2021 VCE Mathematical Methods Trial Examination 1

Detailed Answers

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Question 1

a. $y = \log_e\left(\tan(3x)\right)$ Chain rule

$y = \log_e(u)$, $u = \tan(3x)$

$\frac{dy}{du} = \frac{1}{u}$, $\frac{du}{dx} = \frac{3}{\cos^2(3x)}$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u} \cdot \frac{3}{\cos^2(3x)} = \frac{1}{\tan(3x)} \cdot \frac{3}{\cos^2(3x)}$ M1

$\frac{dy}{dx} = \frac{\cos(3x)}{\sin(3x)} \cdot \frac{3}{\cos^2(3x)}$

$\frac{dy}{dx} = \frac{3}{\sin(3x)\cos(3x)}$ A1

b. $\int \frac{1}{\sin(3x)\cos(3x)}\,dx = \frac{1}{3}\log_e\left(\tan(3x)\right) + c$ A1

Question 2

$X = N(80,36)$, $\mu = 80$, $\sigma = 6$, $Z = N(0,1)$

$\Pr(Z < -2.5) = \Pr(Z > 2.5) = p$ and $\Pr(-2.5 < Z < -1.5) = \Pr(1.5 < Z < 2.5) = q$,

Consider $\Pr(0 < Z < 1.5) + \Pr(1.5 < Z < 2.5) + \Pr(Z > 2.5) = 0.5$

$\Pr(0 < Z < 1.5) + q + p = 0.5$, $\Pr(0 < Z < 1.5) = 0.5 - (p + q)$ M1

$\Pr(X > 71 \mid X < 95) = \Pr\left(Z > \frac{71-80}{6} \mid Z < \frac{95-80}{6}\right)$

$\Pr(X > 71 \mid X < 95) = \Pr(Z > -1.5 \mid Z < 2.5) = \frac{\Pr(-1.5 < Z < 2.5)}{\Pr(Z < 2.5)}$

$\Pr(X > 71 \mid X < 95) = \frac{\Pr(-1.5 < Z < 0) + \Pr(0 < Z < 1.5) + \Pr(1.5 < Z < 2.5)}{1 - \Pr(Z > 2.5)}$ M1

$\Pr(X > 71 \mid X < 95) = \frac{2\Pr(0 < Z < 1.5) + \Pr(1.5 < Z < 2.5)}{1 - \Pr(Z > 2.5)} = \frac{2\left(0.5 - (p+q)\right) + q}{1 - p}$

$\Pr(X > 71 \mid X < 95) = \frac{1 - 2p - q}{1 - p}$ A1

Question 3

$f(x)=\log_3(x+a)+b$

$f(0)=8 \quad \Rightarrow (1) \quad 8=\log_3(a)+b$

$f(2)=9 \quad \Rightarrow (2) \quad 9=\log_3(a+2)+b$

$(2)-(1) \quad 1=\log_3(a+2)-\log_3(a)=\log_3\left(\frac{a+2}{a}\right)=\log_3(3)$ M1

$\frac{a+2}{a}=3$, $a+2=3a$, $a=1$, (1) $b=8-\log_3(1)=8$ A1

$g(x)=p\log_5(q-x)$

$g(6)=0 \Rightarrow 0=p\log_5(q-6)$, $q-6=1$, $q=7$ M1

$g(2)=9 \Rightarrow 9=p\log_5(5)$, $p=9$ A1

Question 4

$f(x)=\log_e(kx)-\frac{kx}{2x+3}$ differentiating using the quotient rule in the second term

$f'(x)=\frac{1}{x}-\left[\frac{k(2x+3)-2kx}{(2x+3)^2}\right]$

$f'(x)=\frac{1}{x}-\left[\frac{2kx+3k-2kx}{(2x+3)^2}\right]$ M1

$f'(x)=\frac{1}{x}-\frac{3k}{(2x+3)^2}=\frac{(2x+3)^2-3kx}{x(2x+3)^2}=\frac{4x^2+12x+9-3kx}{(2x+3)^2}$

$f'(x)=\frac{4x^2+(12-3k)x+9}{(2x+3)^2}=0$

$4x^2+(12-3k)x+9=0$ for stationary points A1

$\Delta=(12-3k)^2-4\times4\times9$

$\Delta=\left(3(4-k)\right)^2-16\times9$ M1

$\Delta=9\left(16-8k+k^2-16\right)=9k(k-8)$

for no stationary points $\Delta<0 \Rightarrow 0<k<8$ A1

Question 5

a.i. Since there can be 0 , 1 , 2 or 3 females, out of 3, the proportion

$\hat{P} = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$ A1

ii. $\Pr\left(\hat{P} \ge \frac{1}{2}\right) = \Pr\left(\hat{P} = \frac{2}{3}\right) + \Pr\left(\hat{P} = 1\right) = \Pr(1M, 2F) + \Pr(3F)$ M1

