

VCE Mathematical Methods Units 3&4

Written Examination 2

Suggested Solutions

SECTION A – MULTIPLE-CHOICE QUESTIONS

1	☒ A	☐ B	☐ C	☐ D	☐ E
2	☐ A	☐ B	☐ C	☒ D	☐ E
3	☒ A	☐ B	☐ C	☐ D	☐ E
4	☐ A	☒ B	☐ C	☐ D	☐ E
5	☐ A	☐ B	☒ C	☐ D	☐ E
6	☐ A	☐ B	☒ C	☐ D	☐ E
7	☐ A	☐ B	☐ C	☒ D	☐ E
8	☐ A	☐ B	☐ C	☐ D	☒ E
9	☐ A	☐ B	☐ C	☐ D	☒ E
10	☐ A	☒ B	☐ C	☐ D	☐ E
11	☐ A	☐ B	☒ C	☐ D	☐ E
12	☐ A	☒ B	☐ C	☐ D	☐ E
13	☐ A	☒ B	☐ C	☐ D	☐ E
14	☐ A	☐ B	☐ C	☒ D	☐ E
15	☐ A	☐ B	☐ C	☐ D	☒ E
16	☐ A	☒ B	☐ C	☐ D	☐ E
17	☐ A	☐ B	☒ C	☐ D	☐ E
18	☐ A	☐ B	☐ C	☒ D	☐ E
19	☐ A	☒ B	☐ C	☐ D	☐ E
20	☒ A	☐ B	☐ C	☐ D	☐ E

