

Question 1 (3 marks)

a. $y = (x^2 + 3x + 2)^5$
 $\frac{dy}{dx} = 5(x^2 + 3x + 2)^4(2x + 3)$ (chain rule)

(1 mark)

b. $h(x) = \frac{\tan(x)}{3x}$
 $h'(x) = \frac{3x \times \sec^2(x) - 3 \tan(x)}{(3x)^2}$ (quotient rule)

(1 mark)

$$h'\left(\frac{\pi}{3}\right) = \frac{\pi \times \left(\frac{1}{\cos^2\left(\frac{\pi}{3}\right)} \right) - 3 \tan\left(\frac{\pi}{3}\right)}{\left(3 \times \frac{\pi}{3}\right)^2}$$

$$= \frac{\pi \times \frac{1}{\left(\frac{1}{2}\right)^2} - 3\sqrt{3}}{\pi^2}$$

$$= \frac{4\pi - 3\sqrt{3}}{\pi^2}$$

(1 mark)

Question 2 (3 marks)

a. $\int \frac{1}{3x+2} dx = \frac{1}{3} \log_e(3x+2) + c$ (Note that “+c” is required here.)

(1 mark)

- b. Using the rule $\int (ax+b)^n dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c$, $n \neq -1$ from the formula sheet,

$$\begin{aligned} g'(x) &= \frac{1}{(2x-1)^2} \\ &= (2x-1)^{-2} \\ g(x) &= \int (2x-1)^{-2} dx \\ &= \frac{1}{2 \times -1} (2x-1)^{-1} + c \\ &= \frac{-1}{2(2x-1)} + c \end{aligned}$$

(1 mark)

Given $g\left(\frac{1}{4}\right) = 5$

then $5 = \frac{-1}{2\left(\frac{1}{2} - 1\right)} + c$

$$c = 4$$

So $g(x) = \frac{-1}{2(2x-1)} + 4$

(1 mark)

Question 3 (3 marks)Method 1

$$\begin{aligned} \sqrt{3} - \cos(2x) &= \cos(2x) \quad x \in R \\ \sqrt{3} &= 2\cos(2x) \end{aligned}$$



$$\cos(2x) = \frac{\sqrt{3}}{2}$$

Cosine is positive in the first and fourth quadrants and the base angle is $\frac{\pi}{6}$.

(1 mark)

1st quadrant solution:

$$2x = \frac{\pi}{6} + 2k\pi, \quad k \in Z$$

$$x = \frac{\pi}{12} + k\pi, \quad k \in Z$$

(1 mark)4th quadrant solution:

$$2x = \left(2\pi - \frac{\pi}{6}\right) + 2k\pi, \quad k \in Z$$

$$= \frac{11\pi}{6} + 2k\pi$$

$$x = \frac{11\pi}{12} + k\pi, \quad k \in Z$$

(1 mark)Method 2

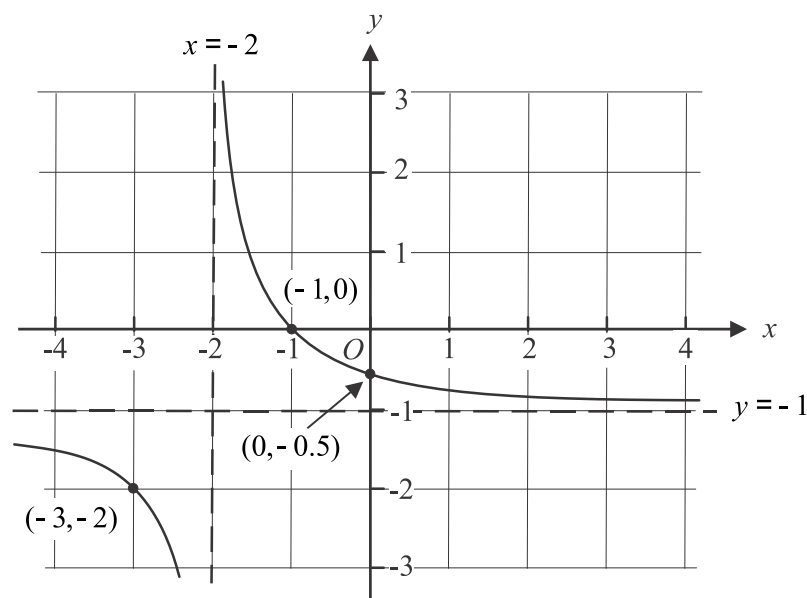
$$\sqrt{3} - \cos(2x) = \cos(2x), \quad x \in R$$