$\Pr\left(\hat{P} \ge \frac{1}{2}\right) = 3 \times \frac{6}{9} \times \frac{3}{8} \times \frac{2}{7} + \frac{3}{9} \times \frac{2}{8} \times \frac{1}{7} = \frac{3}{14} + \frac{1}{84} = \frac{18+1}{84}$

$\Pr\left(\hat{P} \ge \frac{1}{2}\right) = \frac{19}{84}$ A1

b. $\left(p - z\sqrt{\frac{p(1-p)}{n}}, p + z\sqrt{\frac{p(1-p)}{n}}\right) = \left(\frac{316}{625}, \frac{484}{625}\right)$

adding $2p = \frac{316}{625} + \frac{484}{625} = \frac{800}{625}$, $p = \frac{400}{625} = \frac{16}{25}$

subtracting $2z\sqrt{\frac{p(1-p)}{n}} = 2 \times \frac{49}{25} \times \sqrt{\frac{\frac{16}{25} \times \left(1 - \frac{16}{25}\right)}{n}} = \frac{484}{625} - \frac{316}{625}$ M1

$= 2 \times \frac{49}{25} \times \sqrt{\frac{\frac{16}{25} \times \frac{9}{25}}{n}} = \frac{168}{625}$

$\frac{49}{25} \times \frac{4 \times 3}{25\sqrt{n}} = \frac{84}{625}$

$\sqrt{n} = \frac{49 \times 12}{84} = 7$

$n = 49$ A1

Question 6

$f(x)=\sqrt{9-3x}$

a. $9-3x\ge 0$, $x\le 3$

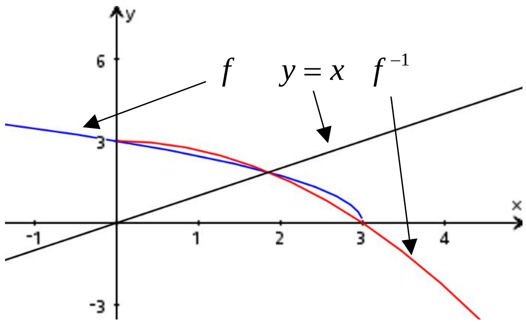
$D=(-\infty,3]=\text{dom } f = \text{ran } f^{-1}$ A1

b. $f^{-1}:\ x=\sqrt{9-3y}$, $x^2=9-3y$ M1

$3y=9-x^2$, $y=f^{-1}(x)=\frac{1}{3}(9-x^2)$

$f^{-1}:[0,\infty)\to R$, $f^{-1}(x)=\frac{1}{3}(9-x^2)$ A1

c. solving $f(x)=f^{-1}(x)$



$\sqrt{9-3x}=\frac{1}{3}(x^2-9)$

$9-3x=\frac{1}{9}(x^4-18x^2+81)$

$81-27x=x^4-18x^2+81$

$x^4-18x^2+27x=0$

$x(x^3-18x+27)=0$

$x(x-3)(x^2+3x-9)=0$

$x=0\ ,\ 3\ ,\ \frac{-3\pm\sqrt{9+36}}{2}=\frac{-3\pm\sqrt{45}}{2}=\frac{3}{2}(-1\pm\sqrt{5})$ M1