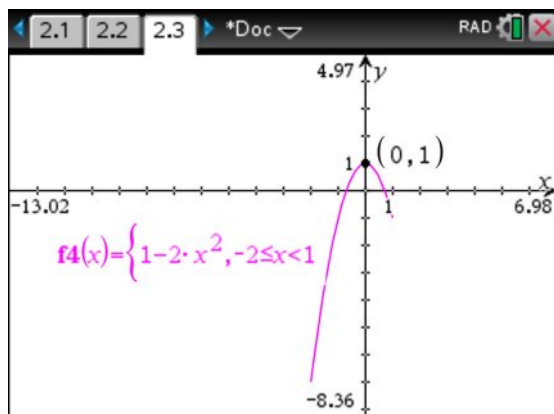
Question 1 A

$$\text{tangentLine}(x^2+2\cdot x-1,x,1) \quad 4\cdot x-2$$

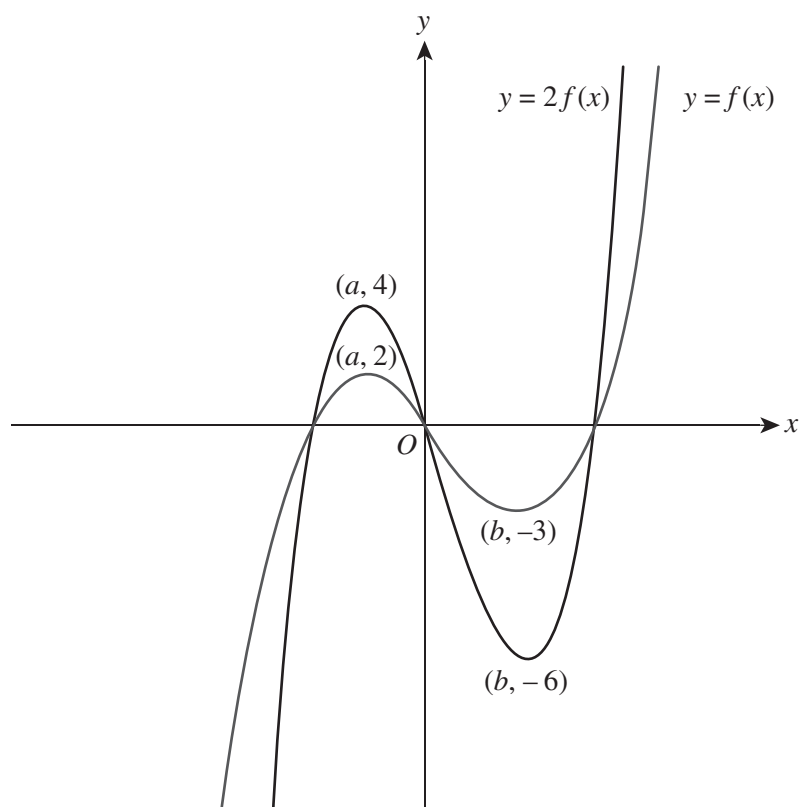
Question 2 D

$$f(-2) = -7$$

$$\therefore \text{range} = [-7, 1]$$

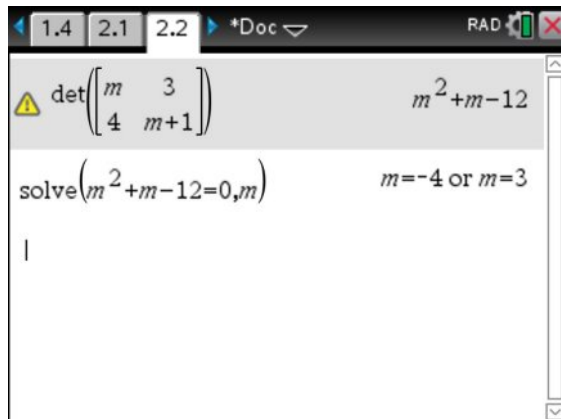
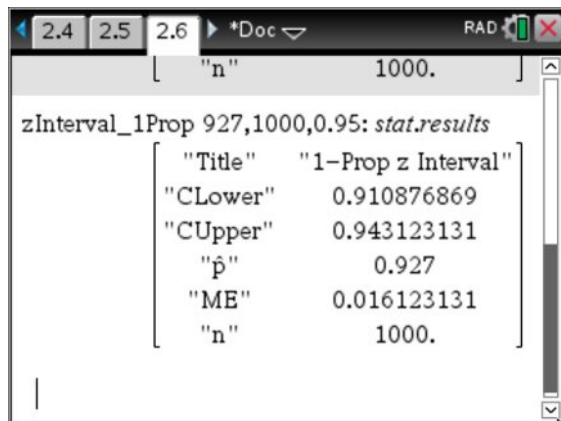
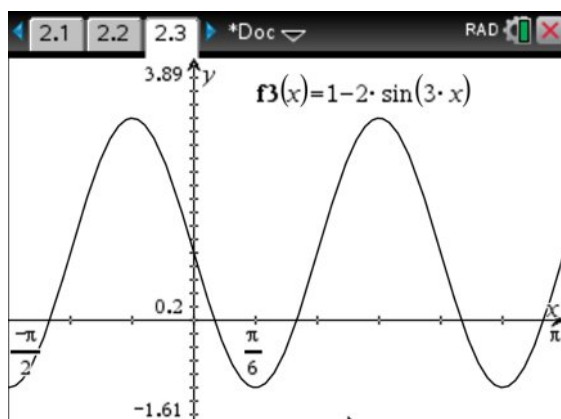
**Question 3 A**

Graphing $y = 2f(x)$ gives turning points that provide the restrictions for c . The graph of $y = c$ is a horizontal line. The solution of $2f(x) = c$ is the intersection point of $y = 2f(x)$ and $y = c$. Therefore, $c < -6$ or $c > 4$ gives only one point of intersection and one solution.



