



YEAR 12 *Trial Exam Paper*

2022

MATHEMATICAL METHODS

Written examination 2

Worked solutions

This book presents:

- worked solutions
- mark allocations
- tips.

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SECTION A – Multiple-choice questions

Question	Answer
1	<i>E</i>
2	<i>D</i>
3	<i>B</i>
4	<i>E</i>
5	<i>C</i>
6	<i>C</i>
7	<i>D</i>
8	<i>A</i>
9	<i>B</i>
10	<i>B</i>
11	<i>D</i>
12	<i>E</i>
13	<i>A</i>
14	<i>B</i>
15	<i>B</i>
16	<i>D</i>
17	<i>B</i>
18	<i>E</i>
19	<i>A</i>
20	<i>D</i>

Question 1**Answer: E****Explanatory notes**

The period of the function is $\frac{11\pi}{12} - \frac{5\pi}{12} = \frac{\pi}{2}$.

Therefore

$$\tan\left(2\left(x - \frac{\pi}{6}\right)\right) = \tan\left(2x - \frac{\pi}{3}\right)$$

**Tips**

- The period of the function can be found quickly.
- Remember that the period of $y = \tan(nx)$ is $\frac{\pi}{n}$.
- If an analytic solution is too difficult or time consuming, you can examine options C, D or E by sketching them using CAS.

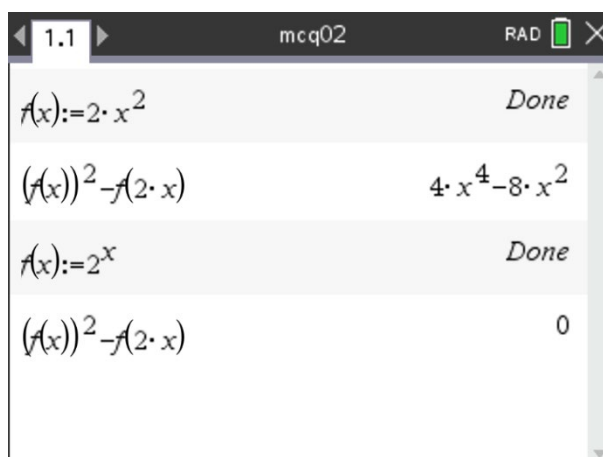
Question 2**Answer: D****Explanatory notes**

Note that $(f(x))^2 = (2^x)^2 = 2^{2x} = f(2x)$.

This may be found by examining each option.

**Tip**

- Defining the function $f(x)$ in CAS and then evaluating the difference $(f(x))^2 - f(2x)$ is an efficient way to find the correct function. If the result is zero, then the equation is satisfied. In the screenshot below, the difference is evaluated for both $f(x) = 2x^2$ and $f(x) = 2^x$. The latter is correct.



Question 3**Answer: B****Explanatory notes**

Use log laws to see that

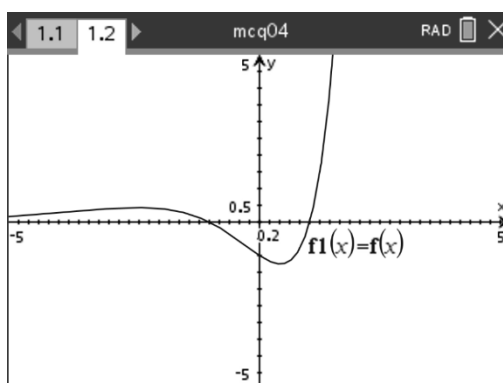
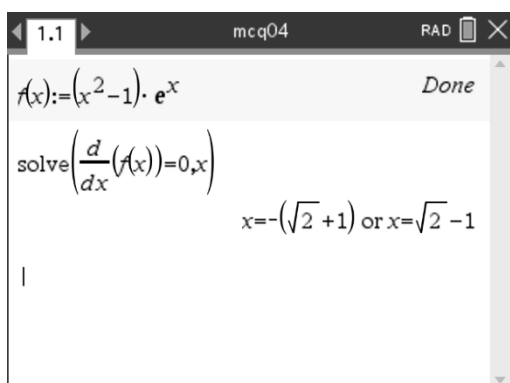
$$\begin{aligned}
 2 + 3 \log_2(x) - \frac{1}{2} \log_2(y) &= \log_2(4) + \log_2(x^3) - \log_2(\sqrt{y}) \\
 &= \log_2\left(\frac{4x^3}{\sqrt{y}}\right)
 \end{aligned}$$

**Tips**

- *Knowing log laws is essential here.*
- *Don't forget that $2 = \log_2(2^2) = \log_2(4)$.*

Question 4**Answer: E****Explanatory notes**

The graph of $f(x) = (x^2 - 1)e^x$ has turning points when $x = -1 - \sqrt{2}$ and $x = -1 + \sqrt{2}$.



If $x \in [-1 - \sqrt{2}, -1 + \sqrt{2}]$, then $f(x)$ will have an inverse function.

In all of the other cases, the function fails to be one-to-one.

**Tip**

- *Use CAS to find the turning points. A quick graph is very useful.*

Question 5**Answer: C****Explanatory notes**

This is a conditional probability problem.

The probability of selecting three red balls (RRR) is

$$\frac{5}{9} \times \frac{4}{8} \times \frac{3}{7} = \frac{5}{42}$$

The probability of selecting two red balls (RRY, RYR, YRR) is

$$\left(\frac{5}{9} \times \frac{4}{8} \times \frac{4}{7}\right) + \left(\frac{5}{9} \times \frac{4}{8} \times \frac{4}{7}\right) + \left(\frac{4}{9} \times \frac{5}{8} \times \frac{4}{7}\right) = 3\left(\frac{5}{9} \times \frac{4}{8} \times \frac{4}{7}\right) = \frac{10}{21}$$

Thus the probability of selecting *at least* two red balls is

$$\frac{5}{42} + \frac{10}{21} = \frac{25}{42}$$

Therefore the probability of selecting three red balls given that at least two red balls were selected is

$$\frac{\frac{5}{42}}{\frac{25}{42}} = \frac{1}{5}$$

**Tip**

- *Considering the various options (RRY, RYR, YRR) is essential.*

Question 6**Answer: C****Explanatory notes**

The average value of $f(2x+1)$ on the interval $[-1,1]$ is $\frac{3}{2}$.

Consequently, the average value of $f(x)$ on the interval $[-1,3]$ is $\frac{3}{2}$.

$$\text{Then } a \int_{-1}^3 \frac{3}{2} dx = 4a \cdot \frac{3}{2} = 8$$

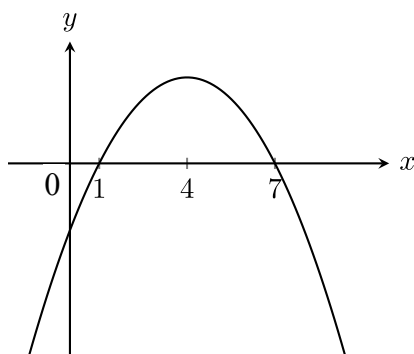
$$\text{Therefore } a = \frac{4}{3}$$

**Tip**

- Since the average value of $f(2x+1)$ on the interval $[-1,1]$ is $\frac{3}{2}$, it is safe to set $f(2x+1) = \frac{3}{2}$. Thus we can set $f(x) = \frac{3}{2}$ on the interval $[-1,3]$.

Question 7**Answer: D****Explanatory notes**

The derivative $f'(x)$ is a quadratic with x -intercepts at $x=1$ and $x=7$. The turning point occurs when $x=4$ (midway between the x -intercepts).



The graph of $y = f'(x)$ is strictly increasing for $x \in (-\infty, 4]$.

**Tips**

- Read the question carefully. We are looking for the interval on which the derivative of ' f ' is strictly increasing.
- The interval on which a function is strictly increasing (or decreasing) may also include endpoints, stationary points (turning points) and stationary points of inflection.

