
Question 1 (3 marks)

a. $y = \cos(1 - x^2)$

$$\begin{aligned}\frac{dy}{dx} &= -\sin(1 - x^2) \times -2x && \text{(chain rule)} \\ &= 2x \sin(1 - x^2)\end{aligned}\quad (1 \text{ mark})$$

b. $f(x) = \frac{\sin(2x)}{1 + e^{2x}}$

$$f'(x) = \frac{(1 + e^{2x}) \times 2\cos(2x) - 2e^{2x} \times \sin(2x)}{(1 + e^{2x})^2} \quad \text{(quotient rule)} \quad (1 \text{ mark})$$

$$\begin{aligned}f'(0) &= \frac{(1 + e^0) \times 2\cos(0) - 2e^0 \times \sin(0)}{(1 + e^0)^2} \\ &= \frac{(1 + 1) \times 2 \times 1 - 2 \times 1 \times 0}{(1 + 1)^2} \\ &= \frac{4 - 0}{2^2} \\ &= 1\end{aligned}\quad (1 \text{ mark})$$

Question 2 (3 marks)

a. $(f \circ g)(x) = \log_e(x^2 + 1)$ (1 mark)

b. $d_{f \circ g} = d_g = R$ (1 mark)

$$r_{f \circ g} = [0, \infty) \quad (1 \text{ mark})$$

Note that the minimum value of x^2 is zero, so the minimum value of $x^2 + 1$ is 1, and $\log_e(1) = 0$.

Question 3 (5 marks)

a. average rate of change = $\frac{f\left(\frac{\pi}{8}\right) - f(0)}{\frac{\pi}{8} - 0}$

$$= \frac{2 - 1}{\frac{\pi}{8}}$$

$$= \frac{8}{\pi} \quad (1 \text{ mark})$$

Note that the value of $f\left(\frac{\pi}{8}\right)$ can be found using the symmetry of the graph, i.e. the graph passes through the point $\left(\frac{\pi}{8}, 2\right)$.

Alternatively, it can be found by evaluating $\tan\left(\frac{\pi}{4}\right) + 1 = 2$.

b. $f(x) = 1 + \frac{1}{\sqrt{3}}$

$$\tan(2x) + 1 = 1 + \frac{1}{\sqrt{3}}$$

$$\tan(2x) = \frac{1}{\sqrt{3}}$$

$$2x = \dots - \frac{5\pi}{6}, \frac{\pi}{6}, \frac{7\pi}{6}, \dots$$

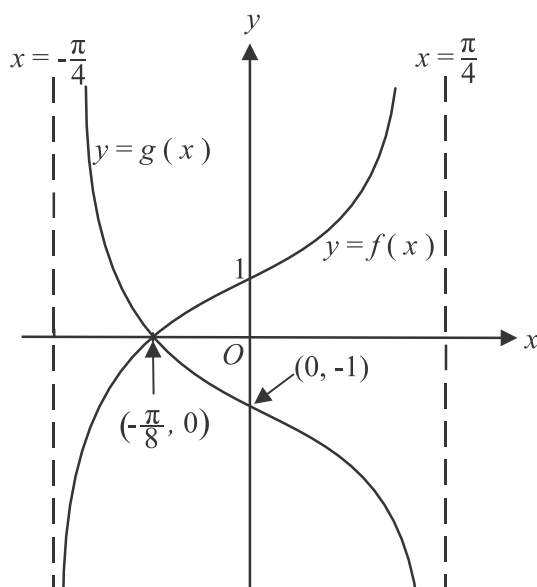
$$x = \dots - \frac{5\pi}{12}, \frac{\pi}{12}, \frac{7\pi}{12}, \dots$$

But $d_f = \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ and from the graph, we see that there would only be one point of intersection between the graph of f and the line $y = 1 + \frac{1}{\sqrt{3}}$. So $x = \frac{\pi}{12}$. (1 mark)



base angle = $\frac{\pi}{6}$ (1 mark)

- c. The graph of f is reflected in the y -axis and then translated 2 units down to obtain the graph of g .



