

2021 VCE Mathematical Methods Trial Examination 1 Detailed Answers



Kilbaha Education

Quality educational content

Kilbaha Education
PO Box 2227
Kew Vic 3101
Australia

Tel: (03) 9018 5376
kilbaha@gmail.com
<https://kilbaha.com.au>

All publications from Kilbaha Education are digital and are supplied to the purchasing school in both WORD and PDF formats with a school site licence to reproduce for students in both print and electronic formats.



Kilbaha Education (Est. 1978) (ABN 47 065 111 373)
PO Box 2227
Kew Vic 3101
Australia

Tel: +613 9018 5376
Email: kilbaha@gmail.com
Web: <https://kilbaha.com.au>

IMPORTANT COPYRIGHT NOTICE FOR KILBAHA PUBLICATIONS

- (1) The material is copyright. Subject to statutory exception and to the provisions of the relevant collective licensing agreements, no reproduction of any part may take place without the written permission of Kilbaha Pty Ltd.
- (2) The contents of these works are copyrighted. Unauthorised copying of any part of these works is illegal and detrimental to the interests of the author(s).
- (3) For authorised copying within Australia please check that your institution has a licence from <https://www.copyright.com.au> This permits the copying of small parts of the material, in limited quantities, within the conditions set out in the licence.
- (4) All pages of Kilbaha files must be counted in Copyright Agency Limited (CAL) surveys.
- (5) Kilbaha files must **not** be uploaded to the Internet.
- (6) Kilbaha files may be placed on a password protected school Intranet.

Kilbaha educational content has no official status and is not endorsed by any State or Federal Government Education Authority.

While every care has been taken, no guarantee is given that the content is free from error. Please contact us if you believe you have found an error.

CAUTION NEEDED!

All Web Links when created linked to appropriate Web Sites. Teachers and parents must always check links before using them with students to ensure that students are protected from unsuitable Web Content. Kilbaha Education is not responsible for links that have been changed in its publications or links that have been redirected.

Question 1

a. $y = \log_e(\tan(3x))$ Chain rule

$$y = \log_e(u) \quad , \quad u = \tan(3x)$$

$$\frac{dy}{du} = \frac{1}{u} \quad , \quad \frac{du}{dx} = \frac{3}{\cos^2(3x)}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u} \cdot \frac{3}{\cos^2(3x)} = \frac{1}{\tan(3x)} \cdot \frac{3}{\cos^2(3x)} \quad \text{M1}$$

$$\frac{dy}{dx} = \frac{\cos(3x)}{\sin(3x)} \cdot \frac{3}{\cos^2(3x)}$$

$$\frac{dy}{dx} = \frac{3}{\sin(3x)\cos(3x)} \quad \text{A1}$$

b. $\int \frac{1}{\sin(3x)\cos(3x)} dx = \frac{1}{3} \log_e(\tan(3x)) + c \quad \text{A1}$

Question 2

$$X = N(80, 36) \quad , \quad \mu = 80, \sigma = 6, Z = N(0, 1)$$

$$\Pr(Z < -2.5) = \Pr(Z > 2.5) = p \quad \text{and} \quad \Pr(-2.5 < Z < -1.5) = \Pr(1.5 < Z < 2.5) = q,$$

$$\text{Consider } \Pr(0 < Z < 1.5) + \Pr(1.5 < Z < 2.5) + \Pr(Z > 2.5) = 0.5$$

$$\Pr(0 < Z < 1.5) + q + p = 0.5 \quad , \quad \Pr(0 < Z < 1.5) = 0.5 - (p + q) \quad \text{M1}$$

$$\Pr(X > 71 | X < 95) = \Pr\left(Z > \frac{71-80}{6} \mid Z < \frac{95-80}{6}\right)$$

$$\Pr(X > 71 | X < 95) = \Pr(Z > -1.5 | Z < 2.5) = \frac{\Pr(-1.5 < Z < 2.5)}{\Pr(Z < 2.5)}$$

$$\Pr(X > 71 | X < 95) = \frac{\Pr(-1.5 < Z < 0) + \Pr(0 < Z < 1.5) + \Pr(1.5 < Z < 2.5)}{1 - \Pr(Z > 2.5)} \quad \text{M1}$$