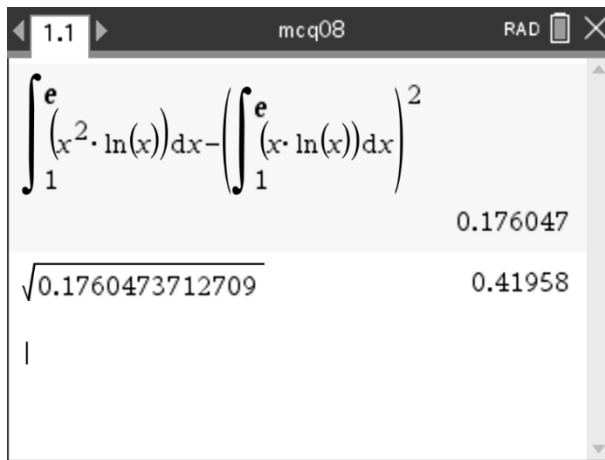
Question 8**Answer: A****Explanatory notes**

The variance is

$$\begin{aligned}\text{Var}(X) &= E(X^2) - (E(X))^2 \\ &= \int_1^e x^2 \log_e(x) dx - \left(\int_1^e x \log_e(x) dx \right)^2 \\ &\approx 0.1760\end{aligned}$$

Therefore the standard deviation is

$$\sigma = \sqrt{\text{Var}(X)} \approx 0.4196$$



The screenshot shows a calculator interface with the following content:

- Top bar: "1.1" on the left, "mcq08" in the center, and "RAD" and a close button on the right.
- Main display area:

$$\int_1^e (x^2 \cdot \ln(x)) dx - \left(\int_1^e (x \cdot \ln(x)) dx \right)^2$$

0.176047

$$\sqrt{0.1760473712709}$$

0.41958

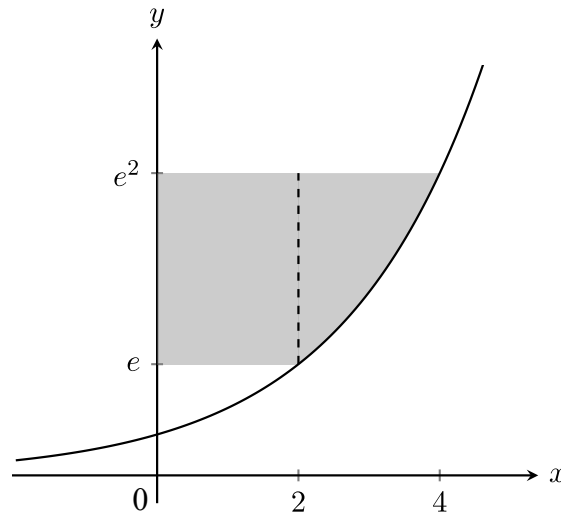
**Tip**

- The formula for the variance of a probability distribution is given on the formula sheet.

Question 9**Answer: B****Explanatory notes**

If $y = f(x) = 2\log_e(x)$, then the inverse function is $f^{-1}(x) = e^{\frac{x}{2}}$.

The area required is equal to the shaded area shown below.



The shaded area can be split into two, with a rectangle on the left. The area of the shaded region is thus

$$\begin{aligned}
 A &= 2(e^2 - e) + \int_2^4 \left(e^2 - e^{\frac{x}{2}} \right) dx \\
 &= 2e^2 - 2e + \int_2^4 \left(e^2 - e^{\frac{x}{2}} \right) dx
 \end{aligned}$$

**Tip**

- *An area bounded by a graph can also be found on a graph of the inverse function. This makes calculation easier and avoids trying to bind to the y-axis.*

Question 10**Answer: B****Explanatory notes**

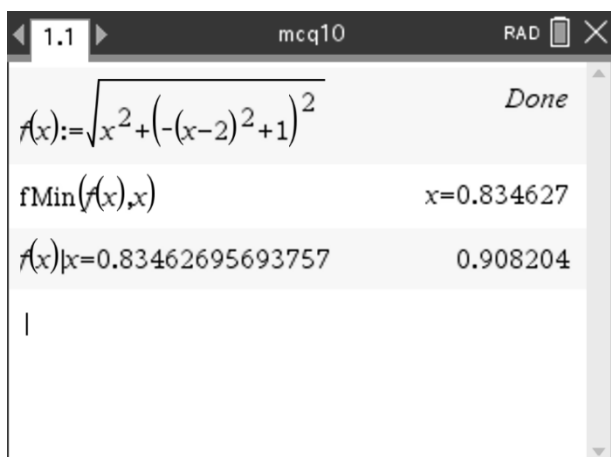
A point on the parabola has coordinates $(x, -(x-2)^2 + 1)$.

The distance from the origin $O(0,0)$ to a point on the parabola is

$$\sqrt{x^2 + (-(x-2)^2 + 1)^2}$$

Using CAS, we find that this distance is least when $x = 0.8346$

The shortest distance is 0.9082

**Tip**

- Use the fMin command on CAS.

Question 11**Answer: D****Explanatory notes**

The domain of f is $(-\infty, 5)$ and the domain of g is $\left[-\frac{3}{4}, \infty\right)$.

The maximal domain of $h = \frac{f}{g}$ is $\left(-\frac{3}{4}, 5\right)$.

**Tip**

- Note that the domain above is the intersection of the two original domains excluding the point $x = -\frac{3}{4}$ where the function $g(x)$ equals zero.

Question 12**Answer: E****Explanatory notes**

The distribution is described as follows:

$$f(x) = \begin{cases} \frac{1}{2} & 0 \leq x \leq 1 \\ -\frac{1}{4}(x-3) & 1 < x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

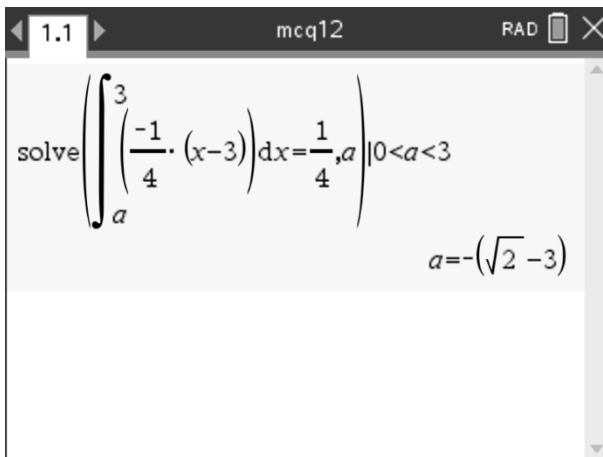
If $\Pr(X < a) = 0.75$, then $\Pr(X > a) = 0.25$

That is

$$\int_a^3 -\frac{1}{4}(x-3)dx = \frac{1}{4}$$

giving $a = 3 - \sqrt{2}$

This equation can be solved quickly using CAS:



The screenshot shows a CAS window titled 'mcq12' with a 'RAD' button and a close button. The input is: $\text{solve}\left(\int_a^3 \left(-\frac{1}{4} \cdot (x-3)\right) dx = \frac{1}{4}, a\right) | 0 < a < 3$. The output is: $a = -(\sqrt{2} - 3)$.

**Tip**

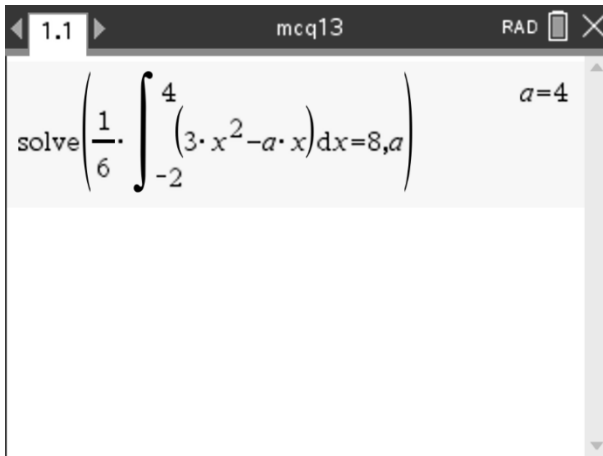
- Use CAS to solve equations like this.

Question 13**Answer: A****Explanatory notes**

Since the average value of $f(x)$ over the interval $[-2, 4]$ is 8, we know that

$$\frac{1}{6} \int_{-2}^4 (3x^2 - ax) dx = 8$$

Solving using CAS gives $a = 4$.

**Tip**

- Use the formula for the average value of a function over a given domain.