but $x\in[0,3]$ $x=0,3,\frac{3}{2}(\sqrt{5}-1)$

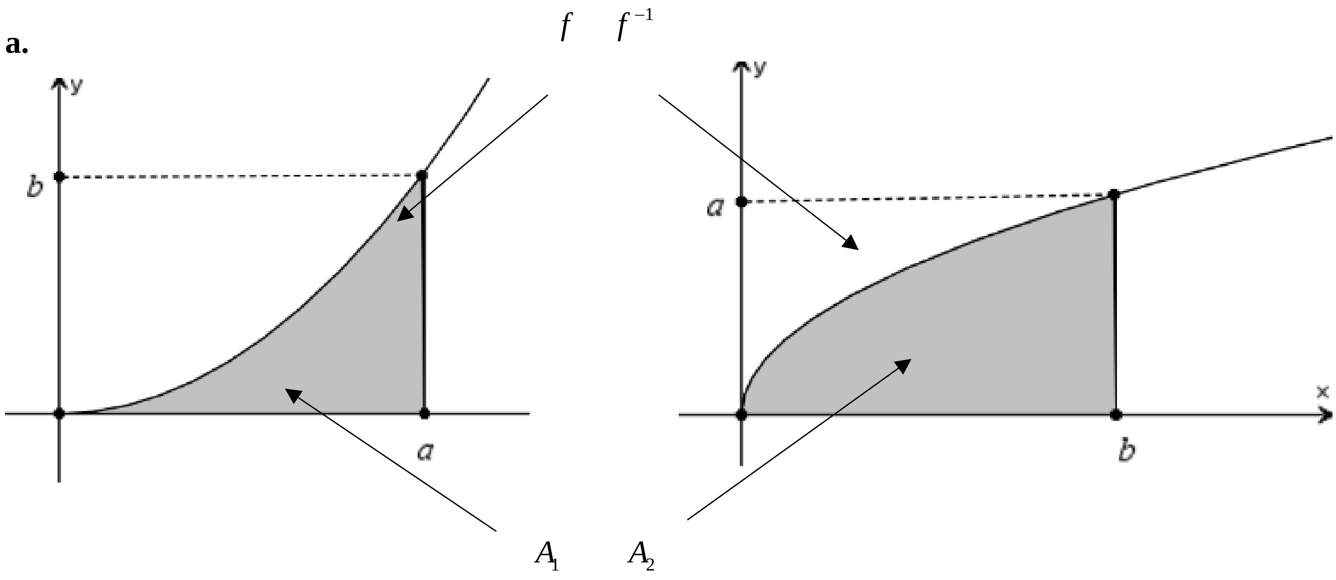
$(0,3)\ ,\ (3,0)\ ,\ \left(\frac{3}{2}(\sqrt{5-1}),\ \frac{3}{2}(\sqrt{5-1})\ \right)$ A1

There are three points of intersection between $f(x)=f^{-1}(x)$ only one of which lies on the line $y=x$.

Note that solving $f(x)=x$ or $f^{-1}(x)=x$ gives $x^2+3x-9=0$ which gives only the one point of intersection on the line $y=x$, that is the point $\left(\frac{3}{2}(\sqrt{5-1}),\ \frac{3}{2}(\sqrt{5-1})\ \right)$

Question 7

a.



$A_1 = \int_0^a f(x)\,dx \quad , \quad A_2 = \int_0^b f^{-1}(x)\,dx$ but $A_1 + A_2 = ab$ the area of the rectangle, hence

$$\int_0^a f(x)\,dx = ab - \int_0^b f^{-1}(x)\,dx \qquad \text{A1}$$

b. Let $A_1 = \int_0^3 \log_e(1+3x)\,dx \quad , \quad f(x) = \log_e(1+3x) \quad , \quad a = 3 \; , \quad b = f(3) = \log_e(10)$

$f: \; y = \log_e(1+3x)$

$f^{-1}: \; x = \log_e(1+3y) \; , \; 1+3y = e^x \quad , \quad y = \frac{1}{3}\left(e^x - 1\right)$ M1

$f^{-1}(x) = \frac{1}{3}\left(e^x - 1\right)$

$$A_2 = \int_0^{\log_e(10)} \frac{1}{3}\left(e^x - 1\right)dx = \frac{1}{3}\left[e^x - x\right]_0^{\log_e(10)} = \frac{1}{3}\left[\left(e^{\log_e(10)} - \log_e(10)\right) - \left(e^0\right)\right]$$

$$A_2 = \frac{1}{3}\left[10 - \log_3(10) - 1\right] = \frac{1}{3}\left(9 - \log_e(10)\right) = 3 - \frac{1}{3}\log_e(10) \qquad \text{A1}$$

$A_1 = ab - A_2$ using **a.**

$$\int_0^3 \log_e(1+3x)\,dx = 3\log_e(10) - \left[3 - \frac{1}{3}\log_e(10)\right] = \left(3 + \frac{1}{3}\right)\log_e(10) - 3$$

$$\int_0^3 \log_e(1+3x)\,dx = \frac{10}{3}\log_e(10) - 3 \quad , \quad p = 10 \; , \quad q = 3 \qquad \text{A1}$$

Question 8

$$\int_0^5 \frac{k}{\sqrt{16-3x}}\,dx = 1 \quad \text{since it is a valid pdf}$$

$$\int_0^5 \frac{k}{\sqrt{16-3x}}\,dx = k\int_0^5 (16-3x)^{-\frac{1}{2}}\,dx = 1$$ A1

$$k\left[\frac{1}{-3\times\frac{1}{2}}(16-3x)^{\frac{1}{2}}\right]_0^5 = -\left[\frac{2k}{3}\sqrt{16-3x}\right]_0^5 = 1$$ A1

$$\left(-\frac{2k}{3}\sqrt{16-15}\right)-\left(-\frac{2k}{3}\sqrt{16}\right) = -\frac{2k}{3}(1-4) = 1$$

