

MATHEMATICAL METHODS

Units 3 & 4 – Written examination 1



2023 Trial Examination

SOLUTIONS

Question 1

a. $y = x^{-2}e^{2x}$

$$\frac{dy}{dx} = -2x^{-3}e^{2x} + 2x^{-2}e^{2x}$$

M1

$$= -2e^{2x}x^{-3}(1 - x)$$

$$= \frac{2}{x^3}(x - 1)e^{2x}$$

A1

b. $f'(x) = (1)\log_e(3x^2) + (x + 1)\frac{6x}{3x^2}$

M1

$$= \log_e(3x^2) + \frac{2}{x}(x + 1)$$

$$f'(2) = \log_e(12) + 3$$

A1

Question 2

$$f(x) = \int \sqrt{x-1} + 2e^x + \cos(x-1) dx$$

$$= \frac{2}{3}(x-1)^{\frac{3}{2}} + 2e^x + \sin(x-1) + c$$

M1

$$0 = \frac{2}{3}(1-1)^{\frac{3}{2}} + 2e^1 + \sin(1-1) + c$$

$$c = -2e$$

$$\text{so } f(x) = \frac{2}{3}(x-1)^{\frac{3}{2}} + 2e^x + \sin(x-1) - 2e$$

A1

Question 3

$$\log_2(2x(x+1)) = 4$$

M1

$$2x^2 + 2x = 16$$

$$x^2 + x - 8 = 0$$

M1

$$x = \frac{-1 \pm \sqrt{33}}{2}$$

$$x = \frac{-1 + \sqrt{33}}{2}$$

A1

Question 4

a. $\left(\frac{2}{3}\right)^4 = \frac{16}{81}$

A1

b. $\Pr(X \geq 3) = \binom{4}{3} \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^1 + \binom{4}{4} \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^0$

M1

$$= 4 \times \frac{2}{81} + \frac{1}{81} = \frac{1}{9}$$

A1

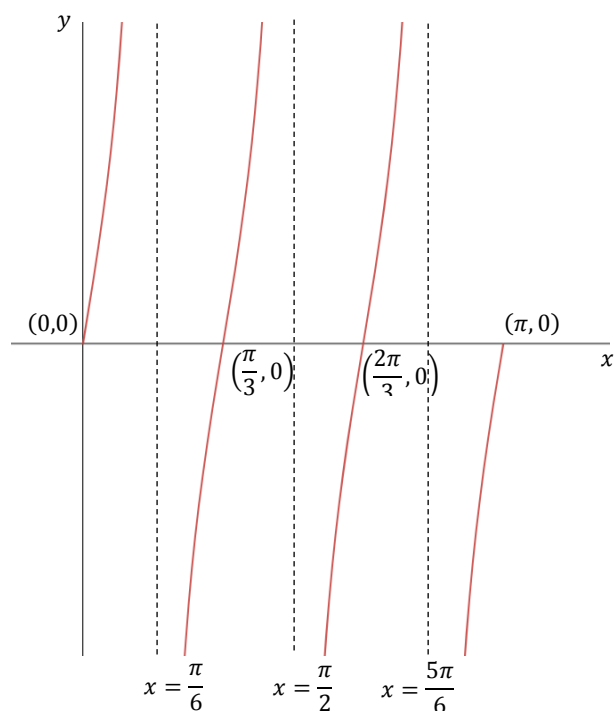
c. $\Pr(X \geq 3 | X > 0) = \frac{\Pr(X \geq 3)}{1 - \Pr(X=0)}$

$$= \frac{\frac{9}{81}}{\frac{16}{81}} = \frac{9}{16}$$

A1

Question 5

a.



A1 shape

A1 asymptotes

A1 intercepts and end points

b. $2 \tan(3x) + 2 = 0$
 $\tan(3x) = -1$

M1

$$\theta_R = \frac{-\pi}{4}$$

M1

$$3x = \frac{-\pi}{4} + n\pi$$

$$x = \frac{-\pi}{12} + \frac{n\pi}{3}$$

$$x = \frac{\pi}{12}(3 + 4n)$$

$$\text{so } x = \frac{\pi}{4}, \frac{7\pi}{12}, \frac{11\pi}{12}$$

A1

Question 6

a. $\frac{1}{2}x^2 - 2x + \frac{3}{2}$
 $= \frac{1}{2}(x^2 - 4x + 3)$
 $= \frac{1}{2}(x^2 - 4x + 4 - 4 + 3)$
 $= \frac{1}{2}((x - 2)^2 - 1)$
 $= \frac{1}{2}(x - 2)^2 - \frac{1}{2}$

A1

- b. Dilation of factor $\frac{1}{2}$ from the x axis followed by translation of 2 units in the positive x direction and $\frac{1}{2}$ unit in the negative y direction.

