



Mathematical Methods

Written Examination 1

2024 Insight Year 12 Trial Exam Paper

Worked Solutions

This book presents:

- worked solutions
- mark allocations
- tips.

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Question 1a.**Worked solution**

$$\begin{aligned}\frac{dy}{dx} &= x(-2\sin(2x)) + \cos(2x) \\ &= \cos(2x) - 2x\sin(2x)\end{aligned}$$

Mark allocation: 1 mark

- 1 answer mark for applying the product rule to find the derivative:

$$\frac{dy}{dx} = \cos(2x) - 2x\sin(2x)$$

**Tip**

- You may find it helpful to begin by writing down the product rule.

Question 1b.**Worked solution**

$$\begin{aligned}f'(x) &= \frac{(e^x - 1)\left(\frac{1}{x}\right) - e^x \log_e(x)}{(e^x - 1)^2} \\ f'(1) &= \frac{(e - 1)\left(\frac{1}{1}\right) - e \log_e(1)}{(e - 1)^2} \\ &= \frac{(e - 1)}{(e - 1)^2} \\ &= \frac{1}{e - 1}\end{aligned}$$

Mark allocation: 2 marks

- 1 answer mark for applying the quotient rule to find the derivative:

$$f'(x) = \frac{(e^x - 1)\left(\frac{1}{x}\right) - e^x \log_e(x)}{(e^x - 1)^2}$$

- 1 answer mark for correctly evaluating the derivative at $x = 1$: $f'(1) = \frac{1}{e - 1}$

**Tip**

- When using the quotient rule, or any other differentiation rule, it can be helpful to write down the rule.

Question 2**Worked solution**

Let $\sin(x) = a$.

$$2a^2 + 3a - 2 = 0$$

$$(2a - 1)(a + 2) = 0$$

$$a = \frac{1}{2} \text{ [Note that } a \neq -2 \text{ because } -1 \leq a \leq 1]$$

$$\therefore \sin(x) = \frac{1}{2}$$

$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$

Mark allocation: 2 marks

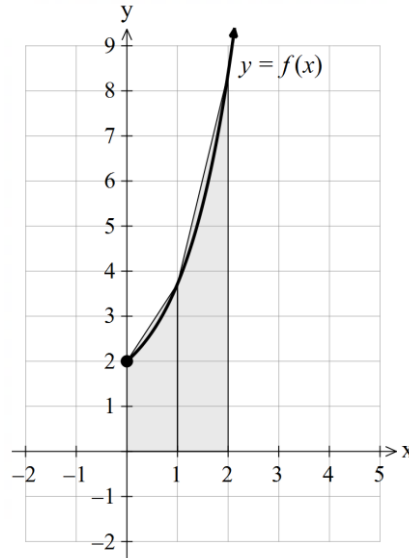
- 1 method mark for using a suitable method for solving the equation, such as factorising
- 1 answer mark for the correct values: $x = \frac{\pi}{6}, \frac{5\pi}{6}$

**Tips**

- *Substitution can be a useful method for producing a quadratic equation that can then be solved by factorising.*
- *Ensure you are familiar with the exact trigonometric values.*
- *Remember to consider whether answers are feasible, e.g. $\cos(x) \neq -2$ because $-1 \leq \cos(x) \leq 1$.*

Question 3a.**Worked solution**

$$\begin{aligned}
 A &= \frac{2-0}{4} [f(0) + 2f(1) + f(2)] \quad \left(\text{or } \frac{1}{2}(1)[f(0) + f(1)] + \frac{1}{2}(1)[f(1) + f(2)] \right) \\
 &= \frac{1}{2} [f(0) + 2f(1) + f(2)] \\
 &= \frac{1}{2} (e^0 + 1 + 2(e^1 + 1) + e^2 + 1) \\
 &= \frac{1}{2} (2e + e^2 + 5)
 \end{aligned}$$

