

The Mathematical Association of Victoria
Trial Examination 2023
MATHEMATICAL METHODS

Trial Written Examination 1 - SOLUTIONS

Question 1

a. $\frac{d}{dx}(x^3 \sin(2x))$
 $= x^3 \times 2 \cos(2x) + \sin(2x) \times 3x^2$
 $= 2x^3 \cos(2x) + 3x^2 \sin(2x)$ **1A**

b. $f(x) = \frac{e^{2x}}{2x+1}$
 $f'(x) = \frac{(2x+1)2e^{2x} - e^{2x} \times 2}{(2x+1)^2}$ **1M**
 $f'(2) = \frac{10e^4 - 2e^4}{5^2}$
 $= \frac{8e^4}{25}$ **1A**

Question 2

$$f: (-2, \infty) \rightarrow R, f(x) = \frac{1}{x+2}, g: (3, \infty) \rightarrow R, g(x) = \frac{1}{x-3}$$

a. $f \circ g$ exists because the range of g , R^+ , is a subset of the domain of f , $(-2, \infty)$. **1A**

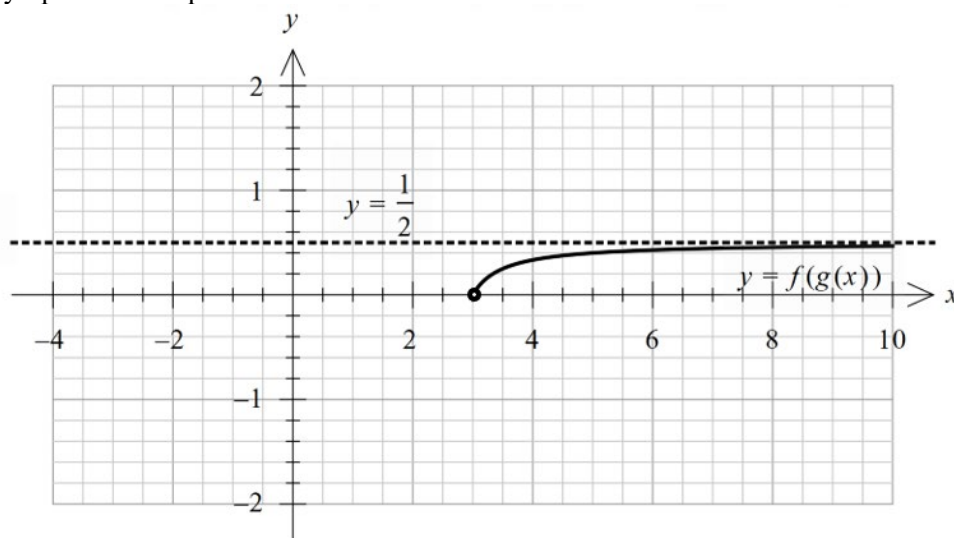
b. $(f \circ g)(x) = \frac{1}{\frac{1}{x-3} + 2}$ **1M**
 $= \frac{x-3}{1+2(x-3)}$
 $= \frac{x-3}{2x-5}$
 $= \frac{x - \frac{5}{2} - \frac{1}{2}}{2x-5}$
 $= \frac{2x-5}{2(2x-5)} - \frac{1}{4x-10}$
 $= \frac{1}{2} - \frac{1}{4x-10}$ **1M show that**

OR

$$\begin{array}{r}
 \frac{1}{2} \\
 2x-5 \overline{) x-3} \\
 \underline{x-\frac{5}{2}} \\
 -\frac{1}{2} \\
 \frac{x-3}{2x-5} = \frac{1}{2} - \frac{\frac{1}{2}}{2x-5} \\
 = \frac{1}{2} - \frac{1}{4x-10}
 \end{array}$$

1M show thatc. The domain of $f \circ g$ is the same as the domain of g .Shape with open circle at $(3, 0)$ **1A**

Asymptote with equation

1A**Question 3**

a. $g(x) = \int \frac{1}{(3x+1)^2} dx$

$$= \int (3x+1)^{-2} dx$$

$$= \frac{(3x+1)^{-1}}{3 \times -1} + c$$

$$= \frac{(3x+1)^{-1}}{-3} + c$$

$$g(x) = -\frac{1}{3(3x+1)} + c$$

1M

given $g(0) = 3$

$$3 = -\frac{1}{3} + c$$

$$c = \frac{10}{3}$$

$$g(x) = -\frac{1}{3(3x+1)} + \frac{10}{3} \quad \mathbf{1A}$$

$$\begin{aligned} \text{b. } & \int_{-\frac{\pi}{18}}^{\frac{\pi}{4}} \left(-\frac{1}{2} \sin(6x) \right) dx \\ &= \left[\frac{1}{12} \cos(6x) \right]_{-\frac{\pi}{18}}^{\frac{\pi}{4}} \\ &= \frac{1}{12} \left(\cos\left(\frac{3\pi}{2}\right) - \cos\left(-\frac{\pi}{3}\right) \right) \quad \mathbf{1M} \\ &= \frac{1}{12} \left(0 - \frac{1}{2} \right) \\ &= -\frac{1}{24} \quad \mathbf{1A} \end{aligned}$$