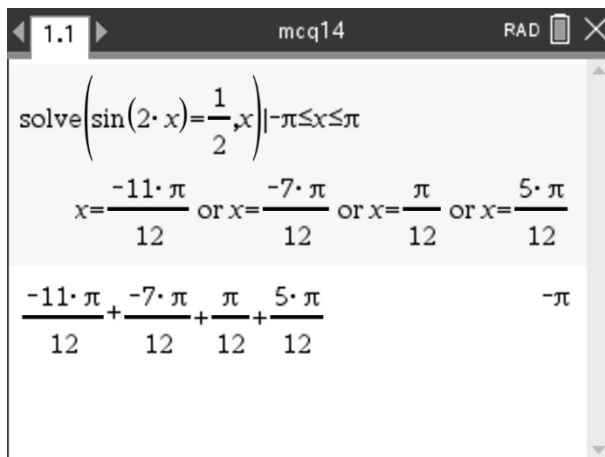
Question 14**Answer: B****Explanatory notes**

Solving $\sin(2x) = \frac{1}{2}$ for $-\pi \leq x \leq \pi$ gives

$$x = -\frac{11\pi}{12}, -\frac{7\pi}{12}, \frac{\pi}{12}, \frac{5\pi}{12}$$

The sum of the solutions is

$$-\frac{11\pi}{12} - \frac{7\pi}{12} + \frac{\pi}{12} + \frac{5\pi}{12} = -\pi.$$



1.1 mcq14 RAD

$$\text{solve}\left(\sin(2 \cdot x) = \frac{1}{2}, x\right) \mid -\pi \leq x \leq \pi$$

$$x = \frac{-11 \cdot \pi}{12} \text{ or } x = \frac{-7 \cdot \pi}{12} \text{ or } x = \frac{\pi}{12} \text{ or } x = \frac{5 \cdot \pi}{12}$$

$$\frac{-11 \cdot \pi}{12} + \frac{-7 \cdot \pi}{12} + \frac{\pi}{12} + \frac{5 \cdot \pi}{12} \quad -\pi$$

**Tip**

- *It is easiest to simply find the solutions and sum them using CAS, as shown above.*

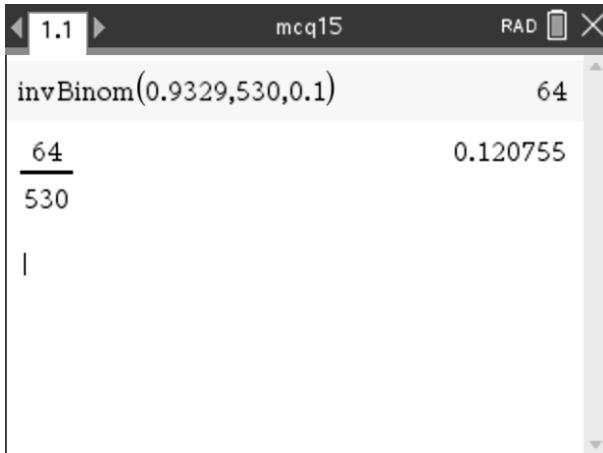
Question 15**Answer: B****Explanatory notes**

Let $X \sim \text{Bi}(0.1, 530)$. If $\Pr(\hat{P} > a) = 0.0671$, then

$$\Pr(X > 530 \times a) = 0.0671$$

$$\Pr(X \leq 530 \times a) = 1 - 0.0671 = 0.9329$$

Use the inverse binomial function on CAS to find the value of $530 \times a$:



$$a \times 530 = 64$$

$$a = \frac{64}{530} = 0.1208$$

Therefore a is closest to 0.12

Question 16**Answer: D****Explanatory notes**

Let $y = f(\log_e(2x))$ and $u = \log_e(2x)$.

Hence $\frac{du}{dx} = \frac{2}{2x} = \frac{1}{x}$ and $\frac{dy}{du} = f'(u) = f'(\log_e(2x))$.

Therefore

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{f'(\log_e(2x))}{x}$$

**Tip**

- This is a straightforward application of the chain rule.

Question 17**Answer: B****Explanatory notes**

Let $X \sim N(85, 5^2)$ and $Z \sim N(0, 1)$. Then

$$\Pr(75 < Z < 100) = \Pr\left(\frac{75-85}{5} < Z < \frac{100-85}{5}\right) = \Pr(-2 < Z < 3).$$

Consider each option in turn:

- A. $\Pr(-3 < Z < 2) = \Pr(-2 < Z < 3)$ ✓
- B. $2 \Pr(0 < Z < 3) + \Pr(2 < Z < 3) \neq \Pr(-2 < Z < 3)$ ✗
- C. $2 \Pr(0 < Z < 2) + \Pr(2 < Z < 3) = \Pr(-2 < Z < 3)$ ✓
- D. $2 \Pr(-3 < Z < 0) - \Pr(-3 < Z < -2) = \Pr(-2 < Z < 3)$ ✓
- E. $\Pr(-2 < Z < 2) + \Pr(2 < Z < 3) = \Pr(-2 < Z < 3)$ ✓

Question 18**Answer: E****Explanatory notes**

Solving $a + \frac{15a+2}{25} + \frac{25a^2+9}{25} + \frac{125a^3+4}{25} = 1$ gives $a = \frac{1}{5}$.

Substituting this value into the equations for x gives

x	1	2	3	4
$\Pr(X = x)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{5}$

Thus the expected value, $E(X)$, $= \frac{1}{5} + \frac{2}{5} + \frac{6}{5} + \frac{4}{5} = \frac{13}{5}$.

**Tips**

- The sum of probabilities equals 1. Use CAS to solve the cubic equation.
- The formula for the expected value is given in the formula sheet.

Question 19**Answer: A****Explanatory notes**

Stationary points occur when $\frac{dy}{dx} = 0$. Differentiating gives

$$\frac{dy}{dx} = 6x^2 + 2(p-1)x + \frac{1}{3}(p+3)$$

For there to be two stationary points, the discriminant of this quadratic must be positive. The discriminant is

$$\begin{aligned}\Delta &= 4(p-1)^2 + 4 \cdot 6 \cdot \frac{1}{3}(p+3) \\ &= 4p^2 - 16p - 20\end{aligned}$$

Solving $4p^2 - 16p - 20 = 0$ gives $p = -1$ or $p = 5$. Thus $\Delta > 0$ if $p \in (-\infty, -1) \cup (5, \infty)$ or $p \in \mathbb{R} \setminus [-1, 5]$.

**Tip**

- Consider drawing a quick sketch of the quadratic for the discriminant.

Question 20**Answer: D****Explanatory notes**

Note that

$$\begin{aligned}3 \cos(2x) + 1 &= 3 \sin\left(2x + \frac{\pi}{2}\right) + 1 \\ &= 3 \sin\left(2x - \frac{3\pi}{2}\right) + 1 \\ &= 3 \sin\left(2\left(x - \frac{3\pi}{4}\right)\right) + 1\end{aligned}$$

That is:

- dilation by a factor of 3 from the x -axis
- dilation by a factor of 2 from the y -axis and
- translation by $\frac{3\pi}{4}$ units to the right and 1 unit up.

**Tip**

- Use the symmetry properties of trigonometric functions to answer this question.

SECTION B

Question 1a.

Worked solution

The function is a cosine function with amplitude 2, shifted up by 5 units.

Maximum: 7

Minimum: 3

Mark allocation: 1 mark

- 1 mark for both answers



Tip

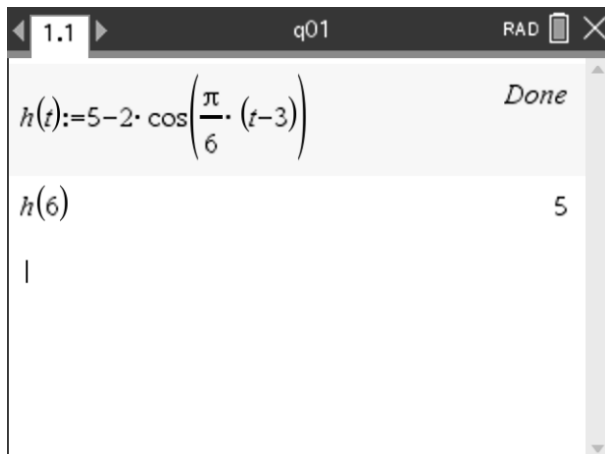
- Note that this particular cosine graph moves 2 units above and 5 units below.

Question 1b.

Worked solution

$$\begin{aligned} h(5) &= 5 - 2 \cos\left(\frac{\pi}{6}(6-3)\right) \\ &= 5 - 2 \cos\left(\frac{\pi}{2}\right) \\ &= 5 \end{aligned}$$

The depth of water is 5 metres.



Mark allocation: 1 mark

- 1 mark for the correct answer



Tip

- While this question can be done by hand, using CAS helps avoid careless errors.

