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NAME: _____

MATHEMATICAL METHODS

Unit 3 & 4 Practice Written Examination 1

Reading time: 15 minutes

Writing time: 1 hour

QUESTION AND ANSWER BOOK

Structure of Book

Number of questions	Number of questions to be answered	Number of marks
9	9	40

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, and rulers.
- Students are NOT permitted to bring into the examination room: any technology (calculators or software), notes of any kind, blank sheets of paper and/or correction fluid/tape.

Materials supplied

- Question and Answer Book of 14 pages.
- Formula Sheet.
- Working space is provided throughout the book.

Instructions

- Write your student number in the space provided above on this page.
- Unless otherwise indicated, the diagrams in this book are not drawn to scale.
- All written responses must be in English.

At the end of the Examination

- You may keep the Formula Sheet.

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Question 1 (3 marks)

Let $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = \frac{\cos(x)}{\sin(2x)}$

a. Find $f'(x)$

1 mark

b. Let $g: \mathbb{R} \rightarrow \mathbb{R} \setminus \{0\}, g(x) = 3 \log_e(x^2)$. For what value of x is $g'(x) = 1$?

2 marks

Question 2 (3 marks)

Given $x \in [0, 8\pi]$, solve the equation:

$$\left(\sin\left(\frac{x}{2}\right)\right)^2 = 2\sin\left(\frac{x}{2}\right) - 1$$

Question 3 (7 marks)

Consider $f(x) = \frac{a}{\sqrt{b-x}}$, $a, b \in \mathbb{R}^{++}$

a. Determine the equation of $f^{-1}(x)$ and state its domain.

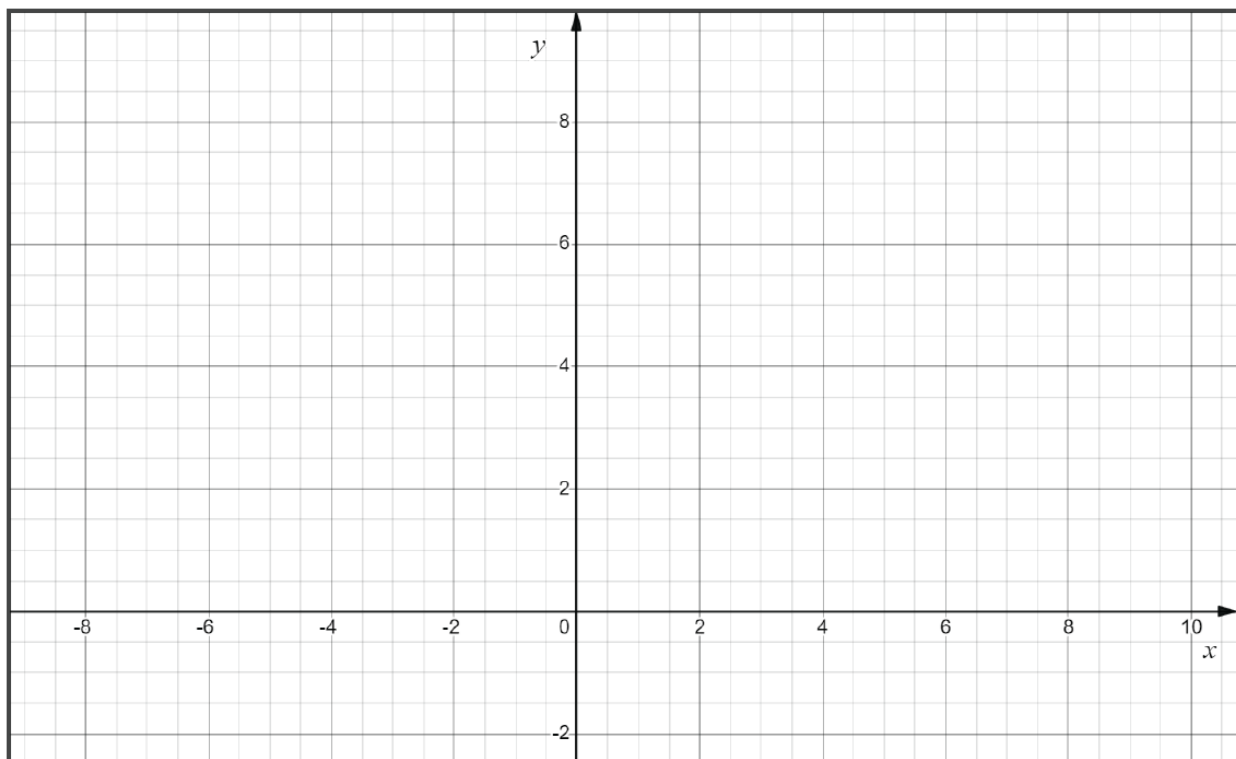
2 marks

- b. If $f^{-1}(1) = 2$ and $f(5) = 2$, find the values of a and b .

2 marks

- c. Hence sketch the graph of $f(x)$ on the axes provided. Include the coordinate of all intercepts and equations of any asymptotes. Hint: The graph of $f^{-1}(x)$ may be of assistance.