$$\sqrt{3} = 2\cos(2x)$$

$$\cos(2x) = \frac{\sqrt{3}}{2} \quad (\text{base angle is } \frac{\pi}{6}) \quad \textbf{(1 mark)}$$

$$2x = 2k\pi \pm \frac{\pi}{6}, \quad k \in Z \quad \textbf{(1 mark)}$$

$$x = k\pi \pm \frac{\pi}{12}, \quad k \in Z \quad \textbf{(1 mark)}$$

Question 4 (5 marks)**a.**x-intercepts occur when $y = 0$

$$0 = \frac{1}{x+2} - 1$$

$$1 = \frac{1}{x+2}$$

$$x+2=1$$

$$x = -1$$

(1 mark) – correct asymptotes with equations **(1 mark)** – correct intercepts**(1 mark)** – correct branches/shapey-intercepts occur when $x = 0$

$$y = \frac{1}{2} - 1$$

$$= -\frac{1}{2}$$

b.

$$f(x) = \frac{1}{x+2} - 1$$

Let $y = \frac{1}{x+2} - 1$

Swap x and y for inverse.

$$x = \frac{1}{y+2} - 1$$

$$x+1 = \frac{1}{y+2}$$

$$(x+1)(y+2) = 1$$

$$y+2 = \frac{1}{x+1}$$

$$y = \frac{1}{x+1} - 2$$

So $f^{-1}(x) = \frac{1}{x+1} - 2$ **(1 mark)**

$$r_f = R \setminus \{-1\} \text{ from part a.}$$

$$d_{f^{-1}} = r_f$$

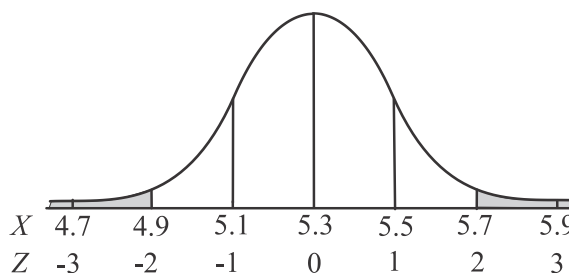
$$= R \setminus \{-1\}$$

1 mark)

Question 5 (3 marks)**a. Method 1**

Draw a diagram.

$$\begin{aligned}\Pr(X < 4.9) &= \Pr(Z < -2) \\ &= \Pr(Z > 2) \\ &\quad \text{by symmetry}\end{aligned}$$

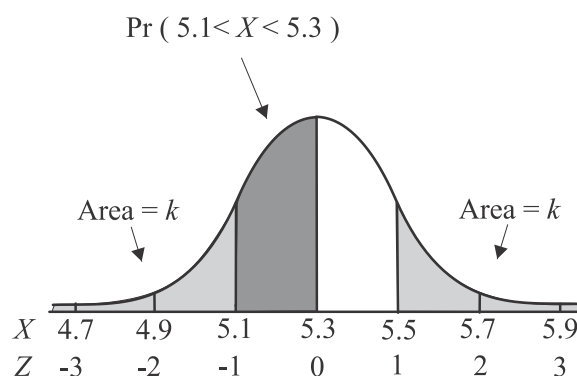
So $a = 2$. **(1 mark)****Method 2**

$$x = 4.9$$

$$\text{so } z = \frac{4.9 - 5.3}{0.2} \quad \text{using the rule } z = \frac{x - \mu}{\sigma}$$

$$= -2$$

$$\begin{aligned}\Pr(X < 4.9) &= \Pr(Z < -2) \\ &= \Pr(Z > 2) \quad \text{by symmetry}\end{aligned}$$

So $a = 2$. **(1 mark)****b.**Note that since $\Pr(Z > 1) = k$, then $\Pr(Z < -1) = k$ by symmetry.