(1 mark) - correct shape
(1 mark) - correct axis intercepts

Question 4 (4 marks)

a. $\int_0^{e-1} \frac{3}{x+1} dx = \left[3 \log_e(x+1) \right]_0^{e-1}$ (1 mark)

$$= 3(\log_e(e-1+1) - \log_e(1))$$

$$= 3(\log_e(e) - 0)$$

$$= 3 \times 1$$

$$= 3$$
 (1 mark)

b. $f'(x) = 2\sin(\pi x)$

$$f(x) = \int 2\sin(\pi x) dx$$

$$= -\frac{2}{\pi} \cos(\pi x) + c$$
 (1 mark)

Since $f\left(\frac{1}{3}\right) = 0$,

$$0 = -\frac{2}{\pi} \cos\left(\frac{\pi}{3}\right) + c$$

$$c = \frac{2}{\pi} \times \frac{1}{2}$$

$$= \frac{1}{\pi}$$

So $f(x) = -\frac{2}{\pi} \cos(\pi x) + \frac{1}{\pi}$. (1 mark)

Question 5 (4 marks)

a. $y = x^2 \log_e(2x)$

$$\begin{aligned}\frac{dy}{dx} &= 2x \log_e(2x) + x^2 \times \frac{2}{2x} \\ &= 2x \log_e(2x) + x\end{aligned}$$

(1 mark)

b. average value = $\frac{1}{1 - \frac{1}{2}} \int_{\frac{1}{2}}^1 f(x) dx$

$$\begin{aligned}&= 2 \int_{\frac{1}{2}}^1 x \log_e(2x) dx \\ &= \int_{\frac{1}{2}}^1 2x \log_e(2x) dx\end{aligned}$$

(1 mark)

From part a. $\frac{dy}{dx} = 2x \log_e(2x) + x$

$$\begin{aligned}\int (2x \log_e(2x) + x) dx &= x^2 \log_e(2x) \\ \int 2x \log_e(2x) dx + \int x dx &= x^2 \log_e(2x) \\ \int 2x \log_e(2x) dx &= x^2 \log_e(2x) - \int x dx \\ &= x^2 \log_e(2x) - \frac{x^2}{2} + c\end{aligned}$$

So average value = $\left[x^2 \log_e(2x) - \frac{x^2}{2} \right]_{\frac{1}{2}}^1$

(1 mark)

$$\begin{aligned}&= \left(\log_e(2) - \frac{1}{2} \right) - \left(\frac{1}{4} \log_e(1) - \frac{1}{8} \right) \\ &= \log_e(2) - \frac{1}{2} - 0 + \frac{1}{8} \\ &= \log_e(2) - \frac{3}{8} \\ &= \log_e(2) - \frac{3}{2^3}\end{aligned}$$

(1 mark)

Question 6 (2 marks)

$$E(\hat{P}) = p$$

$$\Pr(\hat{P}=0) = \binom{4}{0} p^0 (1-p)^4 = \frac{1}{4} \quad (1 \text{ mark})$$

$$1-p = \pm \frac{1}{\sqrt{2}}$$

$$\text{Since } 0 < p < 1, \quad \text{then } p = 1 - \frac{1}{\sqrt{2}}. \quad (1 \text{ mark})$$

Question 7 (4 marks)

a. Stationary points occur when $f'(x) = 0$

$$f(x) = x + \frac{1}{x-2}$$

$$= x + (x-2)^{-1}$$

$$f'(x) = 1 - (x-2)^{-2}$$

$$\text{We require that } 1 - \frac{1}{(x-2)^2} = 0$$

$$1 = \frac{1}{(x-2)^2}$$

$$(x-2)^2 = 1$$

$$x-2 = \pm 1$$

$$x = 1+2 \quad \text{or} \quad x = -1+2$$

$$x = 3 \quad \text{or} \quad x = 1$$

(1 mark)

$$f(1) = 1 + \frac{1}{-1} = 0 \quad \text{and} \quad f(3) = 3 + \frac{1}{3-2} = 4$$

The stationary points are (1, 0) and (3, 4). (1 mark)

b. i. Looking at the stationary points on the graph and using our answers to part a., we note that if the graph of f is translated by **between** 0 and 4 units downwards, then the graph of f will not intersect with the x -axis. Hence there will be no solutions to the equation $f(x) + c = 0$. So we require that $-4 < c < 0$.

(1 mark)

ii. Method 1

In order to become the graph of $y = 1 + f(a - x)$, the graph of $y = f(x)$ has been

- reflected in the y-axis
- translated a units horizontally (to the left if a is negative and to the right if a is positive)
- translated 1 unit vertically upwards

For $y = 1 + f(a - x)$ to have no y-intercepts, we require that its vertical asymptote lies on the y-axis.

The vertical translation has no effect on the horizontal movement of the graph.

The reflection in the y-axis means that the asymptote of $x = 2$ will be relocated to become $x = -2$.