Question 4

a.i. As the y -coordinates of the points of inflection are 1 and the y -coordinates of the maximum turning points are 4, the graph of $y = \cos(x)$ has been dilated by a factor of $4 - 1 = 3$ from the x -axis (amplitude is 3) and translated 1 unit up. So $y_1 = 3 \cos(x) + 1$.

Hence, $A = 3$ and $c = 1$. **1A**

a.ii. Solve $nx + \frac{1}{4} = 0$ for $x = -1$

$$n = \frac{1}{4} \quad \mathbf{1A}$$

OR

As the turning point is at $(-1, 4)$, $y_1 = 3 \cos(x) + 1$ has been translated 1 unit left. $f(x)$ can be written in the form $f(x) = A \cos\left(\frac{1}{4}(x+1)\right) + c$.

$$n = \frac{1}{4} \quad \mathbf{1A}$$

Period $= \frac{2\pi}{\frac{1}{4}} = 8\pi$, so a point of inflection will be at $x = -1 + \frac{\text{period}}{4} = -1 + 2\pi$

Hence the general solution for the x -coordinates of the points of inflection is $x = -1 + 2\pi + 4\pi k$, $k \in \mathbb{Z}$. **1A**

OR

Solve $3 \cos\left(\frac{1}{4}x + \frac{1}{4}\right) + 1 = 1$

$$3 \cos\left(\frac{1}{4}x + \frac{1}{4}\right) = 0$$

$$\cos\left(\frac{1}{4}x + \frac{1}{4}\right) = 0$$

$$\frac{1}{4}x + \frac{1}{4} = \cos^{-1}(0)$$

$$\frac{1}{4}x + \frac{1}{4} = \frac{\pi}{2}, \dots$$

$$x = -1 + 2\pi, \dots$$

Hence the general solution for the x -coordinates of the points of inflection is $x = -1 + 2\pi + 4\pi k$, $k \in \mathbb{Z}$. **1A**

b. $f(x) = 3\cos\left(\frac{1}{4}x + \frac{1}{4}\right) + 1$ to $g(x) = \cos\left(\frac{1}{4}x + \frac{1}{4}\right) + 1$

- Dilate by a factor $\frac{1}{3}$ of from the x -axis
- Translate $\frac{2}{3}$ of a unit up

(Dilate by a factor $\frac{1}{A}$ from the x -axis and translate $c - \frac{c}{A}$ units up)

1H**OR**

- Translate 1 unit down
- Dilate by $\frac{1}{3}$ from x -axis
- Translate 1 unit up

(Translate c units down, dilate by a factor of $\frac{1}{A}$ from the x -axis, translate c units up) **1H**

Question 5

a. $f(x) = x^3 + 2x^2 - x + 1$

$f'(x) = 3x^2 + 4x - 1 = 0$ **1M**

$$x = \frac{-4 \pm \sqrt{16 + 12}}{6}$$

$$x = \frac{-4 \pm 2\sqrt{7}}{6}$$

$$x = \frac{-2 \pm \sqrt{7}}{3} \quad \mathbf{1A}$$

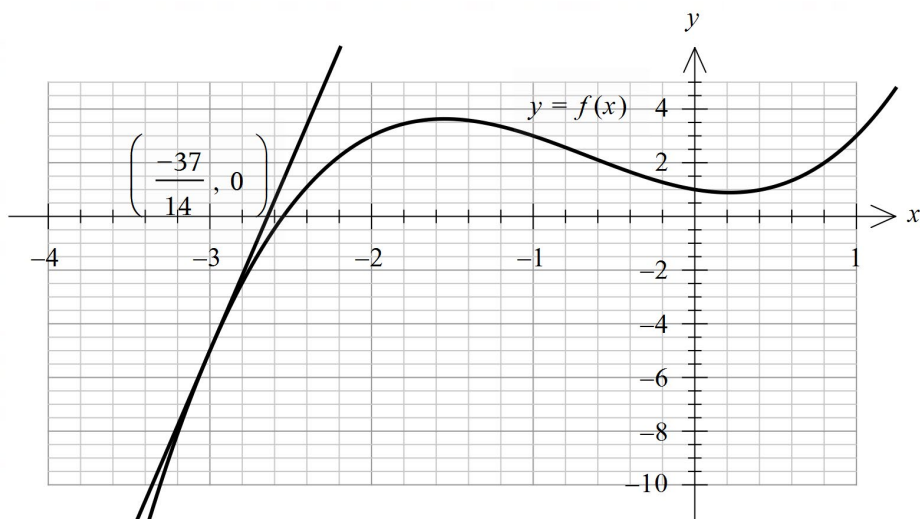
b. $x_1 = -3 - \frac{f(-3)}{f'(-3)}$ **1M**