3 marks



Question 4 (3 marks)

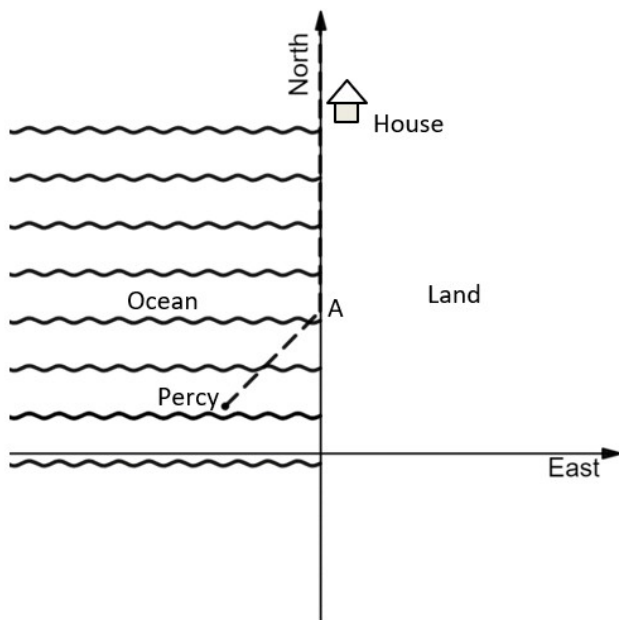
Consider $f(x) = 2\cos(2x)$

Determine the area bound by $f(x)$, the x -axis, $x = 0$ and $x = 3\pi/4$.

Question 5 (6 marks)

Percy was swimming at the beach and suddenly remembered he had an appointment.

His location at that moment was 4 km West and 10 km south of his house, as shown in the diagram below.



Percy needed to get home as quickly as possible. He decided to swim in a straight line towards the beach, then run north along the beach towards home. His swimming speed is 3 km/hr and his running speed along the sandy beach is 4 km/hr.

Let A be the point where he reaches the beach and begins his run. Let x be the distance of A north of his original position.

- a. If L is the total distance Percy travelled to get home, develop an equation for L in terms of x .

2 marks

- b.** Hence, develop an equation for T , the total time taken for Percy to reach the house.

1 mark

- c.** Determine the shortest time that Percy needs to get home. Express your answer in the form

$$\frac{a+b\sqrt{d}}{d}, a, b, c, d \in \mathbb{Z}$$

3 marks

Question 6 (5 marks)

A game of ‘Doubles’ involves a player rolling two regular six-sided dice. The player wins if the number rolled on one die, is double the number rolled on the other die. All other combinations result in a loss.

- a.** What is the probability that a player wins a particular game of Doubles? 2 marks

- b.** In a particular game of Doubles, one of the rolls was greater than 3. What is the probability that the player won the game? 1 mark

- c. A game of Doubles costs \$1 to play. How much should the prize payment be so that the game is fair? A fair game is one where the average amount won or lost is \$0. 2 marks

Question 7 (4 marks)

X is a random variable representing the number of mosquito bites a hiker may receive while traversing the Swamp of Despair. The probability density function of X is provided below.

$$f(x) = ax(b-x), 0 < x < b$$

- a.** Assuming a hiker cannot receive more than 10 mosquito bites, determine the value of the parameters a and b in $f(x)$.

2 marks

- b.** Hence what is the probability that a hiker will receive not more than two mosquito bites when crossing the swamp?

2 marks

Question 8 (3 marks)

50 people from the city of Wombat were surveyed regarding their favourite breed of dog. A certain proportion, less than half, of the respondents indicated that they preferred Cocker Spaniels. Let $\hat{p}$ be the sample proportion of people who prefer Cocker Spaniels.

- a. If the standard deviation of $\hat{p}$ is $\sqrt{42}/100$, how many of the 50 people surveyed preferred cocker spaniels? 2 marks

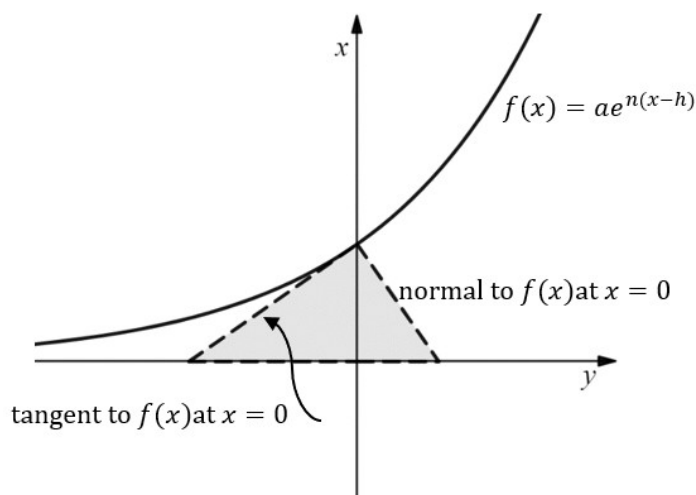
- b. If the standard deviation of $\hat{p}$ was to be halved, how many people would need to be surveyed?