$$\Pr(X > 71 | X < 95) = \frac{2\Pr(0 < Z < 1.5) + \Pr(1.5 < Z < 2.5)}{1 - \Pr(Z > 2.5)} = \frac{2(0.5 - (p + q)) + q}{1 - p}$$

$$\Pr(X > 71 | X < 95) = \frac{1 - 2p - q}{1 - p} \quad \text{A1}$$

Question 3

$$f(x) = \log_3(x+a) + b$$

$$f(0) = 8 \Rightarrow (1) \quad 8 = \log_3(a) + b$$

$$f(2) = 9 \Rightarrow (2) \quad 9 = \log_3(a+2) + b$$

$$(2) - (1) \quad 1 = \log_3(a+2) - \log_3(a) = \log_3\left(\frac{a+2}{a}\right) = \log_3(3) \quad \text{M1}$$

$$\frac{a+2}{a} = 3, \quad a+2 = 3a, \quad a = 1, \quad (1) \quad b = 8 - \log_3(1) = 8 \quad \text{A1}$$

$$g(x) = p \log_5(q-x)$$

$$g(6) = 0 \Rightarrow 0 = p \log_5(q-6), \quad q-6 = 1, \quad q = 7 \quad \text{M1}$$

$$g(2) = 9 \Rightarrow 9 = p \log_5(5), \quad p = 9 \quad \text{A1}$$

Question 4

$$f(x) = \log_e(kx) - \frac{kx}{2x+3} \quad \text{differentiating using the quotient rule in the second term}$$

$$f'(x) = \frac{1}{x} - \left[\frac{k(2x+3) - 2kx}{(2x+3)^2} \right] \quad \text{M1}$$

$$f'(x) = \frac{1}{x} - \left[\frac{2kx + 3k - 2kx}{(2x+3)^2} \right]$$

$$f'(x) = \frac{1}{x} - \frac{3k}{(2x+3)^2} = \frac{(2x+3)^2 - 3kx}{x(2x+3)^2} = \frac{4x^2 + 12x + 9 - 3kx}{(2x+3)^2}$$

$$f'(x) = \frac{4x^2 + (12-3k)x + 9}{(2x+3)^2} = 0$$

$$4x^2 + (12-3k)x + 9 = 0 \quad \text{for stationary points} \quad \text{A1}$$

$$\Delta = (12-3k)^2 - 4 \times 4 \times 9$$

$$\Delta = (3(4-k))^2 - 16 \times 9 \quad \text{M1}$$

$$\Delta = 9(16-8k+k^2-16) = 9k(k-8)$$

$$\text{for no stationary points } \Delta < 0 \Rightarrow 0 < k < 8 \quad \text{A1}$$

Question 5

a.i. Since there can be 0, 1, 2 or 3 females, out of 3, the proportion

$$\hat{P} = \left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\} \quad \text{A1}$$

ii. $\Pr\left(\hat{P} \geq \frac{1}{2}\right) = \Pr\left(\hat{P} = \frac{2}{3}\right) + \Pr(\hat{P} = 1) = \Pr(1M, 2F) + \Pr(3F)$ M1

$$\Pr\left(\hat{P} \geq \frac{1}{2}\right) = 3 \times \frac{6}{9} \times \frac{3}{8} \times \frac{2}{7} + \frac{3}{9} \times \frac{2}{8} \times \frac{1}{7} = \frac{3}{14} + \frac{1}{84} = \frac{18+1}{84}$$

$$\Pr\left(\hat{P} \geq \frac{1}{2}\right) = \frac{19}{84} \quad \text{A1}$$

b. $\left(p - z\sqrt{\frac{p(1-p)}{n}}, p + z\sqrt{\frac{p(1-p)}{n}}\right) = \left(\frac{316}{625}, \frac{484}{625}\right)$

adding $2p = \frac{316}{625} + \frac{484}{625} = \frac{800}{625}$, $p = \frac{400}{625} = \frac{16}{25}$

subtracting $2z\sqrt{\frac{p(1-p)}{n}} = 2 \times \frac{49}{25} \times \sqrt{\frac{\frac{16}{25} \times \left(1 - \frac{16}{25}\right)}{n}} = \frac{484}{625} - \frac{316}{625}$ M1