$$x_1 = -3 - \frac{-5}{14}$$

$$x_1 = -2\frac{9}{14} = -\frac{37}{14} \quad \mathbf{1A}$$

c. Tangent line with the coordinates of the x -intercept **1A**

Must touch at $(-3, -5)$.

**Question 6**

a. $v(t) = (t-8)^{\frac{2}{3}} + 1$

$$a = v'(t) = \frac{2}{3}(t-8)^{-\frac{1}{3}}$$

$$\frac{2}{3}(t-8)^{-\frac{1}{3}} = 1 \quad \mathbf{1M}$$

$$(t-8)^{-\frac{1}{3}} = \frac{3}{2}$$

$$t-8 = \left(\frac{3}{2}\right)^{-3}$$

$$t-8 = \left(\frac{2}{3}\right)^3$$

$$t = \left(\frac{2}{3}\right)^3 + 8$$

$$t = \frac{8}{27} + 8$$

$$t = \frac{224}{27} = 8\frac{8}{27} \quad \mathbf{1A}$$

b. $v(t) = (t-8)^{\frac{2}{3}} + 1$

$$\text{Average rate of change} = \frac{v(35) - v(0)}{35 - 0} \quad \mathbf{1M}$$

$$= \frac{(35-8)^{\frac{2}{3}} + 1 - (-8)^{\frac{2}{3}} - 1}{35}$$

$$= \frac{(27)^{\frac{2}{3}} - (-8)^{\frac{2}{3}}}{35}$$

$$= \frac{9 - 4}{35}$$

$$= \frac{5}{35} = \frac{1}{7}$$

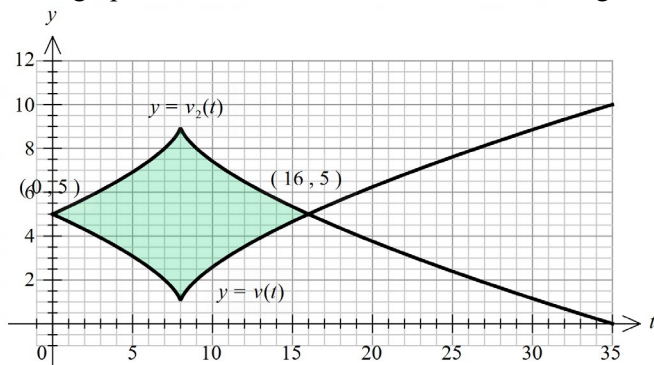
1A

c. The graphs intersect at $t = 0$ and $t = 16$.

The reflection of the graph of v in the x -axis gives the endpoint at $(0, -5)$.

A translation of 10 units up gives the endpoint as $(0, 5)$.

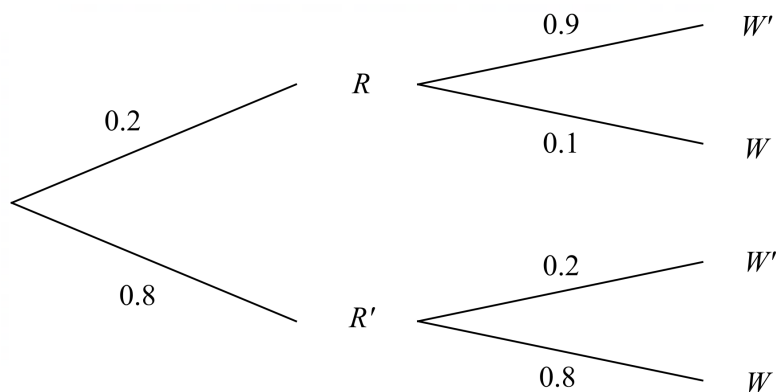
So the graphs intersect at $t = 0$ and 8 units to the right of $t = 8$.