A translation of 2 units to the right will then place the asymptote on the y-axis. So we require that $a = 2$.

(1 mark)

Method 2

$$\begin{aligned} y &= 1 + f(a - x) \\ &= 1 + a - x + \frac{1}{a - x - 2} \end{aligned}$$

Substitute $x = 0$ to get the y-intercept i.e.

$$y = 1 + a + \frac{1}{a - 2}$$

Since $1 + a + \frac{1}{a - 2}$ is undefined when $a = 2$, the graph can have no y-intercepts when $a = 2$.

(1 mark)

Question 8 (7 marks)

- a. Since the probability density function f is continuous at $x=a$, then

$$e^x - 1 = 6e^{-x} \quad \text{at } x=a \quad (1 \text{ mark})$$

$$\text{therefore } e^a - 1 = 6e^{-a}$$

$$e^a - 1 - 6e^{-a} = 0$$

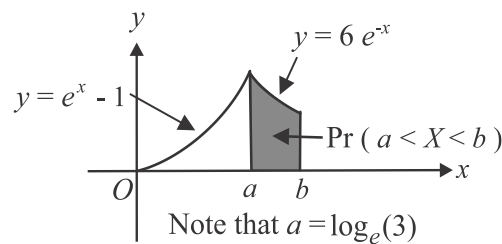
$$e^{2a} - e^a - 6 = 0 \quad (\text{multiply left and right hand sides by } e^a)$$

$$(e^a - 3)(e^a + 2) = 0$$

$$e^a = 3 \quad \text{or } e^a = -2 \quad \text{but } e^a > 0 \text{ so reject this.}$$

$$\text{So } a = \log_e(3). \quad (1 \text{ mark})$$

- b. i. $\Pr(a < X < b) = 1 - \Pr(0 < X < a)$ since $\Pr(0 < X < b) = 1$



$$\begin{aligned} \Pr(0 < X < a) &= \int_0^{\log_e(3)} (e^x - 1) dx \\ &= [e^x - x]_0^{\log_e(3)} \\ &= (e^{\log_e(3)} - \log_e(3)) - (e^0 - 0) \\ &= 3 - \log_e(3) - 1 \\ &= 2 - \log_e(3) \end{aligned}$$

(1 mark)

$$\begin{aligned} \text{So } \Pr(a < X < b) &= 1 - (2 - \log_e(3)) \\ &= \log_e(3) - 1 \end{aligned}$$

(1 mark)

ii. So $\int_{\log_e(3)}^b 6e^{-x} dx = \log_e(3) - 1$ (from part b. i.) **(1 mark)**

Now $\int_{\log_e(3)}^b 6e^{-x} dx = [-6e^{-x}]_{\log_e(3)}^b$ **(1 mark)**

$$\begin{aligned}
 &= (-6e^{-b}) - (-6e^{-\log_e(3)}) \\
 &= (-6e^{-b}) - (-6e^{\log_e(3)^{-1}}) \\
 &= (-6e^{-b}) - \left(-6e^{\log_e\left(\frac{1}{3}\right)}\right) \\
 &= (-6e^{-b}) - \left(-6 \times \frac{1}{3}\right) \\
 &= (-6e^{-b}) - (-2) \\
 &= 2 - 6e^{-b}
 \end{aligned}$$

So $2 - 6e^{-b} = \log_e(3) - 1$

$$-6e^{-b} = \log_e(3) - 3$$

$$6e^{-b} = 3 - \log_e(3)$$

$$e^{-b} = \frac{3 - \log_e(3)}{6}$$

$$-b = \log_e\left(\frac{3 - \log_e(3)}{6}\right)$$

$$b = -\log_e\left(\frac{3 - \log_e(3)}{6}\right)$$

$$b = \log_e\left(\frac{6}{3 - \log_e(3)}\right)$$

(1 mark)

Question 9 (8 marks)

- a.
- x-intercepts
- occur when
- $y=0$

$$0 = \sqrt{3-x}$$

$$x=3$$

A is the point (3, 0).

- y-intercepts
- occur when
- $x=0$

$$y = \sqrt{3-0}$$

$$= \sqrt{3}$$

B is the point (0, $\sqrt{3}$).