Question 4 B

$$\begin{bmatrix} m & 3 \\ 4 & m+1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ m \end{bmatrix}$$

**Question 5 C****Question 6 C**

The local minimum is $\left(\frac{\pi}{6}, -1\right)$ and the local maximum is $\left(\frac{\pi}{2}, 3\right)$. Therefore, the range is $[-1, 3]$.

$$\begin{aligned} \text{period} &= \frac{2\pi}{n} \\ &= \frac{2\pi}{3} \end{aligned}$$

Question 7 D

$$g(x) = 2f(x - 5) + 1$$

Transformations:

- dilation factor of 2 from x -axis $(-1, 2) \rightarrow (-1, 4)$
- translation of 5 units right $(-1, 4) \rightarrow (4, 4)$
- translation of 1 unit up $(4, 4) \rightarrow (4, 5)$

$$\rightarrow g(4) = 5$$

Question 8 E

$$\begin{aligned} \int_5^3 1 - 2f(x) \, dx &= \int_3^5 2f(x) - 1 \, dx \\ &= 2 \int_3^5 f(x) \, dx - \int_3^5 1 \, dx \\ &= 2 \times 10 - 2 \\ &= 18 \end{aligned}$$

Question 9 E

As $0 \leq \Pr(x = 2) \leq 1$, $0 \leq k - \frac{1}{4} \leq 1$, therefore the minimum value of k is equal to $\frac{1}{4}$.

Maximum value of m :

$$\begin{aligned} m &= 1 - \left(\frac{1}{4} + 0 + \left(\frac{1}{4} \right)^2 \right) \\ &= \frac{11}{16} \end{aligned}$$

Question 10 B

The screenshot shows a CAS calculator interface with the following steps:

- Define $f(x) := x^3 + 2 \cdot x$ (Done)
- Define $r(a) := \frac{f(a+1) - f(a)}{a+1-a}$ (Done)
- Simplify $r(a)$ to $3 \cdot a^2 + 3 \cdot a + 3$
- Solve $\left(\frac{d}{da}(r(a)) = 0, a \right)$ to get $a = \frac{-1}{2}$

Question 11 C

Degree must be even as range $\neq \mathbb{R}$.

There is a point of inflection at $x = -a$ and a local maximum at $x = a$.

Therefore, f is at minimum a degree-4 polynomial.

Question 12 B

Let S = success, M = miss, A = Any (success or miss)

$$\Pr(\text{exactly 3}) = \Pr(\text{SSSM}) + \Pr(\text{MSSS})$$

$$\begin{aligned}\Pr(\text{exactly 3}) &= \left(\frac{4}{5}\right)^3 \times \left(\frac{1}{5}\right) + \left(\frac{4}{5}\right)^3 \times \left(\frac{1}{5}\right) \\ &= \frac{27}{128}\end{aligned}$$

Question 13 B

$$\begin{aligned}\text{solve} \left(\int_{\pi}^a \sin(2 \cdot x) \, dx = \frac{1}{4}, a \right) & \mid \pi < a < \frac{3 \cdot \pi}{2} \\ a &= \frac{7 \cdot \pi}{6}\end{aligned}$$

Question 14 D

$$\begin{aligned}\text{solve}(1 - m \cdot x > 0, x) & \mid m > 0 \\ x & < \frac{1}{m} \text{ and } m > 0\end{aligned}$$

Question 15 E

E is correct. This graph has two x -intercepts and has matching gradients with the derivative function f' .

Question 16 B

Differentiating the distance formula from the origin to a point $P(x, c - x^2)$ reveals that the magnitude of the minimum (or maximum) distance occurs at an x -value of $x = \frac{\pm\sqrt{2(2c-1)}}{2}$ if $2c - 1 \geq 0$ or $c \geq \frac{1}{2}$.

If $c \leq \frac{1}{2}$, however, the magnitude of the minimum distance will always be found at the y -intercept, which has a value of c .

Note: A sketch graph could also be used.

TI-84 Plus calculator screen showing the derivation of the minimum distance. The input is $d(x) := \sqrt{x^2 + (c - x^2)^2}$. The solve command is used to solve $\frac{d}{dx}(d(x)) = 0$, resulting in $x = \frac{\sqrt{2 \cdot (2 \cdot c - 1)}}{2}$ and $2 \cdot c - 1 \geq 0$ or $x = \frac{-\sqrt{2 \cdot (2 \cdot c - 1)}}{2}$.