1 mark

Question 9 (6 marks)

Consider $f(x) = ae^{n(x-h)}$, $a, h, n \in \mathbb{R}^{++}$

A triangle can be formed whose vertices correspond to the point of intersection of the tangent and normal lines at $x = 0$, the point of intersection of the tangent line with the x -axis and the point of intersection of the normal line with the x -axis. This triangle is shown in the graph below.



- a. Show that the equation for the area of the triangle formed by the x -axis and tangent and normal lines of $f(x)$ at $x = 0$ is:

$$\text{Area} = \frac{a^3 n^3 + a e^{2nh}}{2 n e^{3nh}}$$

3 marks

- b.** If the area of the triangle is k units squared, and $h = 0$ in $f(x)$, develop an equation that will provide the value of n for any given value of a and k . Include any restrictions on a .

3 marks

END OF QUESTION AND ANSWER BOOK

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MATHEMATICAL METHODS

Written Examinations 1 and 2

FORMULA SHEET

Instructions

This Formula Sheet is provided for your reference.

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Mensuration

area of a trapezium	$\frac{1}{2}(x+b)h$	volume of a pyramid	$\frac{1}{3}Ah$
curved surface area of a cylinder	$2\pi rh$	volume of a sphere	$\frac{4}{3}\pi r^3$
volume of a cylinder	$\pi r^2 h$	area of a triangle	$\frac{1}{2}bc \sin(A)$
volume of a cone	$\frac{1}{3}\pi r^2 h$		

Calculus

$\frac{d}{dx}(x^n)=nx^{n-1}$	$\int x^n dx=\frac{1}{n+1}x^{n+1}+c,n\neq-1$		
$\frac{d}{dx}((ax+b)^n)=an(ax+b)^{n-1}$	$\int \color{red}{\text{u}} \color{red}{\text{u}}$		
$\frac{d}{dx}(e^{ax})=ax^{ax}$	$\int e^{ax} dx=\frac{1}{a}e^{ax}+c$		
$\frac{d}{dx}(\log_e(x))=\frac{1}{x}$	$\int \frac{1}{x} dx=\log_e(x)+c$		
$\frac{d}{dx}(\sin(ax))=a\cos(ax)$	$\int \sin(ax) dx=\frac{-1}{a}\cos(ax)+c$		
$\frac{d}{dx}(\cos(ax))=-a\sin(ax)$	$\int \cos(ax) dx=\frac{1}{a}\sin(ax)+c$		
$\frac{d}{dx}(\tan(ax))=\frac{a}{\cos^2(ax)}=a\sec^2(ax)$			
product rule	$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$	quotient rule	$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$
chain rule	$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$		

Probability

$\Pr(A) = 1 - \Pr(A')$		$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$	
$\Pr(A B) = \frac{\Pr(A \cap B)}{\Pr(B)}$			
mean	$\mu = E(X)$	Variance	$\text{var}(X) = \sigma^2 = E(X^2) - \mu^2$

Probability distribution		Mean	Variance
discrete	$\Pr(X=x) = p(x)$	$\mu = \sum xp(x)$	$\sigma^2 = E((X-\mu)^2) = E(X^2) - \mu^2$
continuous	$\Pr(a < X < b) = \int_a^b f(x) dx$	$\mu = \int_{-\infty}^{\infty} xf(x) dx$	$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx$

Sample proportions

$\hat{p} = \frac{X}{n}$		mean	$E(\hat{p}) = p$
Standard deviation	$sd(\hat{p}) = \sqrt{\frac{p(1-p)}{n}}$	approximate confidence interval	$\left(\hat{p} - z \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + z \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \right)$



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VCE® Mathematical Methods

Unit 3 & 4 Trial Written Examination

EXAMINATION 1

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Solution Pathway

Below are sample answers. Please consider the merit of alternative responses.

Question 1 (3 marks)

$$\text{Let } f: R \rightarrow R, f(x) = \frac{\cos(x)}{\sin(2x)}$$

a. Find $f'(x)$

1 mark

Quotient rule.

$$\text{Let } u = \cos(x), \frac{du}{dx} = -\sin(x)$$

$$\text{Let } v = \sin(2x), \frac{dv}{dx} = 2 \cos(2x)$$

$$f'(x) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

$$\frac{\sin(2x) \times -\sin(x) - \cos(x) \times 2 \cos(2x)}{\sin^2(2x)}$$

$$\frac{-\sin(2x) \sin(x) - \cos(x) 2 \cos(2x)}{\sin^2(2x)}$$

- 1 mark for correct derivative (any mathematically correct form acceptable)

b. Let $g: R \rightarrow R \setminus \{0\}$, $g(x) = 3 \log_e(x)^2$. For what value of x is $g'(x) = 1$

2 marks

$$g(x) = 3 \log_e(x)^2$$

$$6 \log_e(x).$$

$$g'(x) = \frac{6}{x}$$

$$\text{At } g'(x) = 1,$$

$$\frac{6}{x} = 1$$

$$x=6$$

Alternatively:

$$g(x) = 3 \log_e(x)^2$$

$$\text{Let } y = 3 \log_e(x)^2$$

$$\text{Let } u = x^2, y = 3 \log_e(u)$$

$$\frac{du}{dx} = 2x, \frac{dy}{du} = \frac{3}{u}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$2x \times \frac{3}{u}$$

$$2x \times \frac{3}{x^2}$$

$$\frac{6}{x}$$

Then as per above.