$$= 2 \times \frac{49}{25} \times \sqrt{\frac{\frac{16}{25} \times \frac{9}{25}}{n}} = \frac{168}{625}$$

$$\frac{49}{25} \times \frac{4 \times 3}{25\sqrt{n}} = \frac{84}{625}$$

$$\sqrt{n} = \frac{49 \times 12}{84} = 7$$

$n = 49$ A1

Question 6

$$f(x) = \sqrt{9-3x}$$

a. $9-3x \geq 0, x \leq 3$

$$D = (-\infty, 3] = \text{dom } f = \text{ran } f^{-1}$$

A1

b. $f^{-1}: x = \sqrt{9-3y}, x^2 = 9-3y$

M1

$$3y = 9 - x^2, y = f^{-1}(x) = \frac{1}{3}(9 - x^2)$$

A1

$$f^{-1}: [0, \infty) \rightarrow \mathbb{R}, f^{-1}(x) = \frac{1}{3}(9 - x^2)$$

c. solving $f(x) = f^{-1}(x)$

$$\sqrt{9-3x} = \frac{1}{3}(x^2 - 9)$$

$$9-3x = \frac{1}{9}(x^4 - 18x^2 + 81)$$

$$81 - 27x = x^4 - 18x^2 + 81$$

$$x^4 - 18x^2 + 27x = 0$$

$$x(x^3 - 18x + 27) = 0$$

$$x(x-3)(x^2 + 3x - 9) = 0$$

$$x = 0, 3, \frac{-3 \pm \sqrt{9+36}}{2} = \frac{-3 \pm \sqrt{45}}{2} = \frac{3}{2}(-1 \pm \sqrt{5})$$

M1

but $x \in [0, 3] \quad x = 0, 3, \frac{3}{2}(\sqrt{5}-1)$

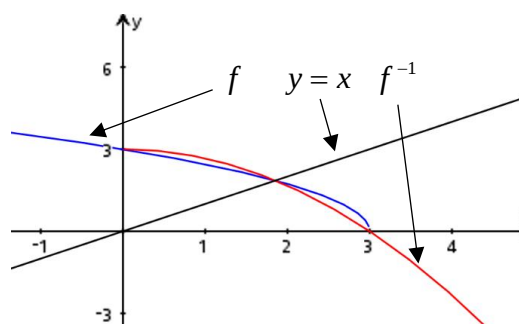
A1

$$(0, 3), (3, 0), \left(\frac{3}{2}(\sqrt{5}-1), \frac{3}{2}(\sqrt{5}-1) \right)$$

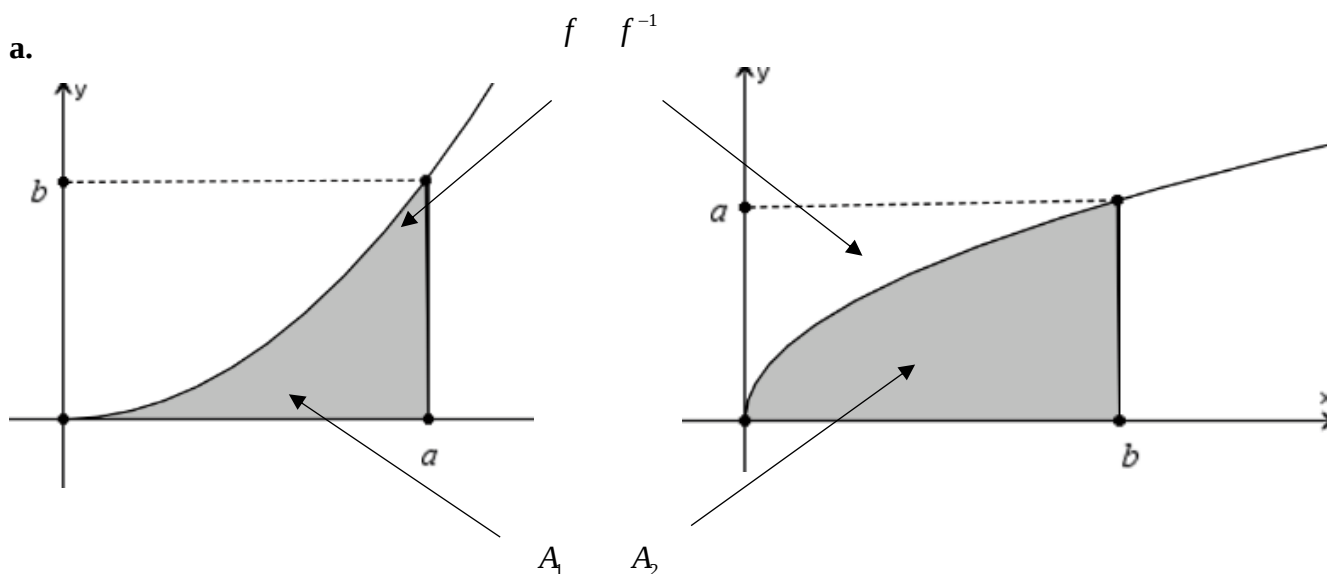
There are three points of intersection between $f(x) = f^{-1}(x)$ only one of which lies on the line $y = x$.