$$\Pr(X > 5.1 | X < 5.3) \quad \text{(conditional probability)}$$

$$= \frac{\Pr(5.1 < X < 5.3)}{\Pr(X < 5.3)}$$

$$= \frac{0.5 - k}{0.5}$$

Note that $\Pr(X < 5.3) = 0.5$

$$= (0.5 - k) \div \frac{1}{2}$$

$$= (0.5 - k) \times 2$$

$$= 1 - 2k$$

(1 mark)**(1 mark)**

Question 6 (4 marks)

- a. Since f is a probability density function, then

$$\int_1^4 k\sqrt{x} \, dx = 1$$

$$k \int_1^4 x^{\frac{1}{2}} \, dx = 1$$

$$k \left[\frac{2}{3} x^{\frac{3}{2}} \right]_1^4 = 1$$

$$\frac{2k}{3} \left(4^{\frac{3}{2}} - 1^{\frac{3}{2}} \right) = 1 \quad \text{Note } 4^{\frac{3}{2}} = (\sqrt{4})^3 = 8$$

$$\frac{2k}{3} (8 - 1) = 1$$

$$14k = 3$$

$$k = \frac{3}{14}$$

as required

(1 mark)

(1 mark)

b. $E(X) = \int_1^4 x f(x) \, dx$

$$= \int_1^4 x \times \frac{3}{14} \sqrt{x} \, dx$$

$$= \frac{3}{14} \int_1^4 x^{\frac{3}{2}} \, dx$$

$$= \frac{3}{14} \left[\frac{2}{5} x^{\frac{5}{2}} \right]_1^4$$

$$= \frac{3}{14} \times \frac{2}{5} \left(4^{\frac{5}{2}} - 1^{\frac{5}{2}} \right)$$

$$= \frac{3}{35} (32 - 1)$$

$$= \frac{93}{35}$$

$$= 2 \frac{23}{35}$$

(1 mark)

(1 mark)

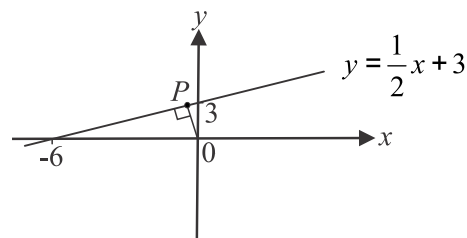
Question 7 (3 marks)Method 1

OP runs perpendicular to $y = \frac{1}{2}x + 3$ (in order to make the shortest distance) and therefore has a gradient of -2 . OP therefore has the equation $y = -2x$. **(1 mark)**

Solving $y = \frac{1}{2}x + 3$ and $y = -2x$ simultaneously gives

$$\begin{aligned} -2x &= \frac{1}{2}x + 3 \\ -\frac{5}{2}x &= 3 \\ x &= -\frac{6}{5} \\ y &= \frac{12}{5} \end{aligned} \quad \textbf{(1 mark)}$$

P is the point $\left(-\frac{6}{5}, \frac{12}{5}\right)$.

**(1 mark)**Method 2 – using the distance formula

We have the two points $O(0,0)$ and $P\left(x, \left(\frac{1}{2}x + 3\right)\right)$.

Let D = distance from O to P

$$\begin{aligned} &= \sqrt{(x-0)^2 + \left(\frac{1}{2}x + 3 - 0\right)^2} \\ &= \sqrt{x^2 + \frac{1}{4}x^2 + 3x + 9} \\ &= \sqrt{\frac{5}{4}x^2 + 3x + 9} \\ &= \left(\frac{5}{4}x^2 + 3x + 9\right)^{\frac{1}{2}} \end{aligned} \quad \textbf{(1 mark)}$$

$$\begin{aligned} \frac{dD}{dx} &= \frac{1}{2} \left(\frac{5}{4}x^2 + 3x + 9\right)^{-\frac{1}{2}} \times \left(\frac{5}{2}x + 3\right) \\ &= \frac{\frac{5}{2}x + 3}{2\sqrt{\frac{5}{4}x^2 + 3x + 9}} \end{aligned}$$

$$\frac{dD}{dx} = 0 \text{ for minimum} \quad \textbf{(1 mark)}$$

So $\frac{5}{2}x + 3 = 0$ (note the denominator

of $\frac{dD}{dx}$ cannot equal zero)

$$x = -\frac{6}{5}$$

Substitute $x = -\frac{6}{5}$ into $y = \frac{1}{2}x + 3$

gives $y = \frac{12}{5}$

P is the point $\left(-\frac{6}{5}, \frac{12}{5}\right)$.