Question 1c.**Worked solution**

$$5 - 2 \cos\left(\frac{\pi}{6}(t-3)\right) = 4$$

$$\cos\left(\frac{\pi}{6}(t-3)\right) = \frac{1}{2}$$

$$\frac{\pi}{6}(t-3) = -\frac{\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$$

$$t-3 = -2, 2, 10, 14$$

$$t = 1, 5, 13, 17$$

The depth of water is four metres at 1.00 am, 5.00 am, 1.00 pm and 5.00 pm.

Mark allocation: 2 marks

- 1 mark for 1.00 am and 5.00 am
- 1 mark for 1.00 pm and 5.00 pm (or the equivalent 24-hour versions)

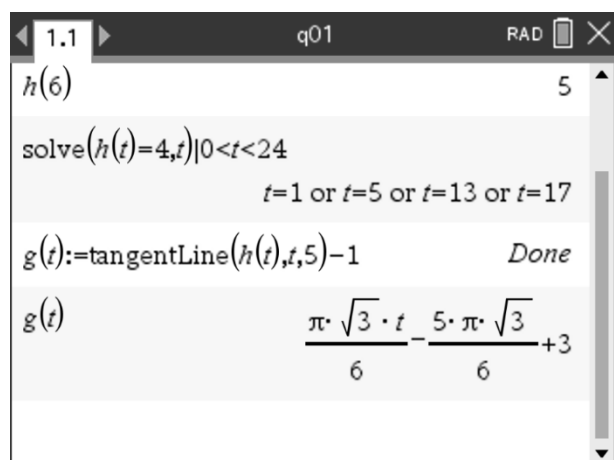
**Tip**

- Use CAS and restrict the domain to ensure that all solutions are obtained.

Question 1d.**Worked solution**

$$\begin{aligned} g(t) &= \frac{\pi\sqrt{3}t}{6} - \frac{5\pi\sqrt{3}}{6} + 3 \\ &= \frac{\pi\sqrt{3}t}{6} - \frac{5\pi\sqrt{3}-18}{6} \end{aligned}$$

This is most easily found by using the tangentLine command on CAS:

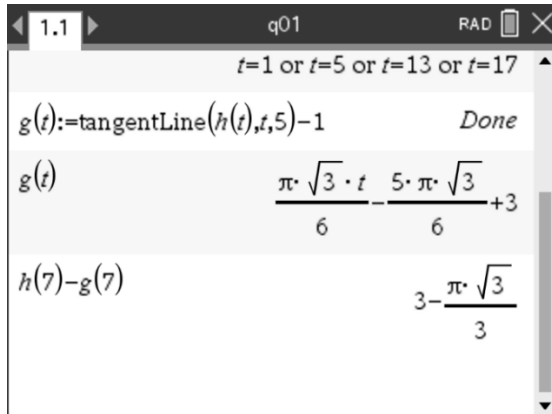
**Mark allocation: 2 marks**

- 1 mark for the correct gradient
- 1 mark for the fully correct answer

Question 1e.**Worked solution**

$$h(7) - g(7) = 3 - \frac{\pi\sqrt{3}}{3}$$

This should be done using CAS:



q01 RAD

$t=1$ or $t=5$ or $t=13$ or $t=17$

$g(t) := \text{tangentLine}(h(t), t, 5) - 1$ Done

$g(t) \quad \frac{\pi \cdot \sqrt{3} \cdot t}{6} - \frac{5 \cdot \pi \cdot \sqrt{3}}{6} + 3$

$h(7) - g(7) \quad 3 - \frac{\pi \cdot \sqrt{3}}{3}$

Mark allocation: 2 marks

- 1 mark for stating the difference between the two functions at $t = 7$
- 1 mark for the correct answer

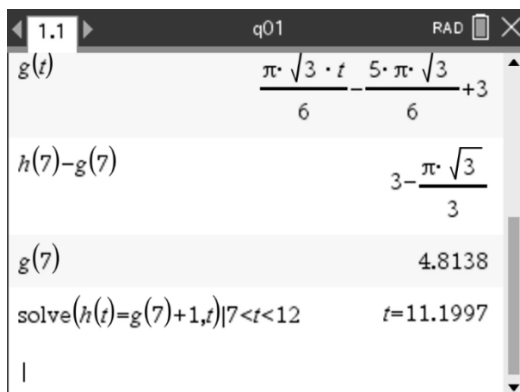
**Tip**

- Remember to give the answer in the form specified in the question.

Question 1f.**Worked solution**

$C(11.20, 4.81)$

Use CAS to find these coordinates:



q01 RAD

$g(t) \quad \frac{\pi \cdot \sqrt{3} \cdot t}{6} - \frac{5 \cdot \pi \cdot \sqrt{3}}{6} + 3$

$h(7) - g(7) \quad 3 - \frac{\pi \cdot \sqrt{3}}{3}$

$g(7) \quad 4.8138$

$\text{solve}(h(t) = g(7) + 1, t) | 7 < t < 12 \quad t = 11.1997$

Mark allocation: 1 mark

- 1 mark for the correct coordinates

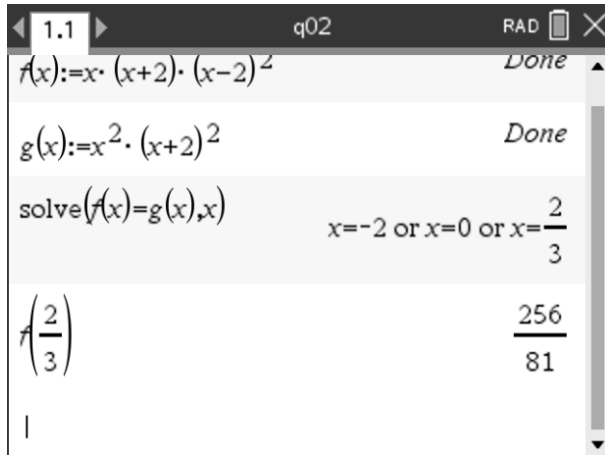
**Tip**

- Remember to give coordinates and not just values.

Question 2a.**Worked solution**

Solve $f(x) = g(x)$ to find the values of a and b .

$$a = \frac{2}{3}, b = \frac{256}{81}$$

**Mark allocation: 2 marks**

- 1 mark for the correct answer for a
- 1 mark for the correct answer for b

**Tip**

- *This question requires an exact answer to be given. Decimal approximations are not acceptable.*

Question 2b.**Worked solution**

The area is found by evaluating two integrals (one for each bounded area):

$$\int_{-2}^0 (g(x) - f(x)) dx + \int_0^{\frac{2}{3}} (f(x) - g(x)) dx = \frac{1136}{81}$$

The screenshot shows a TI-84 Plus calculator interface. At the top, the window is titled '1.1' and 'q02'. The mode is set to 'RAD'. The display shows the function $f\left(\frac{2}{3}\right)$ with the result $\frac{256}{81}$. Below this, the integral expression is entered: $\int_{-2}^0 (g(x) - f(x)) dx + \int_0^{\frac{2}{3}} (f(x) - g(x)) dx$, and the result $\frac{1136}{81}$ is displayed.

Mark allocation: 2 marks

- 1 mark for using two integrals
- 1 mark for the correct answer

**Tip**

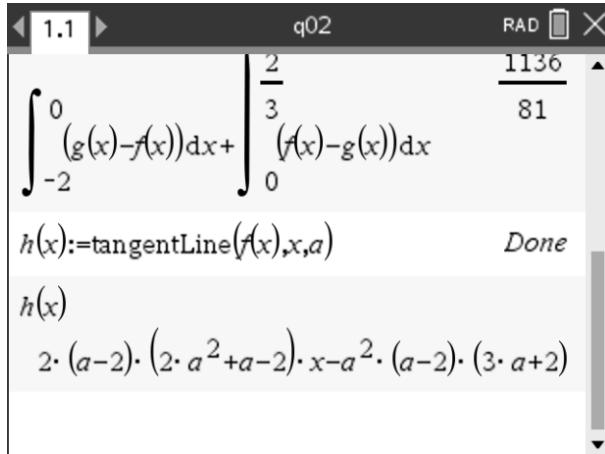
- *Be careful to correctly identify the upper and lower functions.*

Question 2c.**Worked solution**

The equation of the tangent in terms of a is

$$h(x) = -a^2(a-2)(3a+2) + 2(a-2)(2a^2+a-2)x$$

This is found using CAS:

**Mark allocation: 1 mark**

- 1 mark for the correct answer

**Tip**

- The *tangentLine* command allows the use of a variable rather than a specified point.

Question 2d.**Worked solution**

The graphs meet when $x = 1 - a \pm \sqrt{-2a^2 + 2a + 5}$ and when $x = a$ (a general point).