Note: $x = 0$ gives the magnitude of the minimum distance when $c \geq \frac{1}{2}$, but the derivative indicates a local maximum when $c > \frac{1}{2}$.

Question 17 C

$$z = \frac{x - \mu}{\sigma}$$

$$\mu = 0$$

$$\sigma = \frac{x}{z}$$

$$\text{Var}(X) = \sigma^2$$

TI-84 Plus calculator screen showing the calculation of the variance. The steps are: $\text{invNorm}(0.9, 0, 1) = 1.28155157$, then $\frac{1.5}{1.28} = 1.171875$, and finally $(1.171875)^2 = 1.37329102$.

Question 18 D

$$\begin{aligned}
 y &= \sqrt{\frac{1}{f(x)}} \\
 &= [f(x)]^{-\frac{1}{2}} \\
 \frac{dy}{dx} &= -\frac{1}{2} \times [f(x)]^{-\frac{3}{2}} \times f'(x) \\
 \frac{dy}{dx} &= \frac{-f'(x)}{2[\sqrt{f(x)}]^3}
 \end{aligned}$$

Question 19 B

$\sin(\theta)$ is a one-to-one function for $0 \leq \theta \leq \frac{\pi}{2}$.

$$\begin{aligned}
 0 &\leq a^x \leq \frac{\pi}{2} \\
 -\infty &< x \leq \log_a\left(\frac{\pi}{2}\right)
 \end{aligned}$$

Question 20 A

Maximum area occurs when the parabola is tangential to the line at $x = a$.

$$\begin{aligned}
 g(x) &= kx(x - a) \\
 g'(x) &= k(2x - a)
 \end{aligned}$$

When $x = a$, $g'(a) = ak$.

$$m_T = -\frac{b}{a}$$

Let $g'(a) = m_T$.

$$ak = -\frac{b}{a}$$

$$k = -\frac{b}{a^2}$$

$$g(x) = -\frac{b}{a^2}x(x - a)$$

$$\begin{aligned}
 \text{maximum area} &= \int_0^a -\frac{b}{a^2}x(x - a)dx \\
 &= \frac{ab}{6}
 \end{aligned}$$

SECTION B**Question 1** (10 marks)

a. $y = 3x - 2$

A1

$$f(x) := x^3$$

Done

$$\text{tangentLine}(f(x), x, 1)$$

$$3 \cdot x - 2$$

b.
$$A = \int_0^{\frac{2}{3}} x^3 dx + \int_{\frac{2}{3}}^1 x^3 - (3x - 2) dx$$

$$= \frac{1}{12}$$

M2

A1

$$\int_0^{\frac{2}{3}} f(x) dx + \int_{\frac{2}{3}}^1 (f(x) - (3 \cdot x - 2)) dx \quad \frac{1}{12}$$

c. $y = 3x - 2 + k$

M1

Let $y = 0$.

$$3x - 2 + k = 0$$

$$x = \frac{2}{3} - \frac{k}{3}$$

$$\left(\frac{2}{3} - \frac{k}{3}, 0 \right)$$

A1

$$\text{d. area 1} = \int_0^{\frac{2-k}{3}} x^3 + k \, dx + \int_{\frac{2-k}{3}}^1 x^3 + k - (3x - 2 + k) \, dx \quad \text{M1}$$

$$= -\frac{k^2}{6} + \frac{2k}{3} + \frac{1}{12} \quad \text{M1}$$

$$\text{area 2} = -\frac{1}{2} \times (k-2) \times \left(\frac{2-k}{3} - \frac{k}{3} \right)$$