(1 mark)

Question 8 (7 marks)

- a. x-intercepts occur when $y = 0$

$$0 = 3 - e^{kx}$$

$$e^{kx} = 3$$

$$\log_e(3) = kx$$

$$x = \frac{1}{k} \log_e(3) \text{ as required}$$

(1 mark)

b. average value = $\frac{1}{\left(\frac{1}{k} \log_e(3) - 0\right)} \int_0^{\frac{1}{k} \log_e(3)} (3 - e^{kx}) dx$ **(1 mark)**

$$= \frac{k}{\log_e(3)} \left[3x - \frac{1}{k} e^{kx} \right]_0^{\frac{1}{k} \log_e(3)}$$
 (1 mark)

$$= \frac{k}{\log_e(3)} \left\{ \left(\frac{3}{k} \log_e(3) - \frac{1}{k} e^{\log_e(3)} \right) - \left(0 - \frac{1}{k} e^0 \right) \right\}$$

$$= \frac{k}{\log_e(3)} \left(\frac{3}{k} \log_e(3) - \frac{3}{k} + \frac{1}{k} \right) \quad \text{Note that } e^{\log_e(3)} = 3. \quad \textbf{(1 mark)}$$

$$= \frac{k}{\log_e(3)} \left(\frac{3}{k} \log_e(3) - \frac{2}{k} \right)$$

$$= 3 - \frac{2}{\log_e(3)}$$

(1 mark)

c. average rate of change = $\frac{f(1) - f(0)}{1 - 0}$

$$= 3 - e^k - (3 - e^0)$$

$$= 3 - e^k - 3 + 1$$

$$= 1 - e^k$$

(1 mark)

We require $1 - e^k < 0$

$$-e^k < -1$$

$$e^k > 1$$

$$k > 0$$

(1 mark)

Question 9 (9 marks)

a. $f(x) = x \sin(\pi x)$
 $f'(x) = x \times \pi \cos(\pi x) + \sin(\pi x)$ (product rule) (1 mark)

$$\begin{aligned} f'\left(\frac{1}{4}\right) &= \frac{\pi}{4} \cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right) \\ &= \frac{\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}} \\ &= \frac{\pi + 4}{4\sqrt{2}} \\ &= \frac{\sqrt{2}(\pi + 4)}{8} \end{aligned}$$

(1 mark)

b. Area required $= \int_0^1 x \sin(\pi x) dx$

Rearranging $\frac{d}{dx}(x \cos(\pi x)) = \cos(\pi x) - \pi x \sin(\pi x)$

gives $\pi x \sin(\pi x) = \cos(\pi x) - \frac{d}{dx}(x \cos(\pi x))$

so $\pi \int_0^1 x \sin(\pi x) dx = \int_0^1 \cos(\pi x) dx - \int_0^1 \frac{d}{dx}(x \cos(\pi x)) dx$

$$= \left[\frac{1}{\pi} \sin(\pi x) \right]_0^1 - [x \cos(\pi x)]_0^1$$
(1 mark)

$$= \frac{1}{\pi} (\sin(\pi) - \sin(0)) - (\cos(\pi) - 0)$$

$$= \frac{1}{\pi} (0 - 0) - (-1)$$

$$= 1$$

So $\int_0^1 x \sin(\pi x) dx = \frac{1}{\pi}$

Area $= \frac{1}{\pi}$ square units (1 mark)

c. Required area $= \int_0^1 f(x) dx - \int_1^2 f(x) dx + \int_2^3 f(x) dx$

From part b., $\int_0^1 f(x) dx = \frac{1}{\pi}$.

Note that $\int_1^2 f(x-1) dx = \int_0^1 f(x) dx$ because the graph of $y = f(x)$ has been translated 1 unit right to become the graph of $y = f(x-1)$.