If $1 - a + \sqrt{-2a^2 + 2a + 5} = 1 - a - \sqrt{-2a^2 + 2a + 5}$ then

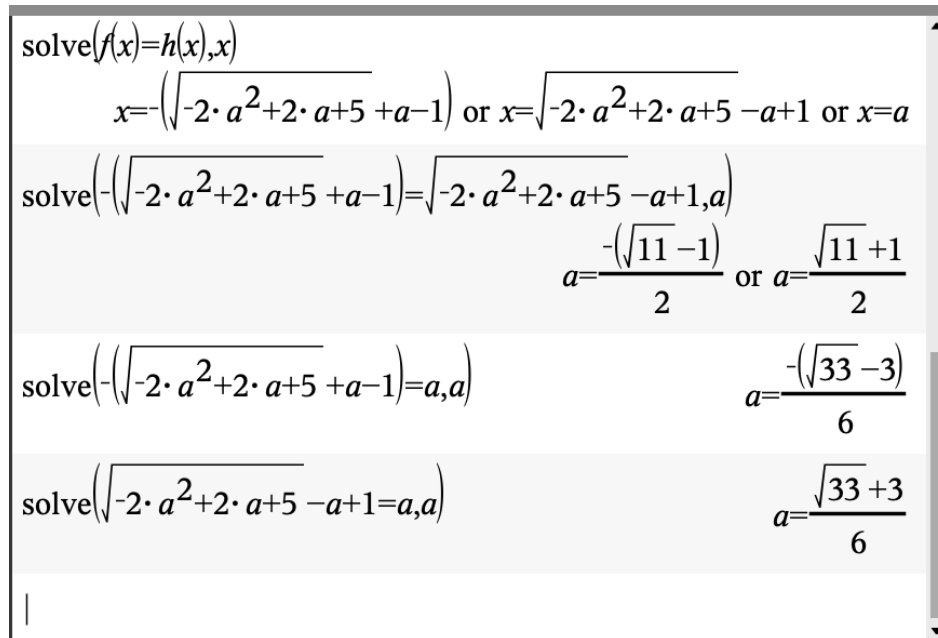
$$a = \frac{1}{2}(1 \pm \sqrt{11})$$

If $1 - a + \sqrt{-2a^2 + 2a + 5} = a$ then

$$a = \frac{3 - \sqrt{33}}{6}$$

If $1 - a - \sqrt{-2a^2 + 2a + 5} = a$ then

$$a = \frac{3 + \sqrt{33}}{6}$$



$\text{solve}(f(x)=h(x),x)$
 $x = -(\sqrt{-2 \cdot a^2 + 2 \cdot a + 5} + a - 1) \text{ or } x = \sqrt{-2 \cdot a^2 + 2 \cdot a + 5} - a + 1 \text{ or } x = a$
 $\text{solve}(-(\sqrt{-2 \cdot a^2 + 2 \cdot a + 5} + a - 1) = \sqrt{-2 \cdot a^2 + 2 \cdot a + 5} - a + 1, a)$
 $a = \frac{-(\sqrt{11} - 1)}{2} \text{ or } a = \frac{\sqrt{11} + 1}{2}$
 $\text{solve}(-(\sqrt{-2 \cdot a^2 + 2 \cdot a + 5} + a - 1) = a, a)$
 $a = \frac{-(\sqrt{33} - 3)}{6}$
 $\text{solve}(\sqrt{-2 \cdot a^2 + 2 \cdot a + 5} - a + 1 = a, a)$
 $a = \frac{\sqrt{33} + 3}{6}$

Mark allocation: 4 marks

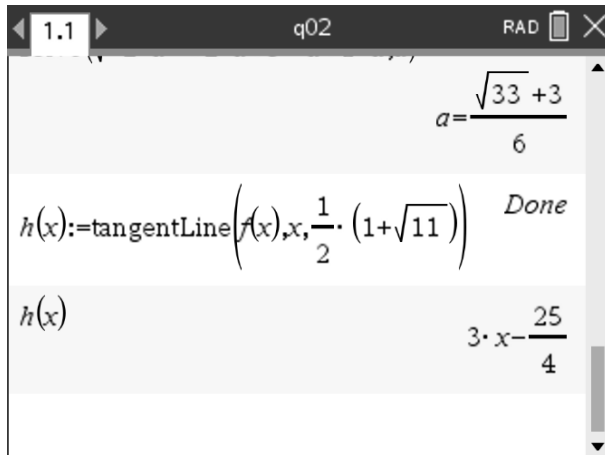
- 1 mark for equating $1 - a + \sqrt{-2a^2 + 2a + 5}$ and $1 - a - \sqrt{-2a^2 + 2a + 5}$
- 1 mark for the solution $a = \frac{1}{2}(1 \pm \sqrt{11})$
- 1 mark for the equations $1 - a + \sqrt{-2a^2 + 2a + 5} = a$ and $1 - a - \sqrt{-2a^2 + 2a + 5} = a$
- 1 mark for the solution $a = \frac{1}{6}(3 \pm \sqrt{33})$

**Tips**

- Use CAS to find where, in terms of 'a', the tangent and the curve meet.
- Consider all the options to find the points where the tangent meets the curve twice.

Question 2e.**Worked solution**

$$h(x) = 3x - \frac{25}{4}$$

**Mark allocation: 1 mark**

- 1 mark for the correct answer

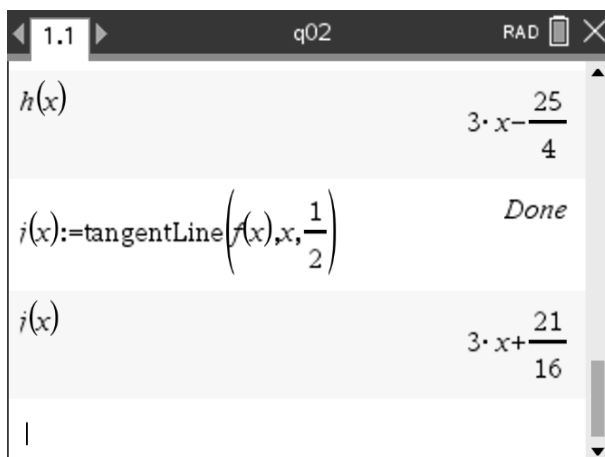
**Tip**

- Having the function $f(x)$ defined makes entry into CAS much easier.

Question 2f.**Worked solution**

The gradient is 3 and $f'(x) = 3$ when $x = \frac{1}{2}$. Therefore

$$j(x) = 3x + \frac{21}{16}$$

**Mark allocation: 1 mark**

- 1 mark for the correct answer

Question 2g.**Worked solution**

The point of intersection of $j(x)$ and $f(x)$ is $\left(\frac{1}{2}, \frac{45}{16}\right)$.

The equation of the line perpendicular to the graph of $y = j(x)$ and which passes through $\left(\frac{1}{2}, \frac{45}{16}\right)$ is

$$\begin{aligned} y &= -\frac{1}{3}\left(x - \frac{1}{2}\right) + \frac{45}{16} \\ &= -\frac{x}{3} + \frac{143}{48} \end{aligned}$$

The screenshot shows a software window with a title bar containing '1.1', 'q02', 'RAD', and a close button. The main area displays the following steps:

- Input: $f\left(\frac{1}{2}\right)$ and $\frac{45}{16}$
- Equation entry: $y(x) := -\frac{1}{3} \cdot \left(x - \frac{1}{2}\right) + \frac{45}{16}$ with a 'Done' button to the right.
- Result: $y(x) = \frac{143}{48} - \frac{x}{3}$

Mark allocation: 1 mark

- 1 mark for the equation of the line

**Tip**

- Use the fact that the product of the gradients of two perpendicular lines is -1 .

Question 2h.**Worked solution**

The line in **part 2g.** meets $h(x)$ at $\left(\frac{443}{160}, \frac{329}{160}\right)$.

