THE HEFFERNAN GROUP
P.O. Box 1180
Surrey Hills North VIC 3127
Phone 03 9836 5021
info@theheffernangroup.com.au
www.theheffernangroup.com.au

Student Name……………………………………

MATHEMATICAL METHODS UNITS 3 & 4

TRIAL EXAMINATION 2

2021

Reading Time: 15 minutes
Writing time: 2 hours

Instructions to students

This exam consists of Section A and Section B.
Section A consists of 20 multiple-choice questions, which should be answered on the detachable answer sheet which can be found on page 25 of this exam.
Section B consists of 5 extended-answer questions.
Section A begins on page 2 of this exam and is worth 20 marks.
Section B begins on page 11 of this exam and is worth 60 marks.
There is a total of 80 marks available.
All questions in Section A and Section B should be answered.
In Section B, where more than one mark is allocated to a question, appropriate working must be shown.
Where a numerical answer is required, an exact value must be given unless otherwise directed.
Diagrams in this exam are not to scale except where otherwise stated.
Students may bring one bound reference into the exam.
Students may bring into the exam one approved technology (calculator or software) and, if desired, one scientific calculator. Calculator memory does not need to be cleared. For approved computer-based CAS, full functionality may be used.
A formula sheet can be found on pages 23 and 24 of this exam.

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SECTION A – Multiple-choice questions

Question 1

Let $p(x) = x^4 - 2x^3 - k$, where $k \in R$.
When $p(x)$ is divided by $x - 3$, the remainder is 15.
The value of *k* is

A. 5
B. 12
C. 18
D. 42
E. 120

Question 2

The function *f* has a maximal domain given by $x \in [-1, \infty)$.
A possible rule for *f* is

A. $f(x) = \dfrac{1}{x+1}$

B. $f(x) = \log_e(x+1)$

C. $f(x) = \dfrac{1}{\sqrt{x+1}}$

D. $f(x) = \sqrt{x+1}$

E. $f(x) = \dfrac{1}{x^2+1}$

Question 3

The equation $2\sin\left(3x + \dfrac{\pi}{6}\right) - \sqrt{3} = 0$ has the solutions given by

A. $x = \dfrac{\pi(12k-2)}{18}$ or $x = \dfrac{\pi(12k+1)}{18}$, $k \in Z$

B. $x = \dfrac{\pi(12k-1)}{18}$ or $x = \dfrac{\pi(12k-3)}{18}$, $k \in Z$

C. $x = \dfrac{\pi(12k-1)}{18}$ or $x = \dfrac{\pi(12k+3)}{18}$, $k \in Z$

D. $x = \dfrac{\pi(12k+1)}{18}$ or $x = \dfrac{\pi(12k-1)}{18}$, $k \in Z$

E. $x = \dfrac{\pi(12k+1)}{18}$ or $x = \dfrac{\pi(12k+3)}{18}$, $k \in Z$

Question 4

The point $P(1,-2)$ lies on the graph of *f*.
The point $Q(3,-6)$ lies on the graph of *h*.
A transformation that maps the graph of *f* to the graph of *h* also maps the point *P* to the point *Q*.
The relationship between *f* and *h* could be given by

A. $h(x)=\frac{1}{3}f(x+2)$
B. $h(x)=2f(x-3)$
C. $h(x)=3f(x-2)$
D. $h(x)=3f(x)$
E. $h(x)=3f(x+2)$

Question 5

The function $h:D\to R,\ h(x)=4x^3-12x+5$ will have an inverse function for

A. $D=(-\infty,0)$
B. $D=[-1,0]$
C. $D=[0,2]$
D. $D=[0,\infty)$
E. $D=R$

Question 6

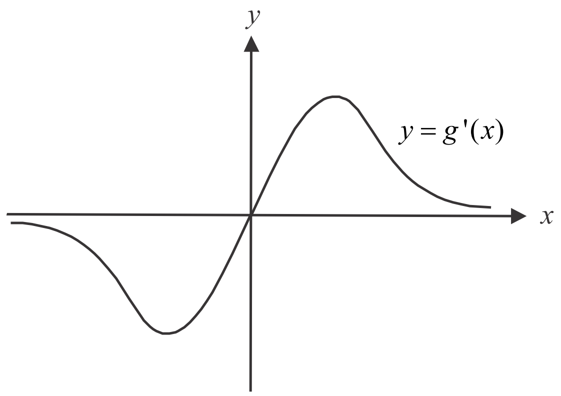
Let $f:R\to R,\ f(x)=x^2$.
The average rate of change of *f* between $x=0$ and $x=4$ is equal to the instantaneous rate of change of *f* when $x=a$.