So
$$\begin{aligned} \int_1^2 f(x) dx &= -3 \int_1^2 f(x-1) dx \\ &= -3 \int_0^1 f(x) dx \\ &= -3 \times \frac{1}{\pi} \\ &= -\frac{3}{\pi} \end{aligned} \quad (1 \text{ mark})$$

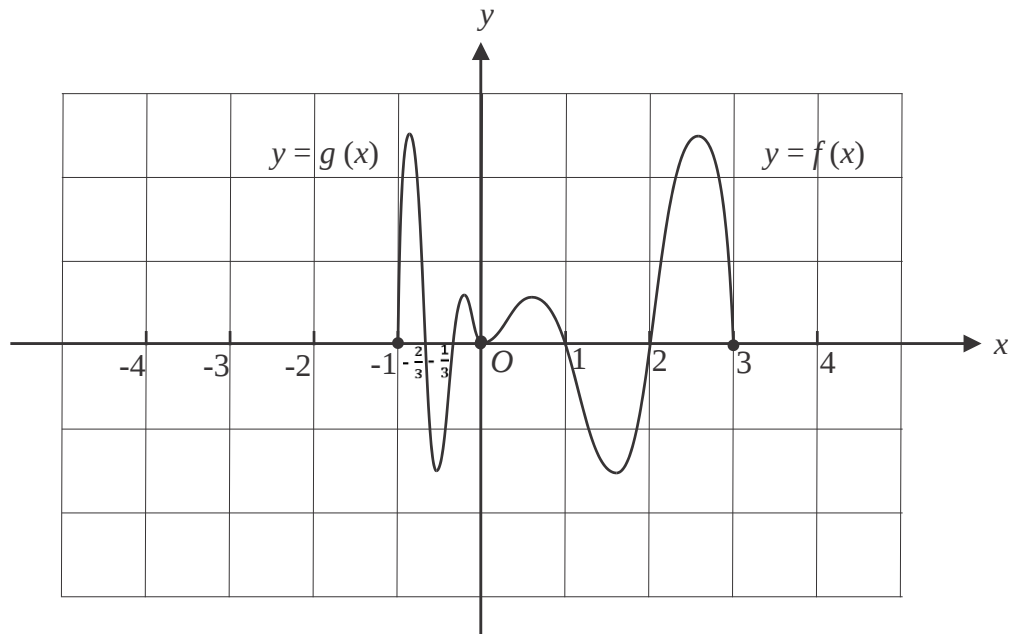
Also
$$\begin{aligned} \int_1^3 f(x) dx &= 2 \int_0^1 f(x) dx \\ &= 2 \times \frac{1}{\pi} \\ &= \frac{2}{\pi} \end{aligned} \quad (1 \text{ mark})$$

So
$$\begin{aligned} \int_2^3 f(x) dx + \int_1^2 f(x) dx &= \frac{2}{\pi} \\ \int_2^3 f(x) dx &= \frac{2}{\pi} + \frac{3}{\pi} \\ &= \frac{5}{\pi} \end{aligned}$$

So area between the graph of f and the x -axis is $\frac{1}{\pi} + \frac{3}{\pi} + \frac{5}{\pi} = \frac{9}{\pi}$.

(1 mark)

- d. i. Method 1 – sketch the graph of g



$$d_g = [-1, 0]$$

(1 mark)

Method 2

$$d_f = [0, 3]$$

If the graph of f is dilated by a factor of $\frac{1}{3}$ from the y -axis (i.e. compressed horizontally) then the new domain will be $x \in [0, 1]$.

If this graph is then reflected in the y -axis to become the graph of g , then $d_g = [-1, 0]$.

(1 mark)

ii. $f(x) = x \sin(\pi x)$

Let $y = x \sin(\pi x)$

After a dilation by a factor of $\frac{1}{3}$ from the y -axis, replace x with $\frac{x}{1/3} = 3x$

and we obtain $y = 3x \sin(3\pi x)$.

After a reflection in the y -axis, replace x with $-x$

and we obtain $y = -3x \sin(-3\pi x)$.

So $g(x) = -3x \sin(-3\pi x)$.

(1 mark)

The equivalent answer of $g(x) = 3x \sin(3\pi x)$ is also acceptable.