The distance between $\left(\frac{1}{2}, \frac{45}{16}\right)$ and $\left(\frac{443}{160}, \frac{329}{160}\right)$ is

$$\sqrt{\left(\frac{443}{160} - \frac{1}{2}\right)^2 + \left(\frac{329}{160} - \frac{45}{16}\right)^2} = \frac{121\sqrt{10}}{160}$$

Mark allocation: 2 marks

- 1 mark for finding the point $\left(\frac{443}{160}, \frac{329}{160}\right)$
- 1 mark for using the distance formula to derive the correct answer

**Tips**

- *The distance is the shortest perpendicular distance.*
- *Use the results from earlier to find the point of intersection of the lines and then use the distance formula.*

Question 3a.**Worked solution**

Suppose $X \sim N(56, 4^2)$

$$\Pr(X < 53) = 0.2266$$

1.1	q03	RAD	✕
normCdf(0,53,56,4)		0.226627	

Mark allocation: 1 mark

- 1 mark for the correct answer

Question 3b.**Worked solution**

$$\Pr(X < m) = 0.75$$

$$m = 58.6980$$

1.1	q03	RAD	✕
normCdf(0,53,56,4)		0.226627	
invNorm(0.75,56,4)		58.698	
58.697958997801		58.698	

Mark allocation: 1 mark

- 1 mark for the correct answer

**Tip**

- The inverse normal function is used here.

Question 3c.**Worked solution**

Let $W \sim \text{Bi}(30, 0.308538)$

$$\Pr(W \leq 10) = 0.6950$$

1.1	q03	RAD	✕
normCdf(0,53,56,4)	0.226627		
invNorm(0.75,56,4)	58.698		
58.697958997801	58.698		
binomCdf(30,0.308538,0,10)	0.694973		

Mark allocation: 2 marks

- 1 mark for using the binomial distribution
- 1 mark for the correct answer

**Tip**

- Note that the distribution is now binomial. The parameters are n and p .

Question 3d.**Worked solution**

$$\begin{aligned}
 \Pr(\hat{P} > 0.15 | \hat{P} < 0.6) &= \Pr\left(\frac{W}{30} > 0.15 \mid \frac{W}{30} < 0.6\right) \\
 &= \Pr(W > 4.5 \mid W < 18) \\
 &= \Pr(W \geq 5 \mid W \leq 17) \\
 &= \frac{\Pr(5 \leq W \leq 17)}{\Pr(W \leq 17)} \\
 &= 0.9758
 \end{aligned}$$

1.1	q03	RAD	X
normCdf(0,53,56,4)	0.226627		
invNorm(0.75,56,4)	58.698		
58.697958997801	58.698		
binomCdf(30,0.308538,0,10)	0.694973		
<u>binomCdf(30,0.308538,5,17)</u>	<u>0.975832</u>		
binomCdf(30,0.308538,0,17)			

Mark allocation: 2 marks

- 1 mark for using the conditional probability formula
- 1 mark for the correct answer

**Tip**

- Note that conditional probability is required here. Be familiar with the conditional probability formula.

Question 3e.**Worked solution**

The median time is m where

$$\int_{50}^m f(x) dx = \frac{1}{2}$$

$$m = 70.04$$

The screenshot shows a CAS window with the following content:

- Top bar: 1.1, q03, RAD, and a close button.
- Function definition: $f(x) := \frac{1}{9000} \cdot (x-50) \cdot (80-x) \cdot e^{\frac{1}{15} \cdot (x-50)}$
- Integration button: $\int$
- Solve command: $\text{solve}\left(\int_{50}^m f(x) dx = \frac{1}{2}, m\right)$
- Result: $m = 70.0426 \text{ or } m = 85.5897$

Mark allocation: 2 marks

- 1 mark for the appropriate integral (with the right-hand side equal to $\frac{1}{2}$)
- 1 mark for the correct answer

**Tip**

- Define the function in CAS, as it will be used several times.

Question 3f.**Worked solution**

Denote the probability distribution by X . Then

$$\begin{aligned}\text{Var}(X) &= E(X^2) - (E(X))^2 \\ &= 33.23\end{aligned}$$

Therefore $\sigma_X = 5.7646$

Handheld calculator screen showing the calculation of variance. The display shows the formula for variance, the integral calculation for $E(X^2)$ and $(E(X))^2$, and the final result 33.23. Below that, it shows the square root of 33.2300449443, resulting in 5.76455.

Mark allocation: 2 marks

- 1 mark for the appropriate use of the variance formula
- 1 mark for the correct answer

**Tip**

- Remember that the standard deviation is the square root of the variance.

Question 3g.**Worked solution**

$$\begin{aligned}\Pr(X < 54 | X < 60) &= \frac{\Pr(X < 54)}{\Pr(X < 60)} \\ &= 0.0657\end{aligned}$$

Handheld calculator screen showing the calculation of conditional probability. The display shows the formula for conditional probability, the integral calculation for $\Pr(X < 54)$ and $\Pr(X < 60)$, and the final result 0.06569.

Mark allocation: 2 marks

- 1 mark for using the conditional probability formula
- 1 mark for the correct answer

Question 3h.**Worked solution**

The value of $\hat{p}$ is $\frac{0.0747 + 0.2453}{2} \approx 0.16$

Solving $\hat{p} + z \sqrt{\frac{\hat{p}(1-\hat{p})}{50}} = 0.2453$ gives $z \approx 1.6453$

Then $\Pr(Z > 1.6453) \approx 0.05$ is the area in each tail of the standard normal distribution, and so we have a 90% confidence interval.

The calculator screen shows the following steps:

- Line 1: $p := \frac{0.0747 + 0.2453}{2}$ results in 0.16
- Line 2: $\text{solve}\left(p + z \cdot \sqrt{\frac{p \cdot (1-p)}{50}} = 0.2453, z\right)$ results in $z = 1.64526$
- Line 3: $\text{normCdf}(z, \infty, 0, 1) | z = 1.6452594689708$ results in 0.049958

Mark allocation: 2 marks

- 1 mark for finding $\hat{p}$ and z
- 1 mark for the correct answer

**Tips**

The value of $\hat{p}$ is the midpoint of the confidence interval.

The formula for calculating a confidence interval is on the formula sheet.

Question 4a.**Worked solution**

The maximum occurs when $x = \frac{a}{3}$.

The coordinates of A are $A\left(0, \frac{2\sqrt{3}}{9} a^{\frac{3}{2}}\right)$.

The screenshot shows a TI-84 Plus calculator interface. At the top, the window title is '1.1' and 'q04'. The mode is set to 'RAD'. The calculator screen displays the following steps:

- Define the function: $f(x) := \sqrt{x} \cdot (a - x)$
- Find the maximum: $fMax(f(x), x)$ results in $x = \frac{a}{3}$ or $x = 0$
- Calculate the function value at $x = \frac{a}{3}$: $f\left(\frac{a}{3}\right)$ results in $\frac{2 \cdot a^{\frac{3}{2}} \cdot \sqrt{3}}{9}$

Mark allocation: 2 marks

- 1 mark for finding $x = \frac{a}{3}$
- 1 mark for the correct answer

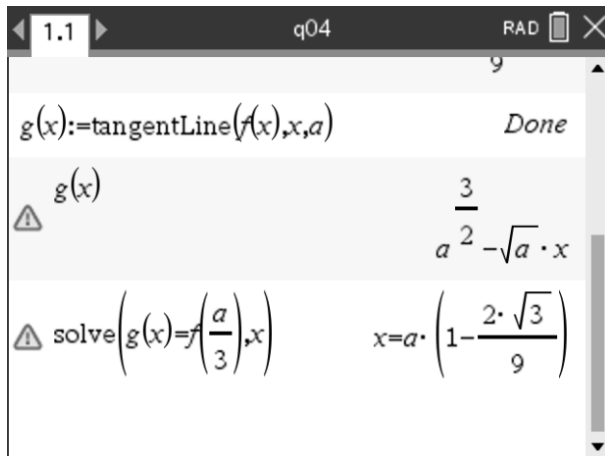
Question 4b.**Worked solution**

The equation of the tangent to f at $x = a$ is

$$g(x) = -\sqrt{a}x + a^{\frac{3}{2}}$$

The coordinates of B are

$$B\left(\frac{(9-2\sqrt{3})a}{9}, \frac{2\sqrt{3}}{9}a^{\frac{3}{2}}\right)$$

**Mark allocation: 2 marks**

- 1 mark for finding the equation of the tangent to f at $x = a$
- 1 mark for the correct coordinates

**Tip**

- To find the coordinates of B , the tangent must be found first. You can find this quickly using CAS.