**Mark allocation: 2 marks**

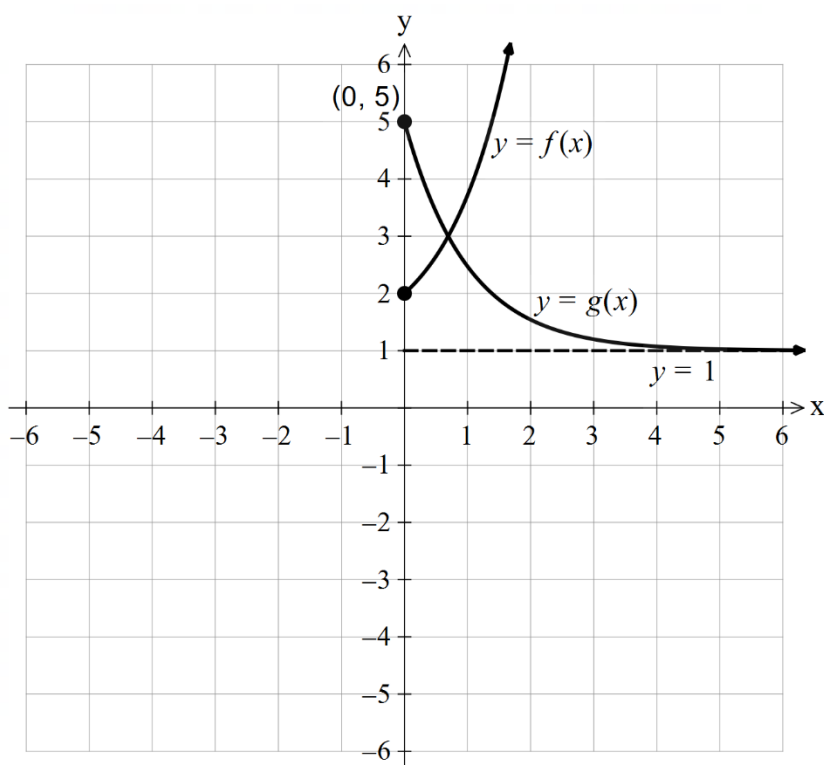
- 1 method mark for setting up the area equation as the area of two trapeziums
- 1 answer mark for the correct answer: $\frac{1}{2}(2e + e^2 + 5)$

Question 3b.**Worked solution**

$$\begin{aligned}
 f(x) &= g(x) \\
 e^x + 1 &= 4e^{-x} + 1 \\
 e^x &= 4e^{-x} \\
 &= \frac{4}{e^x} \\
 e^{2x} &= 4 \\
 2x &= \log_e(4) \\
 x &= \frac{1}{2} \log_e(4) \\
 &= \log_e \left(4^{\frac{1}{2}} \right) \\
 &= \log_e(2)
 \end{aligned}$$

Mark allocation: 2 marks

- 1 answer mark for obtaining $e^{2x} = 4$
- 1 answer mark for the correct value of x : $\log_e(2)$

Question 3c.**Worked solution****Mark allocation: 2 marks**

- 1 answer mark for the correct end point labelled with coordinates $(0, 5)$ and the correct asymptote labelled with its equation $y = 1$ (awarded regardless of the domain that the asymptote is drawn over)
- 1 answer mark for the correct shape of $y = g(x)$ with the point of intersection occurring at $0 < x < 1$

**Tips**

- The value of $\log_e 2$ is between 0 and 1 because $2 < e$. Hence, on your graph the point of intersection of $f(x)$ and $g(x)$ should occur between $x = 0$ and $x = 1$.
- Although you were not asked to show the coordinates of the point of intersection on the graph, it is helpful to work out the y -coordinate of the point of intersection, $y = e^{\log_e(2)} + 1 = 2 + 1 = 3$, in order to sketch a more accurate graph.

Question 4a.**Worked solution**

$$\text{ran } f = [-1, 3]$$

Mark allocation: 1 mark

- 1 answer mark for the correct range: $[-1, 3]$

Question 4b.i.**Worked solution**

We need $\text{ran } g \subseteq \text{dom } h$.