Note that solving $f(x) = x$ or $f^{-1}(x) = x$ gives $x^2 + 3x - 9 = 0$ which gives only

the one point of intersection on the line $y = x$, that is the point $\left(\frac{3}{2}(\sqrt{5}-1), \frac{3}{2}(\sqrt{5}-1) \right)$



Question 7



$A_1 = \int_0^a f(x) dx$, $A_2 = \int_0^b f^{-1}(x) dx$ but $A_1 + A_2 = ab$ the area of the rectangle, hence

$$\int_0^a f(x) dx = ab - \int_0^b f^{-1}(x) dx \quad \text{A1}$$

b. Let $A_1 = \int_0^3 \log_e(1+3x) dx$, $f(x) = \log_e(1+3x)$, $a = 3$, $b = f(3) = \log_e(10)$

$$f: y = \log_e(1+3x)$$

$$f^{-1}: x = \log_e(1+3y) , 1+3y = e^x , y = \frac{1}{3}(e^x - 1) \quad \text{M1}$$

$$f^{-1}(x) = \frac{1}{3}(e^x - 1)$$

$$A_2 = \int_0^{\log_e(10)} \frac{1}{3}(e^x - 1) dx = \frac{1}{3} [e^x - x]_0^{\log_e(10)} = \frac{1}{3} [(e^{\log_e(10)} - \log_e(10)) - (e^0)] \quad \text{A1}$$

$$A_2 = \frac{1}{3} [10 - \log_e(10) - 1] = \frac{1}{3} (9 - \log_e(10)) = 3 - \frac{1}{3} \log_e(10)$$

$$A_1 = ab - A_2 \quad \text{using a.}$$

$$\int_0^3 \log_e(1+3x) dx = 3 \log_e(10) - \left[3 - \frac{1}{3} \log_e(10) \right] = \left(3 + \frac{1}{3} \right) \log_e(10) - 3$$

$$\int_0^3 \log_e(1+3x) dx = \frac{10}{3} \log_e(10) - 3 , \quad p = 10 , \quad q = 3 \quad \text{A1}$$

Question 8

$$\int_0^5 \frac{k}{\sqrt{16-3x}} dx = 1 \quad \text{since it is a valid pdf}$$

$$\int_0^5 \frac{k}{\sqrt{16-3x}} dx = k \int_0^5 (16-3x)^{-\frac{1}{2}} dx = 1 \quad \text{A1}$$

$$k \left[\frac{1}{-3 \times \frac{1}{2}} (16-3x)^{\frac{1}{2}} \right]_0^5 = - \left[\frac{2k}{3} \sqrt{16-3x} \right]_0^5 = 1 \quad \text{A1}$$

$$\left(-\frac{2k}{3} \sqrt{16-15} \right) - \left(-\frac{2k}{3} \sqrt{16} \right) = -\frac{2k}{3} (1-4) = 1$$