Question 4c.**Worked solution**

The area A_1 of the trapezium $OABC$ is

$$\frac{1}{2} \left(\frac{9-2\sqrt{3}}{9}a + a \right) \frac{2\sqrt{3}}{9}a^{\frac{3}{2}} = \frac{6\sqrt{3}-2}{27}a^{\frac{5}{2}}$$

1.1 q04 RAD

$\Delta \text{ solve}\left(g(x)=f\left(\frac{a}{3}\right),x\right) \quad x=a\cdot\left(1-\frac{2\cdot\sqrt{3}}{9}\right)$

$\frac{1}{2}\cdot\left(a\cdot\left(1-\frac{2\cdot\sqrt{3}}{9}\right)+a\right)\cdot f\left(\frac{a}{3}\right)$

$a^{\frac{5}{2}}\cdot\left(\frac{2\cdot\sqrt{3}}{9}-\frac{2}{27}\right)$

Mark allocation: 2 marks

- 1 mark for correctly using the trapezium formula
- 1 mark for the correct answer

**Tips**

- Remember to give your answer in the form specified in the question.
- The formula for the area of a trapezium is found on the formula sheet.

Question 4d.**Worked solution**

There are various ways to approach this problem. One way is to consider that

$$p(x) = bx(x - a)$$

$$p\left(\frac{a}{3}\right) = \frac{2\sqrt{3}}{9}a^{\frac{3}{2}}$$

$$b = -\frac{\sqrt{3}}{\sqrt{a}}$$

$$p(x) = -\frac{\sqrt{3}}{\sqrt{a}}x(x - a)$$

Mark allocation: 2 marks

- 1 mark for providing appropriate working (for example, giving the parabola in intercept form)
- 1 mark for the correct answer

**Tip**

- *The intercept form has been used here. Alternatively, a system of equations could be solved to find the equation of the parabola.*

Question 4e.**Worked solution**

The area of the shaded region is

$$A_2 = \int_0^a p(x) dx = \frac{\sqrt{3}}{6} a^{\frac{5}{2}}$$

1.1 q04 RAD

$$p1(x) = \frac{-\sqrt{3} \cdot x \cdot (x-a)}{\sqrt{a}}$$

$$\int_0^a p1(x) dx = \frac{5}{6} a^2 \cdot \sqrt{3}$$

Mark allocation: 1 mark

- 1 mark for the correct answer

Question 4f.**Worked solution**

The ratio is

$$\begin{aligned} \frac{A_1}{A_2} &= \frac{4}{3} - \frac{4\sqrt{3}}{27} \\ &= \frac{4}{27} (9 - \sqrt{3}) \end{aligned}$$

1.1 q04 RAD

$$\frac{\frac{4}{3} - \frac{4 \cdot \sqrt{3}}{27}}{\frac{5}{6} a^2 \cdot \sqrt{3}}$$

Mark allocation: 1 mark

- 1 mark for the correct answer

Question 5a.**Worked solution**

$$y = e^{\frac{x}{2}} - 3. \text{ Swap } x \text{ and } y:$$

$$x = e^{\frac{y}{2}} - 3$$

$$y = 2 \log_e(x + 3)$$

Therefore $f^{-1}(x) = 2 \log_e(x + 3)$ or $f^{-1}(x) = 2 \ln(x + 3)$.

Mark allocation: 1 mark

- 1 mark for the correct answer

**Tips**

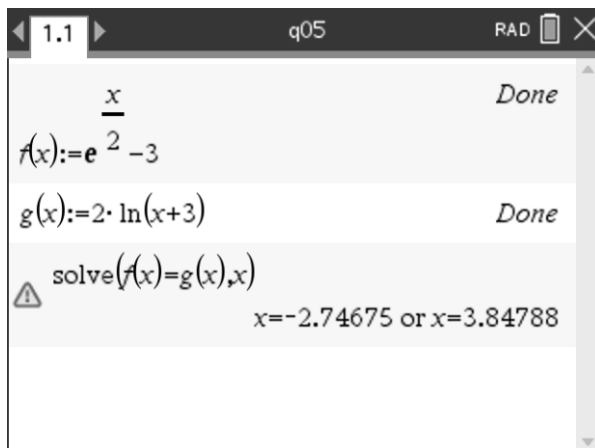
- Remember to write $\log_e$ or $\ln$.
- For a single-mark question where no particular working is required, solving an equation of the form $f(y) = x$ for x on CAS allows the inverse function to be quickly found.

Question 5b.**Worked solution**

The points of intersection are

$(-2.747, -2.747)$ and $(3.848, 3.848)$

Use CAS to find these points:

**Mark allocation: 2 marks**

- 1 mark for each intersection, correct to three decimal places

**Tips**

- Ensure that coordinates are given. Simply writing the x -values will not satisfy the requirements of the question.
- The x -coordinate and y -coordinate are equal, as the intersection points lie on the line $y = x$.

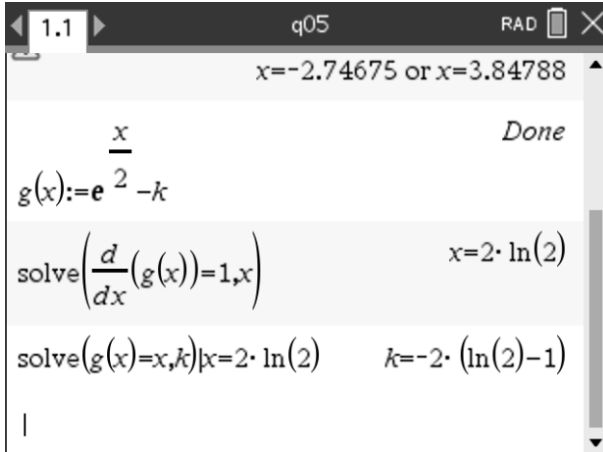
Question 5c.i.**Worked solution**

If the graphs of $y = g(x)$ and $y = g^{-1}(x)$ meet only once, they must meet on the line $y = x$ (which is the tangent to $g(x)$ and $g^{-1}(x)$ at this point). The gradient of the line $y = x$ is 1 and so we solve $g'(x) = 1$.

If $g'(x) = 1$ then $x = 2 \log_e(2)$ and

$$g(2 \log_e(2)) = 2 \log_e(2)$$

$$k = 2 - \log_e(4)$$

**Mark allocation: 1 mark**

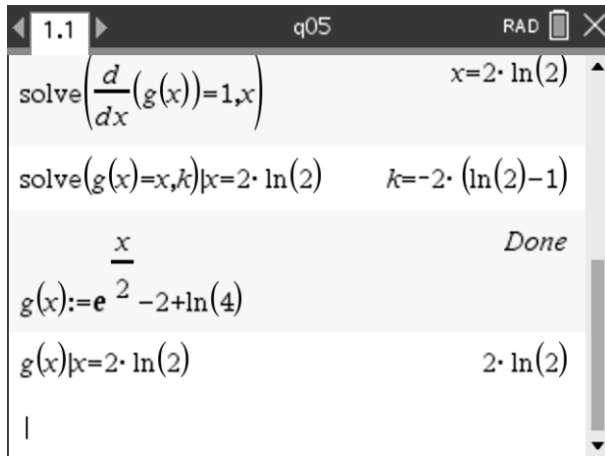
- 1 mark for working that leads to the desired result

**Tip**

- *If the two curves meet only once, then they must meet on the line $y = x$.*

Question 5c.ii.**Worked solution**

$$(2 \log_e(2), 2 \log_e(2)) \text{ or } (\log_e(4), \log_e(4))$$

**Mark allocation: 1 mark**

- 1 mark for the correct coordinates

**Tip**

- As $g(x)$ and $g^{-1}(x)$ meet on the line $y = x$, the x and y values of the point of intersection are equal.