$$= \frac{k^2}{6} - \frac{2k}{3} + \frac{2}{3}$$

$$\text{area 1} = \text{area 2}$$

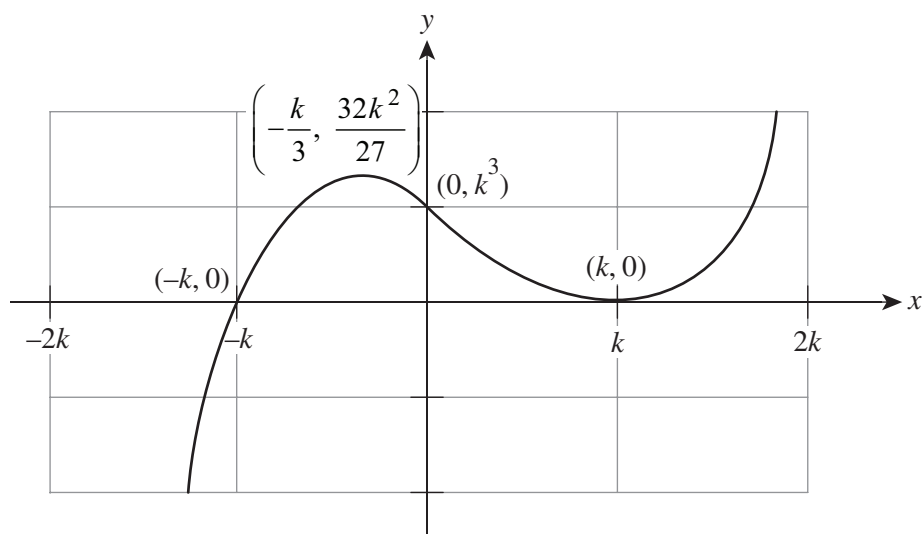
$$-\frac{k^2}{6} + \frac{2k}{3} + \frac{1}{12} = \frac{k^2}{6} - \frac{2k}{3} + \frac{2}{3} \quad \text{M1}$$

$$k = \frac{1}{2} \text{ or } k = \frac{7}{2}$$

$$\text{Since } 0 < k < 2, k = \frac{1}{2} \quad \text{A1}$$

Question 2 (6 marks)

a.



correct intercepts A1
correct turning point A1
correct shape A1

$$\text{b. i. } x^2 - k^2 = \frac{1}{x - k}$$

$$(x - k)(x + k) = \frac{1}{x - k}$$

$$(x - k)^2(x + k) = 1$$

A1

ii. Let $f(x) = (x - k)^2(x + k)$.

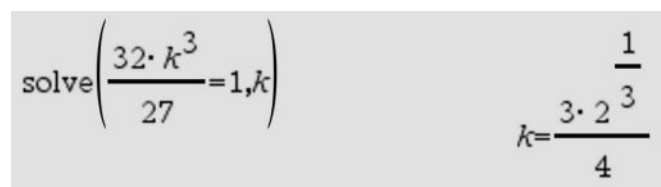
Since $(x - k)^2(x + k) = 1$, $f(x) = 1$.

Two solutions occur when $\frac{32k^3}{27} = 1$, as seen on the graph in **part a**.

M1

$$k = \frac{3 \times 2^{\frac{1}{3}}}{4}$$

A1



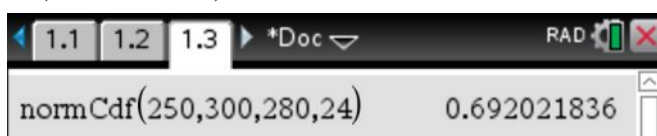
Handwritten solution showing the equation $\text{solve}\left(\frac{32 \cdot k^3}{27} = 1, k\right)$ and the result $k = \frac{3 \cdot 2^{\frac{1}{3}}}{4}$.