(1 mark)

$$\text{gradient} = \frac{\sqrt{3}-0}{0-3}$$

$$= -\frac{\sqrt{3}}{3}$$

Using the coordinates of point A, $y-0 = -\frac{\sqrt{3}}{3}(x-3)$

$$y = -\frac{\sqrt{3}}{3}x + \sqrt{3}$$

(1 mark)

- b.
- Method 1

$$\text{area} = \int_0^3 \left(\sqrt{3-x} - \left(-\frac{\sqrt{3}}{3}x + \sqrt{3} \right) \right) dx$$

$$= \int_0^3 \left(\left((3-x)^{\frac{1}{2}} \right) + \frac{\sqrt{3}}{3}x - \sqrt{3} \right) dx$$

$$= \left[-\frac{2}{3}(3-x)^{\frac{3}{2}} + \frac{\sqrt{3}}{6}x^2 - \sqrt{3}x \right]_0^3$$

(1 mark)

$$= \left(0 + \frac{3\sqrt{3}}{2} - 3\sqrt{3} \right) - \left(-\frac{2}{3} \times 3\sqrt{3} + 0 - 0 \right) \quad \text{since } 3^{\frac{3}{2}} = \sqrt{27} = 3\sqrt{3}$$

$$= \frac{3\sqrt{3}}{2} - 3\sqrt{3} + 2\sqrt{3}$$

$$= \frac{3\sqrt{3}}{2} - \sqrt{3}$$

$$= \frac{\sqrt{3}}{2} \text{ square units}$$

(1 mark)

Method 2 – using $\triangle AOB$

$$\begin{aligned}\text{area} &= \int_0^3 \sqrt{3-x} \, dx - \frac{1}{2} \times 3 \times \sqrt{3} \\ &= \int_0^3 (3-x)^{\frac{1}{2}} \, dx - \frac{3\sqrt{3}}{2} \\ &= \left[-\frac{2}{3} (3-x)^{\frac{3}{2}} \right]_0^3 - \frac{3\sqrt{3}}{2}\end{aligned}$$

(1 mark)

$$\begin{aligned}&= (0) - \left(-\frac{2}{3} \times 3^{\frac{3}{2}} \right) - \frac{3\sqrt{3}}{2} \\ &= \frac{2}{3} \times 3\sqrt{3} - \frac{3\sqrt{3}}{2} \quad \text{i.e. } 3^{\frac{3}{2}} = \sqrt{27} = 3\sqrt{3} \\ &= 2\sqrt{3} - \frac{3\sqrt{3}}{2} \\ &= \frac{\sqrt{3}}{2} \text{ square units}\end{aligned}$$

(1 mark)

c. $f(x) = \sqrt{3-x}$

$$= (3-x)^{\frac{1}{2}}$$

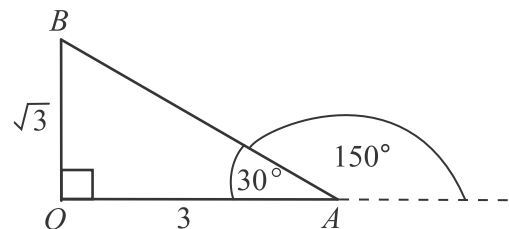
$$f'(x) = \frac{1}{2} (3-x)^{-\frac{1}{2}} \times -1 \quad (\text{chain rule})$$

$$= \frac{-1}{2\sqrt{3-x}}$$

(1 mark)

d. In $\triangle ABO$, $\tan(\angle BAO) = \frac{\sqrt{3}}{3}$

$$\angle BAO = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30^\circ$$



So the line AB makes an angle of 150° with the positive direction of the x -axis. When $\theta = 30^\circ$, the tangent to f at P makes an angle of 120° with the positive direction of the x -axis.

$$\begin{aligned}f'(x) &= \tan(120^\circ) \\ &= -\sqrt{3} \quad (\text{tan is negative in the second quadrant})\end{aligned}$$

(1 mark)



Solve $\frac{-1}{2\sqrt{3-x}} = -\sqrt{3}$ for x (using our result to part **c.**)

$$\frac{1}{2\sqrt{3}} = \sqrt{3-x}$$

$$\frac{1}{4 \times 3} = 3-x$$

$$x = 3 - \frac{1}{12}$$

$$x = \frac{35}{12}$$

(1 mark)

$$f\left(\frac{35}{12}\right) = \sqrt{3 - \frac{35}{12}}$$

$$= \sqrt{\frac{1}{12}}$$

$$= \frac{1}{2\sqrt{3}}$$

P is the point $\left(\frac{35}{12}, \frac{1}{2\sqrt{3}}\right)$ or $\left(\frac{35}{12}, \frac{\sqrt{3}}{6}\right)$.

(1 mark)