Question 5d.**Worked solution**

$$k \in (-\infty, 2 - \log_e(4)) \text{ or } -\infty < k < 2 - \log_e(4)$$

Mark allocation: 1 mark

- 1 mark for the correct interval

Question 5e.**Worked solution**

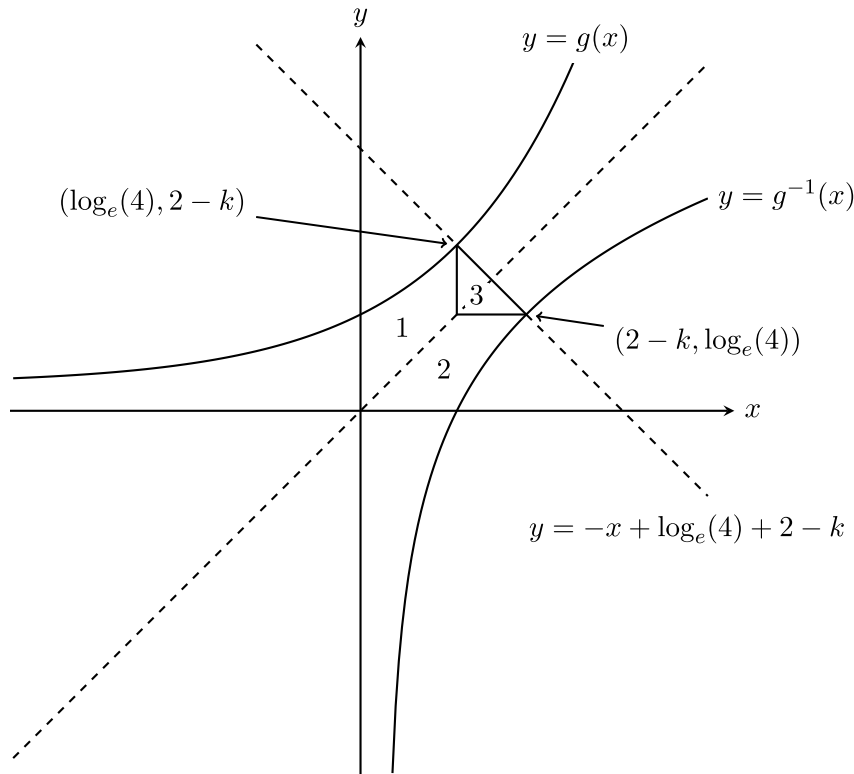
$$\begin{aligned} y &= -(x - \log_e(4)) + g(\log_e(4)) \\ &= -x + \log_e(4) + 2 - k \end{aligned}$$

Mark allocation: 1 mark

- 1 mark for working that leads to the desired result

Question 5f.**Worked solution**

The area can be found by considering the following diagram.



The area labelled 1 is given by the integral $\int_0^{\log_e(4)} (g(x) - x) dx$. The area labelled 2 is (by symmetry) equal to area 1.

The area of the triangle labelled 3 is given by $\frac{1}{2}(2 - k - \log_e(4))^2$. Therefore

$$\begin{aligned} A(k) &= 2 \int_0^{\log_e(4)} (g(x) - x) dx + \frac{1}{2}(2 - k - \log_e(4))^2 \\ &= \frac{1}{2}k^2 - (2 - \log_e(4)) + 6 - k \log_e(4) - \frac{1}{2}(\log_e(4))^2 \end{aligned}$$

$g(x) := e^{\frac{x}{2}} - k$	
$\text{solve}\left(\frac{d}{dx}(g(x))=1, x\right)$	$x = 2 \cdot \ln(2)$
$\text{solve}(g(x)=x, k) _{x=2 \cdot \ln(2)}$	$k = 2 \cdot (\ln(2) - 1)$
$\frac{x}{2}$	Done
$g(x) := e^{\frac{x}{2}} - 2 + \ln(4)$	
$g(x) _{x=2 \cdot \ln(2)}$	$2 \cdot \ln(2)$
$\frac{x}{2}$	Done
$g(x) := e^{\frac{x}{2}} - k$	
$2 \cdot \int_0^{\ln(4)} (g(x) - x) dx + \frac{1}{2} \cdot (2 - k - \ln(4))^2$	$\frac{k^2}{2} + k \cdot (-2 \cdot \ln(2) - 2) - 2 \cdot (\ln(2))^2 - 4 \cdot \ln(2) + 6$
$a(k) := \frac{k^2}{2} + k \cdot (-2 \cdot \ln(2) - 2) - 2 \cdot (\ln(2))^2 - 4 \cdot \ln(2) + 6$	Done

Mark allocation: 2 marks

- 1 mark for the appropriate working
- 1 mark for the correct answer



Tip

- This question requires careful consideration of the areas involved. It may be useful to annotate the graph.

Question 5g.**Worked solution**

The minimum value of $A(k)$ is $4(\log_e(2))^2 - 8\log_e(2) + 4$.

This occurs when the curves $y = g(x)$ and $y = g^{-1}(x)$ meet; that is, when $k = 2 - \log_e(4)$.

1.1 q05 RAD

$$\frac{k^2}{2} + k \cdot (-2 \cdot \ln(2) - 2) - 2 \cdot (\ln(2))^2 - 4 \cdot \ln(2) + 6$$

$$a(k) := \frac{k^2}{2} + k \cdot (-2 \cdot \ln(2) - 2) - 2 \cdot (\ln(2))^2 - 4 \cdot \ln(2) + 6$$

Done

$$a(k)|_{k=2-\ln(4)} \quad 4 \cdot (\ln(2))^2 - 8 \cdot \ln(2) + 4$$

Mark allocation: 2 marks

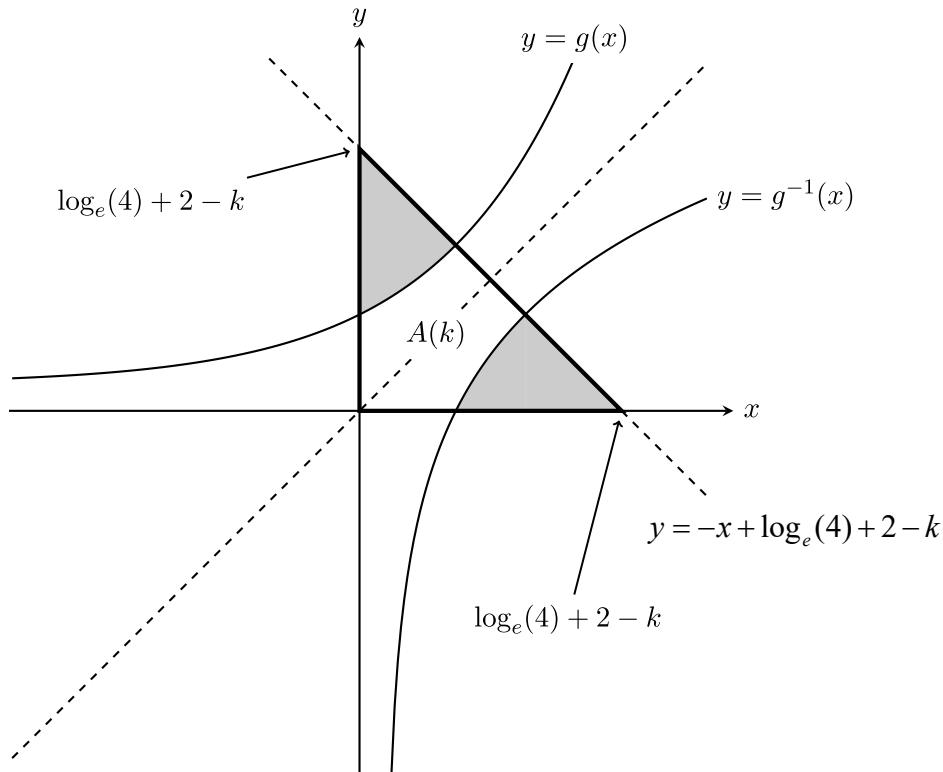
- 1 mark for the correct minimum value of $A(k)$
- 1 mark for the correct value of k

**Tip**

- *Read the question carefully. Two items of information are required.*

Question 5h.**Worked solution**

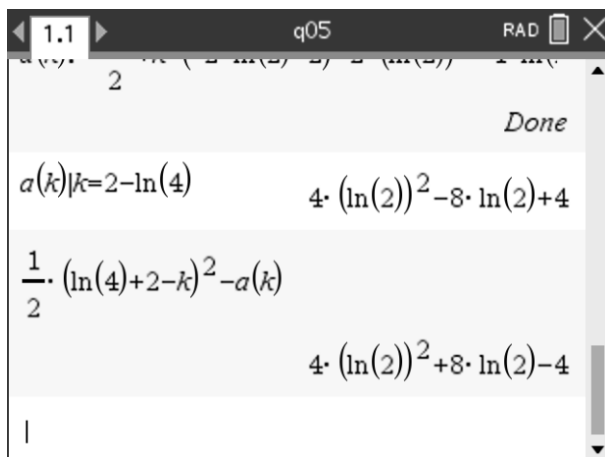
The total shaded area is equal to the difference in the areas of the triangle with vertices at $(0,0)$, $(0, \log_e(4) + 2 - k)$ and $(\log_e(4) + 2 - k, 0)$, and the area $A(k)$ found in **part f**.



The value of B is a constant.

$$B = \frac{1}{2}(\log_e(4) + 2 - k)^2 - A(k)$$

$$= 4(\log_e(2))^2 + 8\log_e(2) - 4$$



Mark allocation: 2 marks

- 1 mark for attempting to calculate the area
- 1 mark for the correct answer

END OF WORKED SOLUTIONS