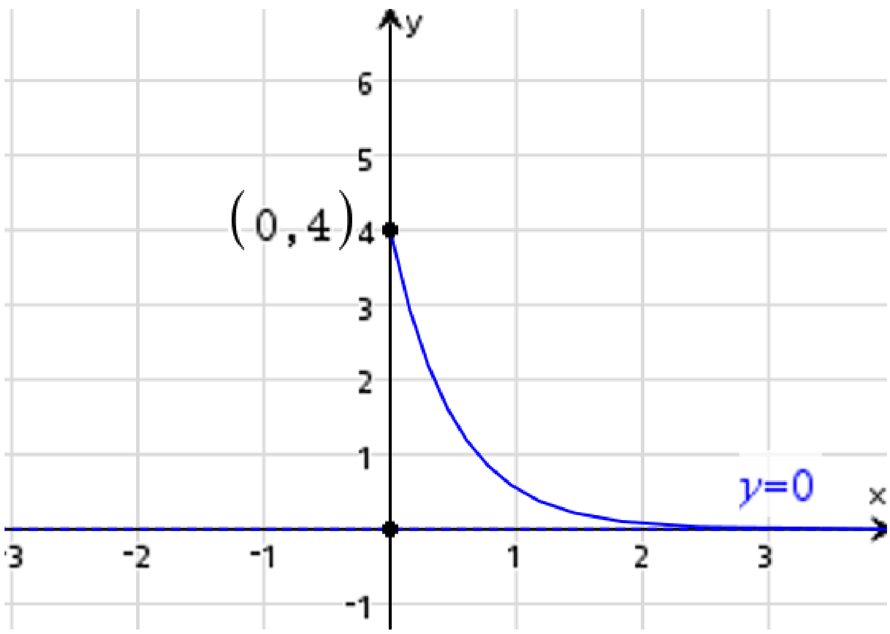
$$2k = 1$$

$$k = \frac{1}{2}$$ A1

Question 9

a) The graph of $f:[0,\infty)\to R$, $f(x)=4e^{-2x}$ crosses the y-axis at $(0,4)$ and has the x-axis $y=0$ as a horizontal asymptote.

A1



b. $$\bar{y} = \frac{1}{3-1}\int_1^3 4e^{-2x}\,dx = 2\int_1^3 e^{-2x}\,dx$$

$$\bar{y} = -\left[e^{-2x}\right]_1^3 = -e^{-6} + e^{-2} = e^{-2} - e^{-6}$$ A1

c. $P\left(u, 4e^{-2u}\right)$, $\dfrac{dy}{dx} = f'(x) = -8e^{-2x}$, $f'(u) = -8e^{-2u}$ M1

$T: \quad y - 4e^{-2u} = -8e^{-2u}(x-u)$

$y = -8e^{-2u}x + 4e^{-2u}(2u+1)$ A1

d. at R, $x = 0$, $y_R = 4(2u+1)e^{-2u}$ $\quad R\left(0, 4e^{-2u}(2u+1)\right)$

at Q, $y = 0$ solving $\quad -8e^{-2u}x + 4e^{-2u}(2u+1) = 0$

$$x_Q = \frac{4e^{-2u}(2u+1)}{8e^{-2u}} = \frac{1}{2}(2u+1)$$

$Q\left(\dfrac{1}{2}(2u+1), 0\right)$, $R\left(0, 4e^{-2u}(2u+1)\right)$ A1

e. area OQR, $A(u) = \dfrac{1}{2} \times \dfrac{1}{2}(2u+1) \times 4(2u+1)e^{-2u} = (2u+1)^2 e^{-2u}$ A1

f. differentiating using the product rule

$$\frac{dA}{du} = e^{-2u}\frac{d}{du}\left((2u+1)^2\right) + (2u+1)^2\frac{d}{du}\left(e^{-2u}\right)$$

$$\frac{dA}{du} = 4e^{-2u}(2u+1) - 2(2u+1)^2 e^{-2u}$$ M1

$$\frac{dA}{du} = 2e^{-2u}(2u+1)\left(2 - (2u+1)\right)$$

$\dfrac{dA}{du} = 2e^{-2u}(2u+1)(1-2u) = 0$ for a maximum or minimum area

$u = \dfrac{1}{2}$ since $u \in [0, \infty)$

$A\left(\dfrac{1}{2}\right) = 4e^{-1} = \dfrac{4}{e}$ is the maximum area A1

now as $u \to \infty$ $A(u) \to 0$ the minimum area is 0 A1

End of detailed answers for the
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