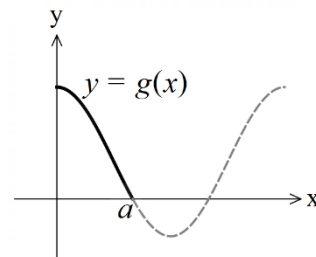
$$\therefore \text{ran } g \subseteq [0, \infty)$$

$$\therefore 2\cos(4x) + 1 \geq 0$$

Consider the graph of $g(x) = 2\cos(4x) + 1$.

a is the first positive value of x for which $2\cos(4x) + 1 = 0$.

$$\text{Solve } \cos(4x) = -\frac{1}{2}.$$



The required angle is in quadrant 2. The related angle in quadrant 1 is $\frac{\pi}{3}$.

$$4x = \pi - \frac{\pi}{3}$$

$$= \frac{2\pi}{3}$$

$$x = \frac{\pi}{6}$$

Therefore, $a = \frac{\pi}{6}$.

Mark allocation: 2 marks

- 1 method mark for recognising that $\text{ran } g \subseteq [0, \infty)$ or solving $2\cos(4x) + 1 = 0$
- 1 answer mark for the correct value of a : $\frac{\pi}{6}$

**Tips**

- When solving inequalities, a graphical approach is often helpful.
- The question asks for the value of a , so ensure your final answer is stated as $a = \frac{\pi}{6}$, not $x = \frac{\pi}{6}$.

Question 4b.ii.**Worked solution**

$$\text{dom } g = \left[0, \frac{\pi}{8}\right]$$

$$g(0) = 2 \cos(0) + 1 = 3$$

$$g\left(\frac{\pi}{8}\right) = 2 \cos\left(\frac{\pi}{2}\right) + 1 = 1$$

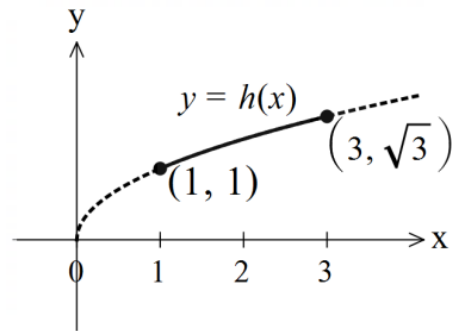
$$\therefore \text{When } \text{dom } g = \left[0, \frac{\pi}{8}\right], \text{ran } g = [1, 3].$$

$\text{ran } g = [1, 3]$ becomes the input for h .

Consider the graph of h .

$$h(1) = 1, h(3) = \sqrt{3}$$

$$\therefore \text{ran } (h \circ g) = [1, \sqrt{3}]$$

**Mark allocation: 2 marks**

- 1 answer mark for finding the range of $g: [1, 3]$
- 1 answer mark for the correct range of $(h \circ g)(x): [1, \sqrt{3}]$

**Tips**

- $\text{ran } g = [1, 3]$ becomes the input for h , so work out $\text{ran } h$ when $x \in [1, 3]$.
- It is not always necessary to find the rule for a composite function. **Part b.** only needed consideration of the domain and range of g and h , so the rule for $(h \circ g)(x)$ was not needed.

Question 5a.**Worked solution**

$$f(x) = x^3 + x^2$$

$$f'(x) = 3x^2 + 2x$$

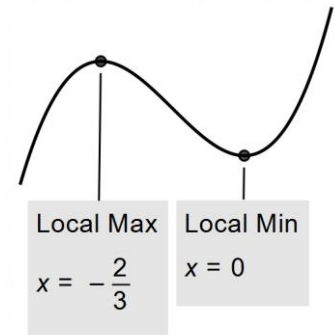
Stationary points occur where $f'(x) = 0$.

$$3x^2 + 2x = 0$$

$$x(3x + 2) = 0$$

$$\therefore x = 0, x = -\frac{2}{3}$$

Since f is a positive cubic function, there is a local maximum at $x = -\frac{2}{3}$ and a local minimum at $x = 0$.