- **1 mark** for correct derivative of $g(x)$.
- **1 mark** for correct value of x for when $g'(x) = 1$

Question 2 (3 marks)

Given $x \in [0, 8\pi]$, solve the equation:

$$\left(\sin\left(\frac{x}{2}\right)\right)^2 = 2\sin\left(\frac{x}{2}\right) - 1$$

$$\left(\sin\left(\frac{x}{2}\right)\right)^2 - 2\sin\left(\frac{x}{2}\right) + 1 = 0$$

$$\text{Let } k = \sin\left(\frac{x}{2}\right)$$

$$\therefore k^2 - 2k + 1 = 0$$

$$(k - 1)^2 = 0$$

$$k = 1$$

$$\therefore \sin\left(\frac{x}{2}\right) = 1$$

$$\therefore \frac{x}{2} = \frac{\pi}{2}$$

$$x = \pi$$

Check for other solutions within the domain $[0, 8\pi]$.

$$\text{Period} = \frac{2\pi}{n} = \frac{2\pi}{\frac{1}{2}} = 4\pi$$

$$x = \pi + 4\pi$$

$$\therefore 5\pi$$

No more solutions since adding 4π to 5π will yield 9π , which is outside the domain.

$$x = \{\pi, 5\pi\}$$

- **1 mark** for a valid approach to solving equation.
- **1 mark** for a valid consideration of the domain
- **1 mark** for correct values of x .

Question 3 (7 marks)

Consider $f(x) = \frac{a}{\sqrt{b-x}}$, $a, b \in \mathbb{R}^{++}$

a. Determine the equation of $f^{-1}(x)$ and state its domain.

2 marks

Let $y = f(x)$

Substitute x and y and rearrange to make y the subject.

$$x = \frac{a}{\sqrt{b-y}}$$

$$x^2 = \frac{a^2}{b-y}$$

$$b-y = \frac{a^2}{x^2}$$

$$y = b - \frac{a^2}{x^2}$$

$$\therefore f^{-1}(x) = b - \frac{a^2}{x^2}$$

Domain of $f^{-1}(x)$ = range of $f(x)$

Range of $f(x) = (0, \infty)$

Domain $f^{-1}(x) = (0, \infty)$

- **1 mark** for correct equation for $f^{-1}(x)$
- **1 mark** for correct domain of $f^{-1}(x)$.

b. If $f^{-1}(1) = 2$ and $f(5) = 2$, find the values of a and b .

2 marks

$$f^{-1}(1) = 2$$

$$\therefore 2 = b - a^2 \quad 1.$$

$$f(5) = 2$$

$$\therefore 2 = \frac{a}{\sqrt{b-5}} \quad 2.$$

$$\text{From 1., } b = a^2 + 2$$

Substitute $a^2 + 2$ for b in 2.

$$2 = \frac{a}{\sqrt{a^2 + 2 - 5}}$$

$$2 = \frac{a}{\sqrt{a^2 - 3}}$$

$$4 = \frac{a^2}{a^2 - 3}$$

$$4a^2 - 12 = a^2$$

$$3a^2 = 12$$

$$a^2 = 4$$

$$a = 2 \text{ or } -2$$

Since $b = a^2 + 2$, $b = 6$

$$a = 2, b = 6$$

- **1 mark** for correct value of a .
- **1 mark** for correct value of b .

- c. Sketch the graph of $f(x)$ on the axes provided. Include the coordinate of all intercepts and equations of any asymptotes. Hint: The graph of $f^{-1}(x)$ may be of assistance. **3 marks**

Consider $f^{-1}(x) = 6 - \frac{4}{x^2}, x > 0$

Truncus reflected about the x -axis, dilated by a factor of 4 parallel to y -axis and translated 6 units up.

Horizontal asymptote at $y = 6$. Vertical asymptote at $x = 0$.

No y -intercept.

