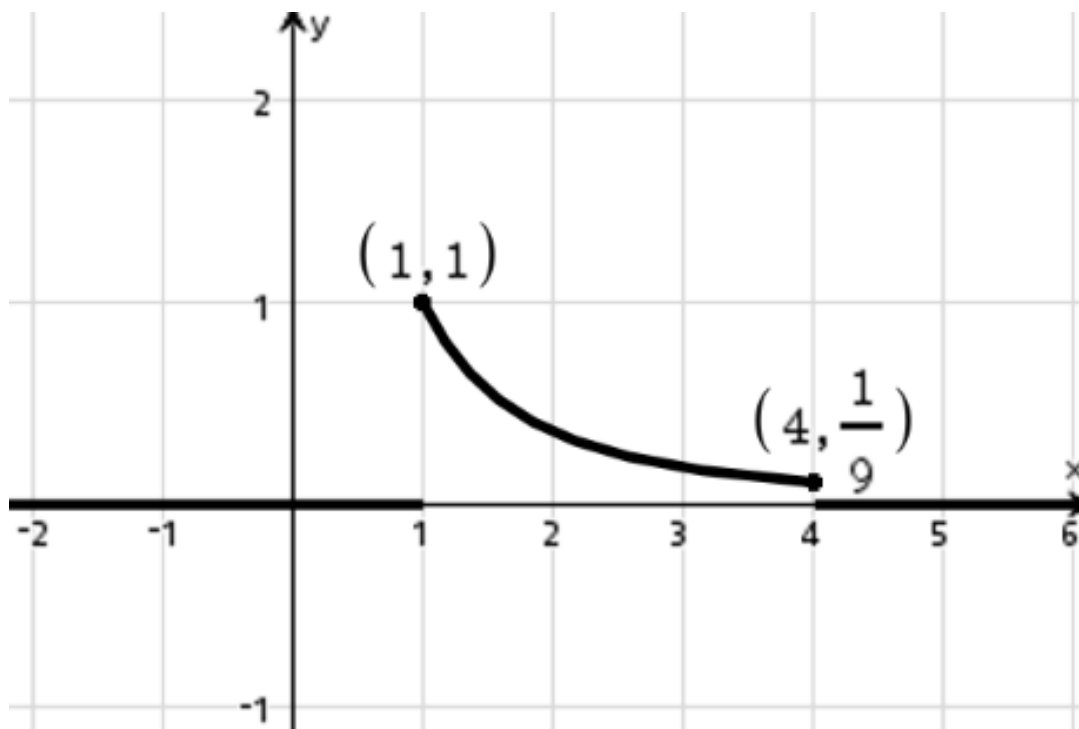
$$f(x) = \begin{cases} \frac{9}{(2x+1)^2} & 1 \leq x \leq 4 \\ 0 & \text{elsewhere} \end{cases}$$

A1

M1

ii. $f(1) = 1, \quad f(4) = \frac{1}{9}$ the graph is part of a truncus graph

G1



$$\text{b. } g(y) = \begin{cases} \frac{b}{2y+1} & 1 \leq y \leq 4 \\ 0 & \text{elsewhere} \end{cases}$$

since it is a probability density function the total area under the curve is 1.

$$\int_1^4 \frac{b}{2y+1} dy = 1$$

$$b \left[\frac{1}{2} \log_e(2y+1) \right]_1^4 = 1$$

$$\frac{b}{2} (\log_e(9) - \log_e(3)) = \frac{b}{2} \log_e(3) = 1$$

$$b = \frac{2}{\log_e(3)}$$

A1

$$E(Y) = \frac{2}{\log_e(3)} \int_1^4 \frac{y}{2y+1} dy$$

$$E(Y) = \frac{1}{\log_e(3)} \int_1^4 \left(\frac{2y}{2y+1} \right) dy = \frac{1}{\log_e(3)} \int_1^4 \left(\frac{2y+1-1}{2y+1} \right) dy$$

M1

$$E(Y) = \frac{1}{\log_e(3)} \int_1^4 \left(1 - \frac{1}{2y+1} \right) dy$$

$$= \frac{1}{\log_e(3)} \left[y - \frac{1}{2} \log_e(2y+1) \right]_1^4$$

$$= \frac{1}{\log_e(3)} \left(4 - \frac{1}{2} \log_e(9) - 1 + \frac{1}{2} \log_e(3) \right) = \frac{1}{\log_e(3)} \left(3 - \frac{1}{2} \log_e(3) \right)$$

A1

$$= \frac{3}{\log_e(3)} - \frac{1}{2}, \quad p = 3, \quad q = -\frac{1}{2}$$

A1

**End of detailed answers for the
2024 Kilbaha VCE Mathematical Methods Trial Examination 1**

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