**Mark allocation: 2 marks**

- 1 answer mark for correct x values: $x = 0, x = -\frac{2}{3}$
- 1 answer mark for the correct nature of each point: a local maximum at $x = -\frac{2}{3}$ and a local minimum at $x = 0$

**Tip**

- *It is helpful to be familiar with the graphical shapes of cubic functions in order to quickly determine the nature of stationary points.*

Question 5b.**Worked solution**

A point of inflection occurs where $f'' = 0$.

$$f''(x) = 6x + 2$$

$$6x + 2 = 0$$

$$x = -\frac{1}{3}$$

Alternatively, for a cubic function with two stationary points, the point of inflection will occur half way between these points; that is, at

$$x = \frac{-\frac{2}{3} + 0}{2} = -\frac{1}{3}.$$

$$\begin{aligned} f\left(-\frac{1}{3}\right) &= \left(-\frac{1}{3}\right)^3 + \left(-\frac{1}{3}\right)^2 \\ &= -\frac{1}{27} + \frac{1}{9} \\ &= \frac{2}{27} \end{aligned}$$

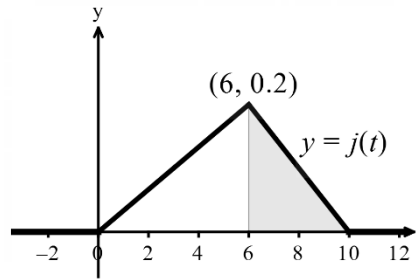
Therefore, the coordinates of the point of inflection are $\left(-\frac{1}{3}, \frac{2}{27}\right)$.

Mark allocation: 2 marks

- 1 mark for the correct x -coordinate: $x = -\frac{1}{3}$
- 1 mark for the correct y -coordinate: $y = \frac{2}{27}$

Question 6a.**Worked solution**

$$\begin{aligned}\Pr(X > 6) &= \frac{1}{2} \times 4 \times 0.2 \\ &= 0.4\end{aligned}$$

**Mark allocation: 1 mark**

- 1 answer mark for the correct probability: 0.4

**Tip**

- The probability can be calculated by finding the relevant area shown on the graph.

Question 6b.**Worked solution**

$$\Pr(X \leq) = \int_0^w \frac{x+1}{12} dx = \frac{1}{3}$$

$$\frac{1}{12} \int_0^w (x+1) dx = \frac{1}{3}$$

$$\frac{1}{12} \left[\frac{x^2}{2} + x \right]_0^w = \frac{1}{3}$$

$$\frac{w^2}{2} + w = 4$$

$$w^2 + 2w - 8 = 0$$

$$(w+4)(w-2) = 0$$

$$\therefore w = -4, w = 2$$

$$w = 2 \text{ (since } 0 \leq w \leq 4\text{)}$$

Mark allocation: 3 marks

- 1 answer mark for setting up the integral expression: $\int_0^w \frac{x+1}{12} dx = \frac{1}{3}$ or equivalent
- 1 method mark for evaluating the definite integral, leading to $\frac{w^2}{2} + w = 4$ (or any multiple of this)
- 1 answer mark for the correct answer: 2

Question 7a.**Worked solution**

$$\begin{aligned}g(x) &= (2x-3)^{\frac{1}{2}} \\g'(x) &= \frac{1}{2}(2)(2x-3)^{-\frac{1}{2}} \\&= (2x-3)^{-\frac{1}{2}} \\&= \frac{1}{\sqrt{2x-3}}\end{aligned}$$

Mark allocation: 1 mark

- 1 answer mark for correct working leading to $g'(x) = \frac{1}{\sqrt{2x-3}}$

**Tips**

- When a question asks you to 'show that' something is the case, clearly show all the steps needed.
- The formula for differentiating expressions of the form $y = (ax+b)^n$ is on the formula sheet. Alternatively, the chain rule can be used.