x -intercept at $f^{-1}(x) = 0$

$$6 - \frac{4}{x^2} = 0$$

$$\frac{4}{x^2} = 6$$

$$x^2 = \frac{2}{3}$$

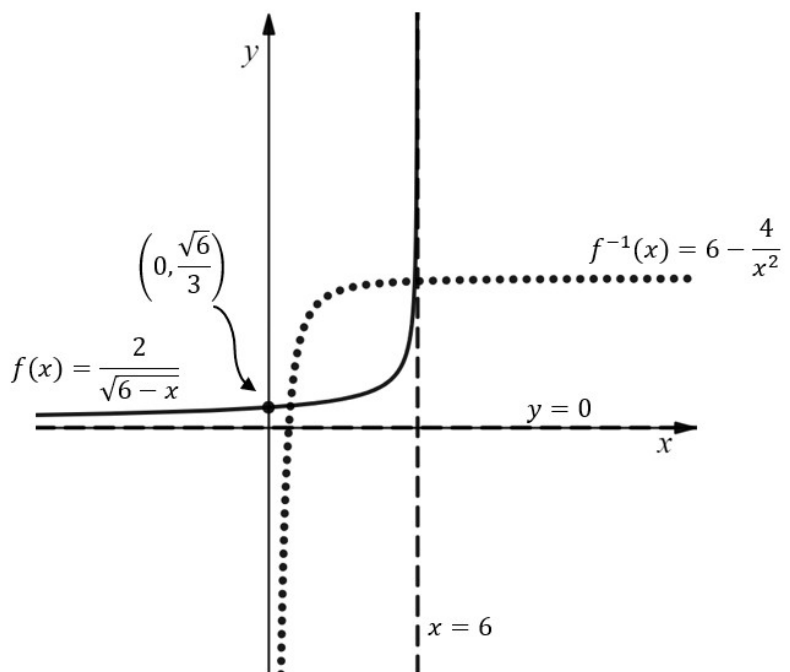
$$x = \frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{6}}{3} \text{ (since } x > 0 \text{)}$$

x -intercept at $\left(\frac{\sqrt{6}}{3}, 0\right)$

$\therefore f(x)$ has a vertical asymptote at $x = 6$ $\wedge$ horizontal asymptote are $y = 0$.

$\therefore f(x)$ has a y -intercept at $\left(0, \frac{\sqrt{6}}{3}\right)$

$f(x)$ is a reflection of $f^{-1}(x)$ about $y = x$.



- **1 mark** for correct position and shape of graph.
- **1 mark** for correct y-intercept.
- **1 mark** for asymptotes shown with correct equations.

Question 4 (3 marks)

Consider $f(x) = 2\cos(2x)$

Determine the area bound by $f(x)$, the x -axis, $x = 0$ and $x = 3\pi/4$.

Need to determine how the relevant area is sectioned within the specified domain.

No horizontal translation of the graph so section between $x = 0$ and the first x -intercept is above the x -axis.

x -intercepts at $f(x) = 0$

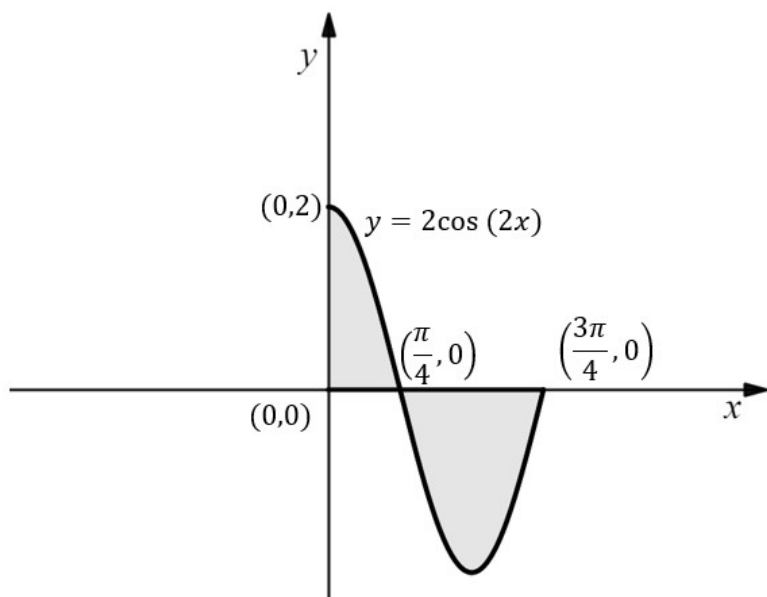
$$2\cos(2x) = 0$$

$$\cos(2x) = 0$$

$$x = \frac{\pi}{4}, \frac{5\pi}{4} \text{ etc}$$

$\therefore$ The section between $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$ is below the x -axis.

The graph below shows the relevant sections.



$$\text{Area} = \int_0^{\frac{\pi}{4}} 2\cos(2x) dx + \int_{\frac{3\pi}{4}}^{\frac{\pi}{4}} 2\cos(2x) dx$$

Note: terminals reversed in second definite integral to make area positive.

$$\int \left[\sin(2x) \right]_0^{\frac{\pi}{4}} + \left[\sin(2x) \right]_{\frac{3\pi}{4}}^{\frac{\pi}{4}}$$