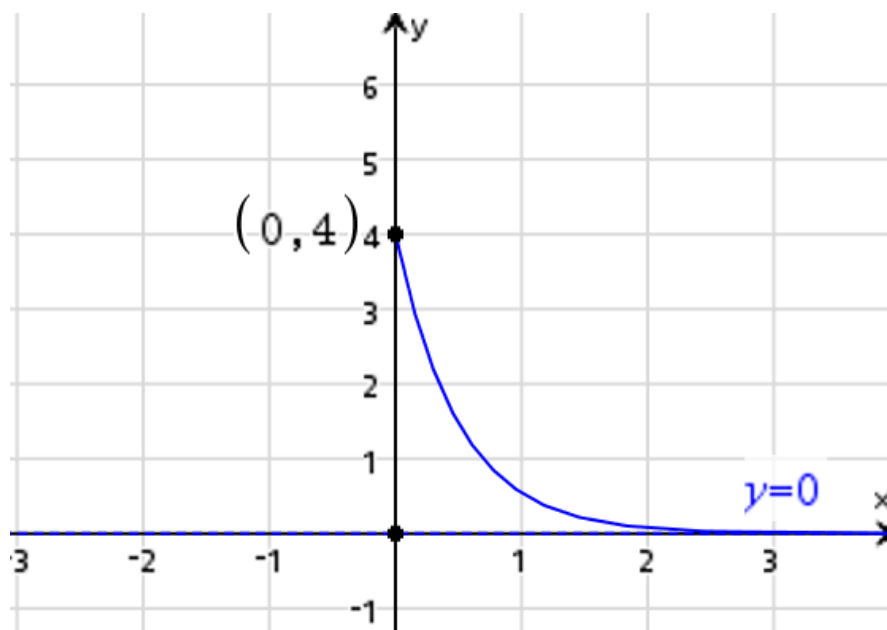
$$2k = 1$$

$$k = \frac{1}{2} \quad \text{A1}$$

Question 9

- a) The graph of $f : [0, \infty) \rightarrow \mathbb{R}$, $f(x) = 4e^{-2x}$ crosses the y-axis at $(0, 4)$ and has the x-axis $y = 0$ as a horizontal asymptote.

A1



b. $\bar{y} = \frac{1}{3-1} \int_1^3 4e^{-2x} dx = 2 \int_1^3 e^{-2x} dx$
 $\bar{y} = -\left[e^{-2x} \right]_1^3 = -e^{-6} + e^{-2} = e^{-2} - e^{-6}$ A1

c. $P(u, 4e^{-2u}), \frac{dy}{dx} = f'(x) = -8e^{-2x}, f'(u) = -8e^{-2u}$ M1

$T: y - 4e^{-2u} = -8e^{-2u}(x - u)$
 $y = -8e^{-2u}x + 4e^{-2u}(2u + 1)$ A1

d. at R, $x = 0, y_R = 4(2u + 1)e^{-2u} \quad R(0, 4e^{-2u}(2u + 1))$
 at Q, $y = 0$ solving $-8e^{-2u}x + 4e^{-2u}(2u + 1) = 0$
 $x_Q = \frac{4e^{-2u}(2u + 1)}{8e^{-2u}} = \frac{1}{2}(2u + 1)$
 $Q\left(\frac{1}{2}(2u + 1), 0\right), R(0, 4e^{-2u}(2u + 1))$ A1

e. area OQR, $A(u) = \frac{1}{2} \times \frac{1}{2}(2u + 1) \times 4(2u + 1)e^{-2u} = (2u + 1)^2 e^{-2u}$ A1

f. differentiating using the product rule

$\frac{dA}{du} = e^{-2u} \frac{d}{du}((2u + 1)^2) + (2u + 1)^2 \frac{d}{du}(e^{-2u})$
 $\frac{dA}{du} = 4e^{-2u}(2u + 1) - 2(2u + 1)^2 e^{-2u}$ M1

$\frac{dA}{du} = 2e^{-2u}(2u + 1)(2 - (2u + 1))$

$\frac{dA}{du} = 2e^{-2u}(2u + 1)(1 - 2u) = 0$ for a maximum or minimum area

$u = \frac{1}{2}$ since $u \in [0, \infty)$

$A\left(\frac{1}{2}\right) = 4e^{-1} = \frac{4}{e}$ is the maximum area A1

now as $u \rightarrow \infty \quad A(u) \rightarrow 0$ the minimum area is 0 A1

**End of detailed answers for the
2021 Kilbaha VCE Mathematical Methods Trial Examination 1**

Kilbaha Education
PO Box 2227
Kew Vic 3101
Australia

Tel: (03) 9018 5376
kilbaha@gmail.com
<https://kilbaha.com.au>