Question 7b.**Worked solution**

$$\begin{aligned}g'(2) &= \frac{1}{\sqrt{4-3}} \\&= 1\end{aligned}$$

$$\therefore \tan \theta = 1$$

$$\theta = 45^\circ$$

Mark allocation: 2 marks

- 1 answer mark for $g'(2) = 1$
- 1 answer mark for the correct angle: 45°

Question 7c.**Worked solution**

$$\tan 30^\circ = \frac{1}{\sqrt{3}}$$

Therefore, the tangent angle is $\geq 30^\circ$ when the gradient of the tangent is $\geq \frac{1}{\sqrt{3}}$, that is,
when $g'(x) \geq \frac{1}{\sqrt{3}}$.

Solving $g'(x) = \frac{1}{\sqrt{3}}$ gives

$$\frac{1}{\sqrt{2x-3}} = \frac{1}{\sqrt{3}}$$

$$2x-3=3$$

$$x=3$$

The graph of $g(x)$ shows that the gradient decreases as x increases. Hence, $g'(x)$ decreases as x increases.

$$\text{dom } g = \left[\frac{3}{2}, \infty \right), \text{ thus } \text{dom } g' = \left(\frac{3}{2}, \infty \right)$$

Therefore the angle is $\geq 30^\circ$ when $\frac{3}{2} < x \leq 3$

$$\therefore \frac{3}{2} < k \leq 3$$

Alternative method for solving $g'(x) \geq \frac{1}{\sqrt{3}}$

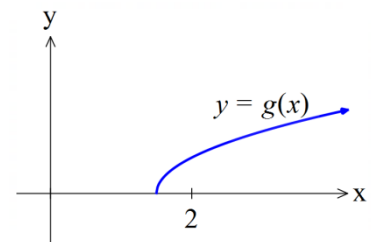
$$\frac{1}{\sqrt{2x-3}} \geq \frac{1}{\sqrt{3}}$$

$$2x-3 \leq 3$$

$$x \leq 3$$

$$\text{dom } g = \left[\frac{3}{2}, \infty \right), \text{ hence } \text{dom } g' = \left(\frac{3}{2}, \infty \right) \text{ if } \frac{3}{2} < x \leq 3.$$

$$\therefore \frac{3}{2} < k \leq 3$$



Mark allocation: 3 marks

- 1 method mark for deriving $g'(x) = \frac{1}{\sqrt{3}}$ or $g'(x) \geq \frac{1}{\sqrt{3}}$
- 1 answer mark for a final answer containing $k \leq 3$ or $x \leq 3$
- 1 answer mark for the fully correct answer: $\frac{3}{2} < k \leq 3$ or $k \in \left(\frac{3}{2}, 3\right]$

**Tips**

- *Non-linear inequalities can be tricky to solve algebraically. A graphical approach is often easiest.*
- *If an algebraic approach is used for solving the inequality, you need to recognise that the fraction with the smaller denominator is actually the larger number.*
- *Consideration of the domains of g and g' is important. Since $\text{dom } g = \left[\frac{3}{2}, \infty\right)$, then $\text{dom } g' = \left(\frac{3}{2}, \infty\right)$. Once $x \leq 3$ has been obtained, consideration of $\text{dom } g'$ leads to $\frac{3}{2} < x \leq 3$.*

Question 8a.**Worked solution**

$$\begin{aligned}\frac{d}{dx}(x^2 \log_e(x)) &= x^2 \left(\frac{1}{x}\right) + 2x \log_e(x) \\ &= 2x \log_e(x) + x\end{aligned}$$

Mark allocation: 1 mark

- 1 mark for a working that leads to $2x \log_e(x) + x$

**Tips**

- *The product rule can be used to differentiate $x^2 \log_e(x)$.*
- *The product rule does not have to be stated, but it is essential to include some working because the question requires you to 'show' how the result is obtained.*

Question 8b.**Worked solution**

Note: we are ultimately going to be calculating a definite integral, so in the working below the constant of integration, $+c$, is not relevant. Hence, at each step an antiderivative, where c is zero, is used. An alternative working can be used that includes ' $+c$ '.