A2

c. $a = 2$

A1

- d. $h(g(x))$ is defined if $\text{ran } g \subseteq \text{dom } h$
 $\text{ran } g = \text{dom } h = \left[-\frac{1}{2}, \infty\right)$ hence $h(g(x))$ is defined.

A1

e. $h(g(x)) = \sqrt{1 + 2\left(\frac{1}{2}(x - 2)^2 - \frac{1}{2}\right)} = \sqrt{(x - 2)^2} = x - 2$

A1

Question 7

- a. Translation of 2 units in the negative x direction

A1

- b. $y_2(x) = -3x^2(x^2 - 4)$

$$\begin{aligned}\frac{dy_2}{dx} &= -6x(x^2 - 4) - 6x^3 = -12x^3 + 24x \\ &= -12x(x^2 - 2) = 0\end{aligned}$$

M1

$$\begin{aligned}x &= 0, \pm\sqrt{2} \\ y_2(0) &= 0 \\ y_2(\sqrt{2}) &= y_2(-\sqrt{2}) = 12\end{aligned}$$

So stationary points at $(0, 0)$, $(-\sqrt{2}, 12)$ and $(\sqrt{2}, 12)$

A1

- c. Area bounded by $y_1(x)$ and the axes is given by $\int_0^2 y_1(x) dx + \int_2^4 y_1(x) dx$

Which is equivalent to $\int_{-2}^0 y_2(x) dx + \int_0^2 y_2(x) dx$

And due to symmetry is equivalent to $2 \int_0^2 y_2(x) dx$

Hence $a = b = 2$

A2

- d. $Area = 2 \int_0^2 -3x^4 + 12x^2 dx = 6 \int_0^2 -x^4 + 4x^2 dx$

$$= 6 \left[-\frac{1}{5}x^5 + \frac{4}{3}x^3 \right]_0^2$$

M1

$$\begin{aligned}&= 6 \left(-\frac{1}{5}(2)^5 + \frac{4}{3}(2)^3 \right) \\ &= 64 - \frac{192}{5} = \frac{128}{5} \text{ sq units}\end{aligned}$$

A1

Question 8

a.

$$\frac{d}{dx}(x^n \log_e(x))$$

$$\text{let } u = x^n, u' = nx^{n-1}, \quad v = \log_e(x), \quad v' = \frac{1}{x}$$

$$\frac{d}{dx}(x^n \log_e(x)) = (x^n) \left(\frac{1}{x} \right) + (nx^{n-1})(\log_e(x))$$

M1

$$= x^{n-1} + nx^{n-1} \log_e(x)$$

$$= x^{n-1}(n \log_e(x) + 1)$$

M1

$$\text{Hence } \int x^{n-1}(n \log_e(x) + 1) dx = x^n \log_e(x)$$

$$\text{b. } \int x^{n-1}(n \log_e(x) + 1) dx = x^n \log_e(x)$$

$$\int x^{n-1} n \log_e(x) dx + \int x^{n-1} dx = x^n \log_e(x)$$

$$\int x^{n-1} n \log_e(x) dx = x^n \log_e(x) - \frac{1}{n} x^n$$

$$\int x^{n-1} \log_e(x) dx = \frac{1}{n} x^n \log_e(x) - \frac{1}{n^2} x^n$$

M1

$$\int_0^1 ax^2 \log_e(x) dx = 1, \quad \text{so } n = 3$$

$$= a \left[\frac{1}{3} x^3 \log_e(x) - \frac{1}{9} x^3 \right]_0^1 = 1$$

M1

$$a \left(\left(0 - \frac{1}{9} \right) - (0 - 0) \right) = 1$$

$$a = -9$$

A1

$$\text{c. } E(X) = \int_0^1 -9x^3 \log_e(x) dx$$

M1

$$\int x^{n-1} \log_e(x) dx = \frac{1}{n} x^n \log_e(x) - \frac{1}{n^2} x^n \quad \text{where } n = 4,$$

$$\int x^3 \log_e(x) dx = \frac{1}{4} x^4 \log_e(x) - \frac{1}{16} x^4$$

$$\text{So } E(X) = -9 \left[\frac{1}{4} x^4 \log_e(x) - \frac{1}{16} x^4 \right]_0^1$$

M1

$$= -9 \left(\left(0 - \frac{1}{16} \right) - (0 - 0) \right)$$

$$E(X) = \frac{9}{16}$$

A1