Question 3 (17 marks)

a. Let $B \sim N(280, 24^2)$

$$\Pr(250 < B < 300) = 0.692$$

A1



TI-Nspire calculator screenshot showing the command `normCdf(250,300,280,24)` and the result `0.692021836`.

b. $\Pr(\text{elite}) = \Pr(B < 240)$

$$= 0.04779\dots$$

M1

$$\Pr(\text{purebred} \cap \text{elite}) = 0.35 \times 0.04779\dots$$

$$= 0.017$$

A1

c. i. $X \sim \text{Bi}(20, 0.35)$

A1

$$\Pr(X = 6) = 0.171$$

A1



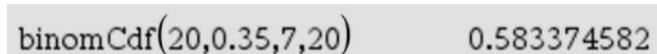
TI-Nspire calculator screenshot showing the command `binomPdf(20,0.35,6)` and the result `0.171229673`.

ii. $\Pr(\hat{P} > 33\%) = \Pr(X \geq 7)$

M1

$$= 0.583$$

A1



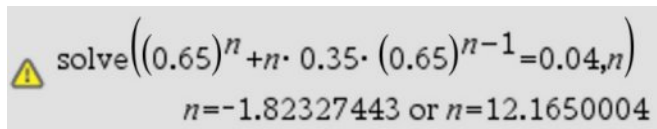
TI-Nspire calculator screenshot showing the command `binomCdf(20,0.35,7,20)` and the result `0.583374582`.

iii. $\Pr\left(\hat{P} \geq \frac{2}{n}\right) = 1 - \Pr(X = 0) - \Pr(X = 1)$

M1

$$0.65^n + n \times 0.35^1 \times 0.65^{n-1} < 0.04$$

Let $0.65^n + n \times 0.35^1 \times 0.65^{n-1} = 0.04$ to solve on CAS.



TI-Nspire calculator screenshot showing the command `solve((0.65)^n + n * 0.35 * (0.65)^(n-1) = 0.04, n)` and the results `n = -1.82327443 or n = 12.1650004`.

As $n > 12.165\dots$, $n = 13$.

A1

d. i. $\hat{p} = \frac{0.1184 + 0.3047}{2}$
 $= 0.21155$ A1

ii. Confidence interval formula: $\left(\hat{p} - z \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + z \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \right)$
 $0.21155 - z \sqrt{\frac{0.21155(1-0.21155)}{52}} = 0.1184$
 $z = 1.6438...$ A1
 $\Pr(-1.64 < Z < 1.64) = 0.9$
 $= 90\% \text{ confidence}$ A1

e. i. Let $y = (t^2 + 2t + 2)e^{-t}$.
 $\frac{dy}{dt} = -t^2 e^{-t}$ A1

$$\frac{d}{dx} \left((x^2 + 2 \cdot x + 2) \cdot e^{-x} \right) \quad -x^2 \cdot e^{-x}$$

$$\int -t^2 e^{-t} dt = (t^2 + 2t + 2)e^{-t}$$

$$\int \frac{mt^2}{e^t} dt = -m(t^2 + 2t + 2)e^{-t}$$

$$\int_0^\infty \frac{mt^2}{e^t} dt = 1 \text{ for a probability density function.}$$

$$\begin{aligned} \left[(t^2 + 2t + 2)e^{-t} \right]_0^\infty &= \lim_{x \rightarrow \infty} ((t^2 + 2t + 2)e^{-t}) - \left(\frac{0^2 + 2(0) + 2}{e^0} \right) \\ &= 0 - 2 \\ &= -2 \end{aligned} \quad \text{M1}$$

$$\begin{aligned} m \times -2 &= -1 \\ &= \frac{1}{2} \end{aligned} \quad \text{A1}$$

ii. mean age = 3 years

A1

$$\Pr(Q > 3) = \frac{17}{2e^3}$$

A1

The screenshot shows a TI-84 Plus calculator interface. At the top, it says '1.1' and '*Doc'. The main display shows the function $q(t) := \frac{t^2 \cdot e^{-t}}{2}$. Below this, it shows the integral $\int_0^{\infty} (t \cdot q(t)) dt$ with the result '3'. At the bottom, it shows the integral $\int_3^{\infty} q(t) dt$ with the result $\frac{17 \cdot e^{-3}}{2}$. The word 'Done' is visible in the top right corner.