$$A = \int_1^{\frac{3}{2}} (x \log_e(x) + 1) dx$$

We know from **part a.** that

$$\int (2x \log_e(x) + x) dx = x^2 \log_e(x)$$

Hence

$$\begin{aligned} \int 2x \log_e(x) dx + \int x dx &= x^2 \log_e(x) \\ \int 2x \log_e(x) dx &= x^2 \log_e(x) - \int x dx \\ 2 \int x \log_e(x) dx &= x^2 \log_e(x) - \frac{1}{2} x^2 \\ \int x \log_e(x) dx &= \frac{1}{2} x^2 \log_e(x) - \frac{1}{4} x^2 \end{aligned}$$

$$\begin{aligned} A &= \int_1^{\frac{3}{2}} (x \log_e(x) + 1) dx \\ &= \left[\frac{1}{2} x^2 \log_e(x) - \frac{1}{4} x^2 + x \right]_1^{\frac{3}{2}} \\ &= \frac{1}{2} \left(\frac{9}{4} \right) \log_e \left(\frac{3}{2} \right) - \frac{1}{4} \left(\frac{9}{4} \right) + \frac{3}{2} - \left(\frac{1}{2} \log_e(1) - \frac{1}{4} + 1 \right) \\ &= \frac{9}{8} \log_e \left(\frac{3}{2} \right) - \frac{9}{16} + \frac{3}{2} - 0 - \frac{3}{4} \\ &= \left(\frac{9}{8} \log_e \left(\frac{3}{2} \right) + \frac{3}{16} \right) \text{ square units} \end{aligned}$$

Mark allocation: 3 marks

- 1 answer mark for a correct integral for the required area: $A = \int_1^{\frac{3}{2}} (x \log_e(x) + 1) dx$
- 1 method mark for using the answer to **part a.** to anti-differentiate $x \log_e(x)$ or $x \log_e(x) + 1$

Possible expressions include:

$$\int x \log_e(x) dx = \frac{1}{2} x^2 \log_e(x) - \frac{1}{4} x^2$$

$$\int (x \log_e(x) + 1) dx = \frac{1}{2} x^2 \log_e(x) - \frac{1}{4} x^2 + x$$

$$\int_1^{\frac{3}{2}} x \log_e(x) dx = \left[\frac{1}{2} x^2 \log_e(x) - \frac{1}{4} x^2 \right]_1^{\frac{3}{2}}$$

$$\int_1^{\frac{3}{2}} (x \log_e(x) + 1) dx = \left[\frac{1}{2} x^2 \log_e(x) - \frac{1}{4} x^2 + x \right]_1^{\frac{3}{2}}$$

- 1 answer mark for the correct area: $\frac{9}{8} \log_e\left(\frac{3}{2}\right) + \frac{3}{16}$

**Tips**

- *The use of the word 'hence' requires that you use the result of the previous part of the question, so your antiderivative should include $x^2 \log_e(x)$. Alternative methods of integrating $x \log_e(x)$ should not be used.*
- *Accurate arithmetic involving fractions is an important skill often tested in the Mathematical Methods 1 exam.*

Question 9a.**Worked solution**

$$\begin{aligned}\Pr(O \cap D) &= \frac{2}{3} \times \frac{1}{4} \\ &= \frac{1}{6}\end{aligned}$$

Mark allocation: 1 mark

- 1 answer mark for the correct answer: $\frac{1}{6}$

**Tip**

- *For independent events:* $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$.