The value of *a* is

A. -1
B. 0
C. 1
D. 2
E. 4

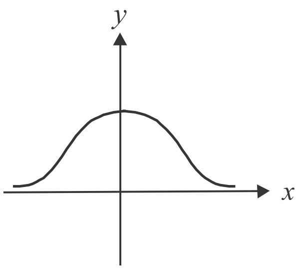
Question 7

Part of the graph of $y = g'(x)$ is shown below.

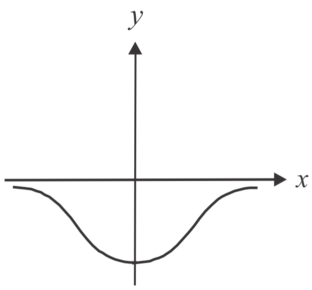


That part of the graph of $y = g(x)$ that corresponds to this graph, could be represented by

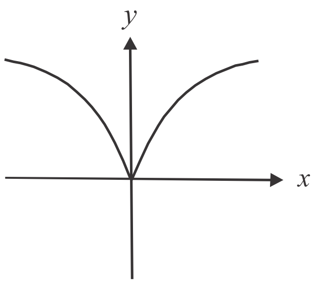
A.



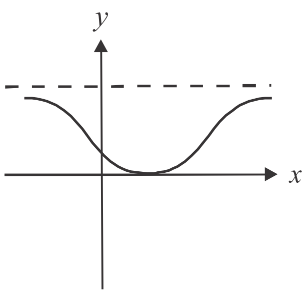
B.



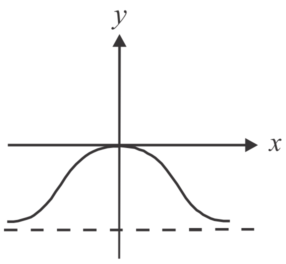
C.



D.



E.



Question 8

A and *B* are two independent events from a sample space such that $\Pr(A) = p$ and $\Pr(A \cap B) = p - q$ where $p, q > 0$.

$\Pr(B)$ is equal to

A. $\frac{p-q}{p}$

B. $\frac{p-q}{q}$

C. q

D. $1-p$

E. $1-p-2q$

Question 9

A continuous random variable *X* has a normal distribution with a mean of 7 and a standard deviation of 0.4.
The random variable *Z* has the standard normal distribution.
$\Pr(6.6 < X < 8.2)$ is equal to

A. $\Pr(-3 < Z < 1)$

B. $\Pr(-3 < Z < 2)$

C. $\Pr(-2 < Z < 1)$

D. $\Pr(-2 < Z < 3)$

E. $\Pr(-1 < Z < 2)$

Question 10

A bag contains two white balls and eight blue balls.
Let $\hat{P}$ be the random variable that represents the proportion of white balls present in samples of four balls taken from the bag.

$\Pr(\hat{P} = 0.25)$ is equal to

A. $\frac{1}{5}$

B. $\frac{1}{4}$

C. $\frac{4}{15}$

D. $\frac{8}{15}$

E. $\frac{3}{20}$

Question 11

The function *f* has the property $f(x^2) = 2f(x)$ for all positive real values of *x*.
The rule for *f* could be