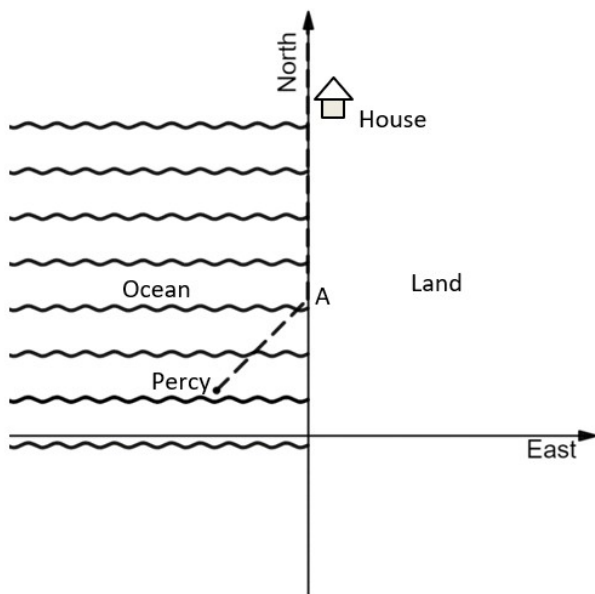
$$\int (1-0) + (1-(-1))$$

3 units squared

- **1 mark** for correct terminals.
- **1 mark** for correct equation for area involving definite integrals.
- **1 mark** for correct area.

Question 5 (6 marks)

Percy was swimming at the beach and suddenly remembered he had an appointment. His location at that moment was 4 km West and 10 km south of his house, as shown in the diagram below.



Percy needed to get home as quickly as possible. He decided to swim in a straight line towards the beach, then run north along the beach towards home. His average swimming speed is 3 km/hr and his running speed along the sandy beach is 4 km/hr.

Let A be the point where he reaches the beach and begins his run. Let x be the distance of A north of his original position.

a. If L is the total distance Percy travelled to get home, develop an equation for L in terms of x .

2 marks

Let L_1 be the straight-line distance to point A from his initial location. L_1 can be determined using Pythagoras' Theorem.

$$L_1^2 = x^2 + 4^2$$

$$L_1 = \sqrt{x^2 + 16}$$

Let L_2 be the distance from point A to the house.

$$L_2 = 10 - x$$

$$\therefore L = L_1 + L_2$$

$$L = \sqrt{x^2 + 16} + 10 - x$$

- **1 mark** for correct expressions for the component distances.
- **1 mark** for a correct expression for the total distance.

- b. Hence, develop an equation for T , the total time taken for Percy to reach the house.

1 mark

$$\text{speed} = \frac{\text{distance travelled}}{\text{time}}$$

$$\text{time} = \frac{\text{distance travelled}}{\text{speed}}$$

$$\therefore T = \frac{\sqrt{x^2+16}}{3} + \frac{10-x}{4}$$

- **1 mark** for a correct equation for the time taken.

- c. Determine the shortest time that Percy needs to get home. Express your answer in the form

$$\frac{a+b\sqrt{d}}{d}, a, b, c, d \in \mathbb{Z} \text{ 3 marks}$$

$$\text{Solve } \frac{dT}{dx} = 0$$

$$T = \frac{\sqrt{x^2+16}}{3} + \frac{10-x}{4}$$

$$\text{Let } y = \frac{\sqrt{x^2+16}}{3}$$

$$\text{Let } u = x^2 + 16 \quad y = \frac{\sqrt{u}}{3} = \frac{1}{3} u^{\frac{1}{2}}$$

$$\frac{du}{dx} = 2x \quad \frac{dy}{du} = \frac{1}{6} u^{-\frac{1}{2}} = \frac{1}{6\sqrt{u}}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\therefore \frac{1}{6\sqrt{u}} \times 2x$$

$$\therefore \frac{x}{3\sqrt{x^2+16}}$$

$$\therefore \frac{dT}{dx} = \frac{x}{3\sqrt{x^2+16}} - \frac{1}{4}$$

$$\frac{x}{3\sqrt{x^2+16}} - \frac{1}{4} = 0$$

$$\frac{x}{3\sqrt{x^2+16}} = \frac{1}{4}$$

$$4x = 3\sqrt{x^2+16}$$

$$16x^2 = 9(x^2+16)$$

$$16x^2 = 9x^2 + 144$$

$$7x^2 = 144$$

$$x^2 = \frac{144}{7}$$

$$x = \frac{12}{\sqrt{7}} = \frac{12\sqrt{7}}{7}$$

Point A located on the beach $\frac{12\sqrt{7}}{7}$ km north of his original position.

$$T = \frac{\sqrt{x^2+16}}{3} + \frac{10-x}{4}$$

$$i \frac{\sqrt{\left(\frac{12\sqrt{7}}{7}\right)^2+16}}{3} + \frac{10-\frac{12\sqrt{7}}{7}}{4}$$

$$i \frac{\sqrt{\frac{144}{7}+16}}{3} + \frac{\frac{70}{7}-\frac{12\sqrt{7}}{7}}{4}$$

$$i \frac{\sqrt{\frac{144}{7}+\frac{112}{7}}}{3} + \frac{\frac{70-12\sqrt{7}}{7}}{4}$$

$$i \frac{\sqrt{\frac{256}{7}}}{3} + \frac{\frac{70-12\sqrt{7}}{7}}{4}$$

$$i \frac{16\sqrt{\frac{1}{7}}}{3} + \frac{70-12\sqrt{7}}{4}$$

$$i \frac{16}{3\sqrt{7}} + \frac{70-12\sqrt{7}}{28}$$

$$i \frac{16\sqrt{7}}{21} + \frac{35-6\sqrt{7}}{14}$$

$$i \frac{32\sqrt{7}}{42} + \frac{105-18\sqrt{7}}{42}$$

$$i \frac{105+14\sqrt{7}}{42}$$

$$i \frac{15+2\sqrt{7}}{6} \text{ hours}$$

- **1 mark** for correct derivative.
- **1 mark** for correct value of x .
- **1 mark** for correct time expressed in the required form.