The bounded area $= \int_0^{16} (v_2(t) - v(t)) dt$ or $2 \int_0^{16} (5 - v(t)) dt$ or $4 \int_0^8 (5 - v(t)) dt$ (other forms)

$$= \int_0^{16} \left(-(t-8)^{\frac{2}{3}} + 9 - \left((t-8)^{\frac{2}{3}} + 1 \right) \right) dt \text{ or } 2 \int_0^{16} \left(5 - \left((t-8)^{\frac{2}{3}} + 1 \right) \right) dt \text{ or } 4 \int_0^8 \left(5 - \left((t-8)^{\frac{2}{3}} + 1 \right) \right) dt$$

$$= \int_0^{16} \left(-2(t-8)^{\frac{2}{3}} + 8 \right) dt \text{ or } 2 \int_0^{16} \left(4 - (t-8)^{\frac{2}{3}} \right) dt \text{ or } 4 \int_0^8 \left(4 - (t-8)^{\frac{2}{3}} \right) dt \quad \mathbf{1A}$$

Question 7**a.**

$$\Pr(R|W)$$

$$= \frac{\Pr(R \cap W)}{\Pr(W)}$$

$$= \frac{0.2 \times 0.1}{0.2 \times 0.1 + 0.8 \times 0.8}$$

1M

$$= \frac{0.02}{0.02 + 0.64}$$

$$= \frac{0.02}{0.66} = \frac{1}{33} \quad \mathbf{1A}$$

$$\mathbf{b.} \quad f(t) = \begin{cases} \frac{3}{5}\sqrt{t} & 0 \leq t \leq 1 \\ \frac{3e}{5}e^{-t} & 1 < t < \infty \\ 0 & t < 0 \end{cases}$$

$$\Pr(T \leq 1) = \frac{3}{5} \int_0^1 \left(t^{\frac{1}{2}}\right) dt$$

$$= \frac{3}{5} \times \frac{2}{3} \left[t^{\frac{3}{2}}\right]_0^1$$

$$= \frac{3}{5} \times \frac{2}{3} (1 - 0)$$

$$= \frac{2}{5} \quad \mathbf{1M} \text{ Show that}$$

$$\mathbf{c.} \quad \frac{3}{5} \int_0^1 \left(t^{\frac{1}{2}}\right) dt + \frac{3}{5} e \int_1^m (e^{-t}) dt = \frac{1}{2}$$

$$\frac{3}{5} e \int_1^m (e^{-t}) dt = \frac{1}{2} - \frac{2}{5} = \frac{1}{10} \quad \mathbf{1M}$$

$$-\frac{3}{5} e [e^{-t}]_1^m = \frac{1}{10}$$

$$e^{-m} - e^{-1} = -\frac{1}{6e} \quad \mathbf{1M}$$

$$e^{-m} = \frac{5}{6e}$$

$$m = -\log_e \left(\frac{5}{6e}\right) = 1 - \log_e \left(\frac{5}{6}\right) = 1 + \log_e \left(\frac{6}{5}\right) \quad \mathbf{1A} \text{ (any form)}$$

OR

$$\frac{3}{5} e \int_m^\infty (e^{-t}) dt = \frac{1}{2} \quad \mathbf{1M}$$

$$-\frac{3}{5} e [e^{-t}]_m^\infty = \frac{1}{2}$$

$$e^{-\infty} - e^{-m} = -\frac{5}{6e} \quad \mathbf{1M}$$

$$-e^{-m} = -\frac{5}{6e}$$

$$m = -\log_e \left(\frac{5}{6e}\right) = 1 - \log_e \left(\frac{5}{6}\right) = 1 + \log_e \left(\frac{6}{5}\right) \quad \mathbf{1A} \text{ (any form)}$$

Question 8

$$g(x) = a \log_2(x+b), \quad g(2) = 6, \quad g(6) = 9$$

$$a \log_2(2+b) = 6 \dots (1)$$

$$a \log_2(6+b) = 9 \dots (2)$$

$$(2) \div (1)$$

$$\frac{\log_2(6+b)}{\log_2(2+b)} = \frac{3}{2} \quad \mathbf{1M}$$

$$\log_2(6+b) = \frac{3}{2} \log_2(2+b)$$

$$6+b = (2+b)^{\frac{3}{2}} \quad \mathbf{1M}$$

Square both sides

$$(6+b)^2 = (2+b)^3$$

$$36 + 12b + b^2 = b^3 + 6b^2 + 12b + 8$$

$$b^3 + 5b^2 - 28 = 0$$

$$\text{When } b = 2, b^3 + 5b^2 - 28 = 8 + 20 - 28 = 0$$

$$(b-2)(b^2 + 7b + 14) = 0$$

$$b^2 + 7b + 14 = 0$$

$$\Delta = 49 - 56 = -7, \text{ no real solution}$$

$$b = 2 \quad \mathbf{1A}$$

$$a \log_2(2+2) = 6$$

$$2a = 6$$

$$a = 3 \quad \mathbf{1H}$$

Question 9

$$\mathbf{a.} \quad h: (-\infty, a) \rightarrow \mathbb{R}, h(x) = \frac{1}{(x-a)^2}$$