Question 4 (7 marks)

a. $f(x) = e^{2x} - 2e^x$

$$f'(x) = 2e^{2x} - 2e^x$$

$$= 2e^x(e^x - 1)$$

Let $f'(x) = 0$

$$\therefore e^x - 1 = 0$$

M1

$$x = 0$$

$$f(0) = -1$$

Turning point: $(0, -1)$

A1

b. $a = 0$, as the turning point is at $x = 0$ and g must be a one-to-one function.

A1

c. domain $g^{-1} = \text{range } g$

$$\text{domain } g^{-1} = [-1, \infty)$$

A1

$$\text{Let } y = e^{2x} - 2e^x.$$

For inverse, swap x and y .

$$x = e^{2y} - 2e^y$$

$$= (e^y - 1)^2 - 1$$

M1

$$e^y = \sqrt{x+1} + 1$$

$$f^{-1}(x) = \log_e(\sqrt{x+1} + 1)$$

A1

d. (0.86, 0.86)

A1

$$\text{solve}(\ln(\sqrt{x+1} + 1) = x, x) \quad x = 0.86032924$$

$$\text{solve}(e^{2 \cdot x} - 2 \cdot e^x = x, x) \\ x = -0.731054944 \text{ or } x = 0.86032924$$

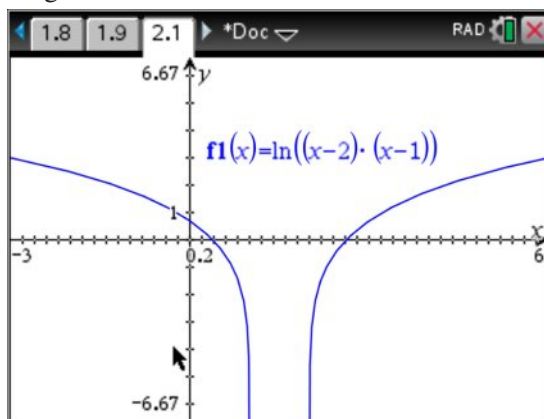
$$\text{solve}(\ln(\sqrt{x+1} + 1) = e^{2 \cdot x} - 2 \cdot e^x, x) \\ x = 0.86032924$$

Question 5 (20 marks)a. i. domain: $R \setminus [1, 2]$

A1

range: R

A1



ii. $\frac{d[g(f(x))]}{dx} = \frac{2x-3}{(x-2)(x-1)}$

A1

$$\text{Let } \frac{d[g(f(x))]}{dx} = 0.$$

$$\frac{2x-3}{(x-2)(x-1)} = 0$$

$$x = \frac{3}{2}$$

However, domain is equal to $R \setminus [1, 2]$ and therefore there are no valid solutions and no stationary points.

A1

- b. i.** $h(x) = f(g(x))$
 $= (\log_e(x) - 1)(\log_e(x) - 2)$ M1
 $= (\log_e(x))^2 - 3\log_e(x) + 2$
 $= \left(\log_e(x) - \frac{3}{2}\right)^2 - \frac{9}{4} + 2$
 $= \left(\log_e(x) - \frac{3}{2}\right)^2 - \frac{1}{4}$ A1
- ii.** $\left(\log_e(x) - \frac{3}{2}\right)^2 \geq 0$
range: $\left[-\frac{1}{4}, \infty\right)$ A1
- c. i.** $h'(x) = \frac{2\log_e(x) - 3}{x}$ A1
 $\frac{d}{dx}(h(x)) \qquad \frac{2 \cdot \ln(x) - 3}{x}$
- ii.** $2\log_e(x) - 3 = 0$
 $x_M = e^{\frac{3}{2}}$ A1
- d.** x_P is the x -coordinate of the intersection of the tangent lines.
 $y_1 = 1 - \frac{x}{e}$ and $y_2 = \frac{x}{e^2} - 1$. M1
 $1 - \frac{x}{e} = \frac{x}{e^2} - 1$
 $2 = \frac{x}{e^2} + \frac{x}{e}$
 $2 = x\left(\frac{1}{e^2} + \frac{1}{e}\right)$
 $2 = x\left(\frac{1+e}{e^2}\right)$
 $x_P = \frac{2e^2}{e+1}$ A1

e. If $x_P < x_M$, then $\frac{2e^2}{e+1} < e^{\frac{3}{2}}$.