Question 6 (5 marks)

A game of ‘Doubles’ involve a player rolling two regular six-sided dice. If the number rolled on one die is double the number rolled on the other, the player wins. All other combinations result in a loss.

- a.** What is the probability that a player wins a particular game of Doubles? **2 marks**

The winning roles are:

(1,2), (2,1), (2,4), (4,2), (3,6) and (6, 3). Thus, there are six ways to win.

The total number of possible roles is 36.

$$Pr(\text{Win}) = \frac{6}{36} = \frac{1}{6}$$

- **1 mark** for determination of all winning combinations.
- **1 mark** for correct probability.

- b.** In a particular game of Doubles, one of the rolls was greater than 3. What is the probability that the player won the game? **1 mark**

This is a conditional probability problem. Let the die that produced a number greater than 3 be ‘Roll 1’.

$$Pr(\text{Win} | \text{Roll 1} > 3) = \frac{Pr(\text{Win} \cap \text{Roll 1} > 3)}{Pr(\text{Roll 1} > 3)}$$

$$\frac{\frac{2}{27}}{\frac{1}{2}} = \frac{4}{27}$$

- **1 mark** for correct probability.

- c. A game of doubles costs \$1 to play. How much should the prize payment be so that the game is fair?

A fair game is one where the average amount won or lost is \$0.

2 marks

For a game to be fair, expected value of prize amount is 0. Let X be the random variable that corresponds to the amount won or lost in a game. Let P be the prize amount and C be the cost to play.

$$E(X) = C \times \Pr(\text{losing}) + P \times \Pr(\text{winning})$$

$$0 = -1 \times \frac{5}{6} + P \times \frac{1}{6}$$

$$P \times \frac{1}{6} = \frac{5}{6}$$

$$P = \$5$$

- **1 mark** for an Expected value equation or equivalent.
- **1 mark** for the correct prize amount.

Question 7 (4 marks)

X is a random variable representing the number of mosquito bites a hiker may receive while traversing the Swamp of Despair. The probability density function of X is provided below.

$$f(x) = ax(b-x), 0 < x < b$$

- a.** Assuming a hiker cannot receive more than 10 mosquito bites, determine the value of the parameters a and b in $f(x)$. **2 marks**

Since the maximum number of mosquito bites is 10, $b = 10$.

For $f(x)$ to be a pdf, area under $f(x)$ must equal one.

$$\text{Area} = \int_0^{10} ax(10-x) dx$$

$$1 = \int_0^{10} 10ax - ax^2 dx$$

$$\int \left[5ax^2 - \frac{1}{3}ax^3 \right]_0^{10}$$

$$\int \left(500a - \frac{1000a}{3} \right) - 0$$

$$\int \frac{1500a}{3} - \frac{1000a}{3}$$

$$\int \frac{500a}{3}$$

$$500a = 3$$

$$a = \frac{3}{500}$$

$$\therefore a = \frac{3}{500}, b = 10$$

- **1 mark** for a correct definite integral.
- **1 mark** for correct parameter values.

- b.** Hence what is the probability that a hiker will receive not more than two mosquito bites when crossing the swamp? **2 marks**

Evaluate:

$$Pr(X \leq 2) = \int_0^2 \frac{3}{500} x(10 - x) dx$$

$$\int \left[\frac{15x^2 - x^3}{500} \right]_0^2$$

$$\int \left(\frac{60 - 8}{500} \right) - 0$$

$$\int \frac{52}{500}$$

$$\int \frac{13}{125}$$

- **1 mark** for a valid definite integral.
- **1 mark** for correct probability.

Question 8 (3 marks)

50 people from the city of Wombat were surveyed regarding their favourite breed of dog. A certain proportion, less than half, of the respondents indicated that they preferred cocker spaniels. Let $\hat{p}$ be the sample proportion of people who prefer cocker spaniels.

- a. The standard deviation of $\hat{p}$ is $\sqrt{42}/100$, how many of the 50 people surveyed preferred cocker spaniels? **2 marks**

$$sd(\hat{p}) = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

$$\frac{\sqrt{42}}{100} = \sqrt{\frac{\hat{p}(1-\hat{p})}{50}}$$

$$\frac{42}{10000} = \frac{\hat{p}(1-\hat{p})}{50}$$

$$\frac{21}{100} = \hat{p} - \hat{p}^2$$

$$\hat{p}^2 - \hat{p} + \frac{21}{100} = 0$$

$$100\hat{p}^2 - 100\hat{p} + 21 = 0$$

$$(10\hat{p} - 7)(10\hat{p} - 3) = 0$$

$$\hat{p} = \frac{7}{10} \text{ or } \frac{3}{10}$$

$$\hat{p} = \frac{3}{10}, \text{ since it is known that less than half prefer cocker spaniels.}$$

- **1 mark** for correctly relating given information to equation for standard deviation of $\hat{p}$.
- **1 mark** for correct value of $\hat{p}$.

b. If the standard deviation of $\hat{p}$ was to be halved, how many people would need to be surveyed?