$$\text{Let } y = \frac{1}{(x-a)^2}$$

Inverse swap x and y

$$x = \frac{1}{(y-a)^2}$$

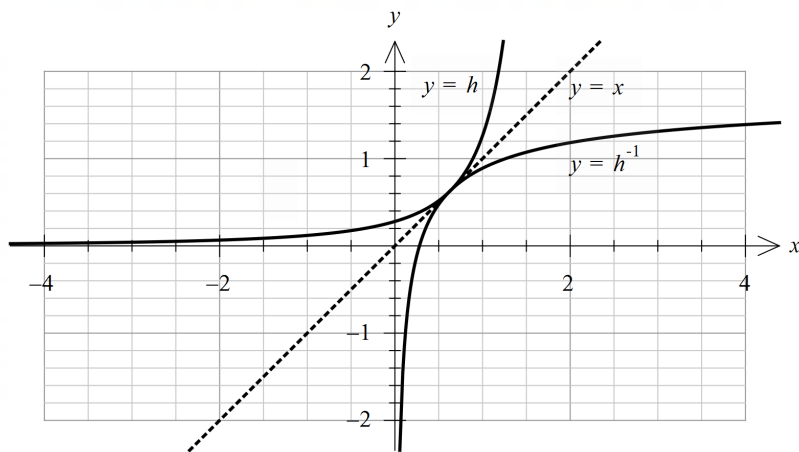
$$(y-a)^2 = \frac{1}{x}$$

$$y = -\sqrt{\frac{1}{x}} + a, \quad y \neq \sqrt{\frac{1}{x}} + a \text{ as the domain of } h \text{ is } (-\infty, a)$$

$$h^{-1}(x) = -\sqrt{\frac{1}{x}} + a \quad \mathbf{1A}$$

b. The graphs will touch along the line $y = x$.

The gradient equals 1.



$$h'(x) = \frac{-2}{(x-a)^3} = 1 \quad \mathbf{1M}$$

$$-2 = (x-a)^3$$

$$x = \sqrt[3]{-2} + a$$

$$h^{-1'}(x) = \frac{1}{2x^{\frac{3}{2}}} = 1$$

$$x = \left(\frac{1}{2}\right)^{\frac{2}{3}}$$

$$\sqrt[3]{-2} + a = \left(\frac{1}{2}\right)^{\frac{2}{3}} \quad \mathbf{1M}$$

$$a = 2^{-\frac{2}{3}} + 2^{\frac{1}{3}}$$

$$= \frac{1}{2^{\frac{2}{3}}} + 2^{\frac{1}{3}} \times \frac{2^{\frac{2}{3}}}{2^{\frac{2}{3}}}$$

$$= \frac{1}{2^{\frac{2}{3}}} + \frac{2}{2^{\frac{2}{3}}}$$

$$= \frac{3}{2^{\frac{2}{3}}} \quad \mathbf{1A}$$

OR

The point of intersection is at $(\sqrt[3]{-2} + a, \sqrt[3]{-2} + a)$.

Substitute into $h(x) = \frac{1}{(x-a)^2}$.

$$\sqrt[3]{-2} + a = \frac{1}{(\sqrt[3]{-2} + a - a)^2} \quad \mathbf{1M}$$

$$\sqrt[3]{-2} + a = \frac{1}{(\sqrt[3]{-2})^2}$$

$$a = \frac{1}{(\sqrt[3]{-2})^2} - \sqrt[3]{-2}$$

$$= \frac{1}{\frac{2}{2^{\frac{2}{3}}}} + 2^{\frac{1}{3}}$$

$$= \frac{1}{\frac{2}{2^{\frac{2}{3}}}} + 2^{\frac{1}{3}} \times \frac{2^{\frac{2}{3}}}{2^{\frac{2}{3}}}$$

$$= \frac{1}{\frac{2}{2^{\frac{2}{3}}}} + \frac{2}{2^{\frac{2}{3}}}$$

$$= \frac{3}{2^{\frac{2}{3}}} \text{ or } \frac{3}{4^{\frac{1}{3}}} \quad \mathbf{1A}$$

END OF SOLUTIONS