$$\frac{2e^{\frac{1}{2}}}{e+1} \leq 1$$

$$2e^{\frac{1}{2}} < e+1$$

M1

$$4e < (e+1)^2$$

$$4e < e^2 + 2e + 1$$

$$0 < e^2 - 2e + 1$$

$$0 < (e-1)^2$$

As $e > 1$ and $(e-1)^2 > 0$, then $x_P < x_M$ must be true.

A1

f. i. $h'(e) = -\frac{1}{e}$ and $h'(e^2) = \frac{1}{e^2}$.

M1

$$\angle APB = \pi + \tan^{-1}\left(-\frac{1}{e}\right) - \tan^{-1}\left(\frac{1}{e^2}\right)$$

$$= \tan^{-1}(e^2) + \tan^{-1}(e)$$

A1

$$\pi + \tan^{-1}\left(-e^{-1}\right) - \tan^{-1}\left(e^{-2}\right) \quad \tan^{-1}\left(e^2\right) + \tan^{-1}(e)$$

ii. $\tan^{-1}(1) = \frac{\pi}{4}$ and $e > 1$.

$$\therefore \tan^{-1}(e) > \frac{\pi}{4} \text{ and } \tan^{-1}(e^2) > \frac{\pi}{4}.$$

$$\angle APB = \tan^{-1}(e^2) + \tan^{-1}(e)$$

$$\angle APB > \frac{\pi}{4} + \frac{\pi}{4}$$

$$> \frac{\pi}{2}$$

A1

$\therefore \angle APB$ is obtuse

g. area of triangle $ABP = \frac{1}{2} \times (e^2 - e) \times (-y_P)$

$$x_P = \frac{2e^2}{e+1}$$

$$y_1 = 1 - \frac{x}{e}$$

$$y_P = \frac{2}{e+1} - 1$$

$$A_1 = \frac{1}{2} \times (e^2 - e) \times \left(1 - \frac{2}{e+1}\right) \quad \text{M1}$$

Area bound by $y = h(x)$ and x -axis:

$$A_2 = -\int_e^{e^2} h(x) dx \quad \text{M1}$$

$$= 3e - e^2$$

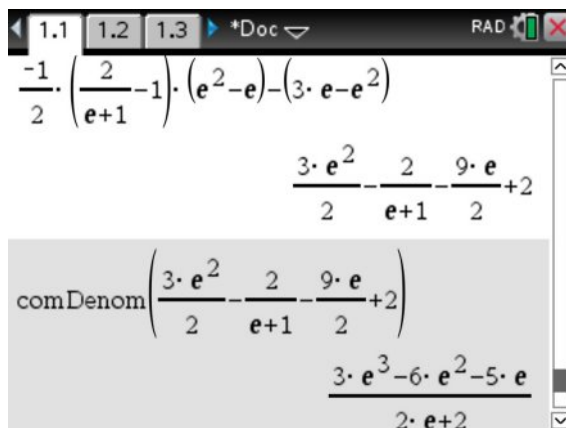


Handwritten calculation for A_2 : $-\int_e^{e^2} h(x) dx = 3 \cdot e - e^2$

$$\text{total area} = A_1 - A_2$$

$$= \frac{1}{2} \times (e^2 - e) \times \left(1 - \frac{2}{e+1}\right) - (3e - e^2) \quad \text{M1}$$

$$= \frac{3e^3 - 6e^2 - 5e}{2e + 2} \quad \text{A1}$$



CASIO calculator screen showing the final result: $\frac{3 \cdot e^3 - 6 \cdot e^2 - 5 \cdot e}{2 \cdot e + 2}$