1 mark

$$sd(\hat{p}) = \frac{\sqrt{42}}{200}$$

$$sd(\hat{p}) = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

$$\frac{\sqrt{42}}{200} = \sqrt{\frac{\frac{3}{10}\left(1-\frac{3}{10}\right)}{n}}$$

$$\frac{42}{40000} = \frac{\frac{21}{100}}{n}$$

$$\frac{21}{20000} = \frac{21}{100n}$$

$$\frac{1}{200} = \frac{1}{n}$$

$$n = 200$$

200 people would need to be surveyed to halve the standard deviation of $\hat{p}$.

- **1 mark** for correct value of $\hat{p}$.

Question 9 (6 marks)

Consider $f(x) = ae^{n(x-h)}$, $a, h, n \in \mathbb{R}$

- a. Show that the equation for the area of the triangle formed by the x -axis and tangent and normal lines of $f(x)$ at $x = 0$ is: **3 marks**

$$\text{Area} = \frac{a^3 n^3 + a e^{2nh}}{2 n e^{3nh}}$$

Find equation of the tangent at $x = 0$

$$f'(x) = a n e^{n(x-h)}$$

$$f'(0) = a n e^{-nh} = \frac{an}{e^{nh}}$$

$$f(0) = a e^{-nh} = \frac{a}{e^{nh}} = y_1$$

$$y - y_1 = m(x - x_1)$$

$$y - \frac{a}{e^{nh}} = \frac{an}{e^{nh}} x$$

$$y = \frac{an}{e^{nh}} x + \frac{a}{e^{nh}}$$

Find x -intercept of tangent line.

$$\frac{an}{e^{nh}} x + \frac{a}{e^{nh}} = 0$$

$$\frac{an}{e^{nh}} x = -\frac{a}{e^{nh}}$$

$$anx = -a$$

$$x = \frac{-1}{n}$$

Find equation of the normal at $x = 0$

$$f'_N(0) = \frac{-1}{f'(0)}$$

$$-\frac{e^{nh}}{an}$$

$$f(0) = ae^{-nh} = \frac{a}{e^{nh}} = y_1$$

$$y - y_1 = m(x - x_1)$$

$$y - \frac{a}{e^{nh}} = \frac{-e^{nh}}{an} x$$

$$y = \frac{-e^{nh}}{an} x + \frac{a}{e^{nh}}$$

Find x -intercept of normal line.

$$\frac{-e^{nh}}{an} x + \frac{a}{e^{nh}} = 0$$

$$\frac{e^{nh}}{an} x = \frac{a}{e^{nh}}$$

$$x e^{nh} = \frac{a^2 n}{e^{nh}}$$

$$x = \frac{a^2 n}{e^{2nh}}$$

$$\text{Base of triangle} = \frac{1}{n} + \frac{a^2 n}{e^{2nh}}$$

$$\frac{e^{2nh} + a^2 n^2}{ne^{2nh}}$$

$$\text{Height of triangle} = \frac{a}{e^{nh}}$$

$$\text{Area} = \frac{1}{2} bh$$

$$\frac{1}{2} \left(\frac{e^{2nh} + a^2 n^2}{ne^{2nh}} \right) \frac{a}{e^{nh}}$$

$$\frac{ae^{2nh} + a^3 n^2}{2ne^{3nh}}$$

$$\frac{a^3 n^2 + ae^{2nh}}{2ne^{3nh}}$$

- **1 mark** for correct tangent and normal lines

- **1 mark** for correct x -intercepts of tangent and normal lines.
- **1 mark** for simplification of expression for area to that required.

- b.** If the area of the triangle is k units squared, and $h = 0$ in $f(x)$, develop an equation that will provide the value of n for any given value of a and k . Include any restrictions on a . **3 marks**

$$\text{Area} = \frac{a^3 n^2 + a e^{2nh}}{2 n e^{3nh}}$$

$$k = \frac{a^3 n^2 + a}{2 n}$$

$$2kn = a^3 n^2 + a$$

$$a^3 n^2 - 2kn + a = 0$$

Solve using quadratic formula.

$$n = \frac{2k \pm \sqrt{4k^2 - 4a^4}}{2a^3}$$

$$i \frac{2k \pm 2\sqrt{k^2 - a^4}}{2a^3}$$

$$i \frac{k \pm \sqrt{k^2 - a^4}}{a^3}, a < \sqrt{k}$$

- **1 mark** for an equation relating k to a and n .
- **1 mark** for a valid solution to the equation (n is the subject).
- **1 mark** for specification of conditions.