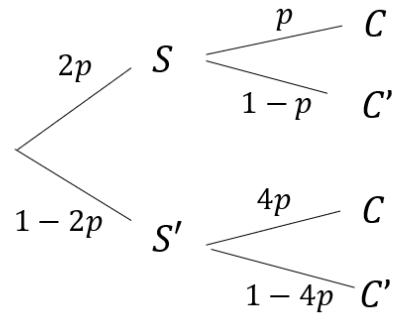
Question 9b.**Worked solutions**

$$\text{Probability} = f(p) = \Pr(S \cap C') + \Pr(S' \cap C)$$

$$= 2p(1-p) + (1-2p)4p$$

$$= 2p - 2p^2 + 4p - 8p^2$$

$$= -10p^2 + 6p$$



The maximum occurs when $f'(p) = 0$.

$$f'(p) = -20p + 6$$

$$-20p + 6 = 0$$

$$p = \frac{3}{10}$$

Alternative methods: complete the square or note that the turning point

is at $p = -\frac{b}{2a}$.

$$\text{maximum} = f\left(\frac{3}{10}\right)$$

$$= -10\left(\frac{3}{10}\right)^2 + 6\left(\frac{3}{10}\right)$$

$$= -\frac{9}{10} + \frac{18}{10}$$

$$= \frac{9}{10}$$

Mark allocation: 3 marks

- 1 method mark for obtaining the probability in terms of p : $-10p^2 + 6p$
- 1 answer mark for determining the value of p that maximises the probability: $p = \frac{3}{10}$
- 1 answer mark for correctly determining the maximum: $\frac{9}{10}$

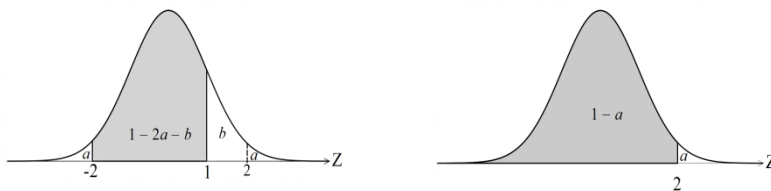
**Tip**

- *Drawing a tree diagram is a useful strategy for answering questions of this type.*

Question 9c.**Worked solution**

$$\begin{aligned}
 \Pr(V > 202 \mid V < 211) &= \Pr\left(Z > \frac{202-205}{3} \mid Z < \frac{211-205}{3}\right) \\
 &= \Pr(Z > -1 \mid Z < 2) \\
 &= \frac{\Pr(Z > -1 \cap Z < 2)}{\Pr(Z < 2)} \\
 &= \frac{\Pr(-1 < Z < 2)}{\Pr(Z < 2)}
 \end{aligned}$$

The following diagrams will help express this in the form required.



$$\Pr(V > 202 \mid V < 211) = \frac{1-2a-b}{1-a} \left(\text{or } 1 - \frac{a+b}{1-a} \text{ or } 1 + \frac{a+b}{a-1} \right)$$

Alternative method

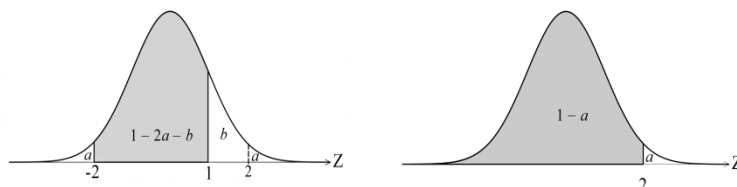
$$\begin{aligned}
 \Pr(V > 202 \mid V < 211) &= \frac{\Pr(-1 < Z < 2)}{\Pr(Z < 2)} \\
 &= \frac{1 - \Pr(Z < -1) - \Pr(Z > 2)}{1 - a} \\
 &= \frac{1 - (a+b) - a}{1 - a} \\
 &= \frac{1 - 2a - b}{1 - a}
 \end{aligned}$$

Mark allocation: 2 marks

- 1 method mark for using any suitable method, such as:
 - simplified conditional probability expressed using Z :

$$\Pr(V > 202 | V < 211) = \frac{\Pr(-1 < Z < 2)}{\Pr(Z < 2)}$$

- or drawing two bell-shaped curves and representative areas (as shown below)



- 1 answer mark for the correct answer: $\frac{1-2a-b}{1-a}$ (or $1 - \frac{a+b}{1-a}$ or $1 + \frac{a+b}{a-1}$)

**Tip**

- Sketching diagrams and using the symmetry properties of the normal distribution are useful techniques you can use.