A. $f(x) = x + 1$

B. $f(x) = \frac{1}{\sqrt{x}}$

C. $f(x) = e^{\frac{x}{2}}$

D. $f(x) = \frac{1}{x-2}$

E. $f(x) = \log_e(x)$

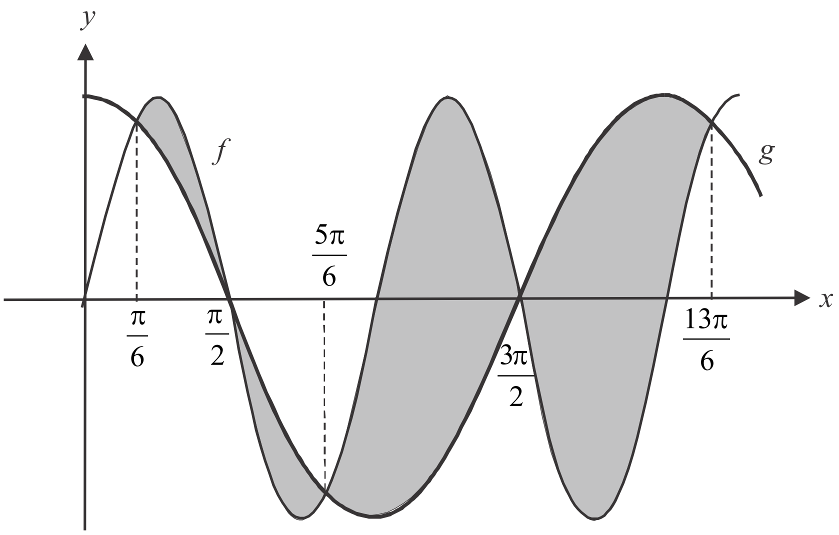
Question 12

If $\int_2^7 g(x)\,dx = 6$ and $\int_2^4 g(x)\,dx = -1$, then $\int_4^7 (g(x) - 1)\,dx$ is equal to

A. 2
B. 3
C. 4
D. 8
E. 10

Question 13

The graphs of $f: R \to R,\ f(x) = 3\sin(2x)$ and $g: R \to R,\ g(x) = 3\cos(x)$ are shown below.



The total area of the shaded regions is given by

A. $\displaystyle\int_{\frac{\pi}{6}}^{\frac{13\pi}{6}} (g(x) - f(x))\,dx$

B. $\displaystyle 2\int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} (g(x) - f(x))\,dx + 2\int_{\frac{3\pi}{2}}^{\frac{13\pi}{6}} (g(x) - f(x))\,dx$

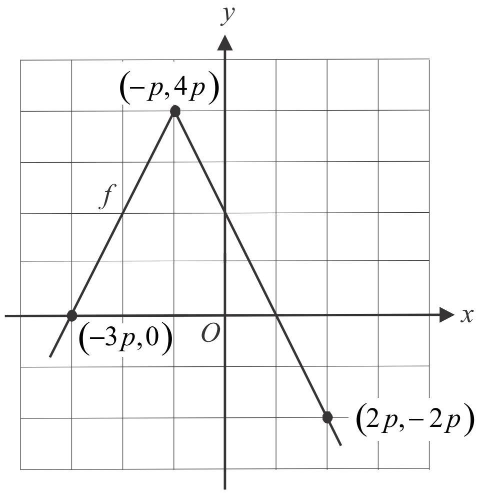
C. $\displaystyle 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (f(x) - g(x))\,dx + \int_{\frac{5\pi}{6}}^{\frac{3\pi}{2}} (f(x) - g(x))\,dx - \int_{\frac{3\pi}{2}}^{\frac{13\pi}{6}} (g(x) - f(x))\,dx$

D. $\displaystyle 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (f(x) - g(x))\,dx + 2\int_{\frac{5\pi}{6}}^{\frac{3\pi}{2}} (g(x) - f(x))\,dx$

E. $\displaystyle \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (f(x) - g(x))\,dx + \int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} (g(x) - f(x))\,dx + 2\int_{\frac{5\pi}{6}}^{\frac{3\pi}{2}} (g(x) - f(x))\,dx$

Question 14

Part of the graph of function *f*, where $p > 0$, is shown below.



The average value of function *f* over the interval $[-3p,\ 2p]$ is

A. $\dfrac{p}{5}$

B. $\dfrac{3p}{5}$

C. $\dfrac{7p}{5}$

D. $\dfrac{13p}{5}$

E. $\dfrac{33p}{5}$

Question 15

Forty percent of Australian school leavers have had a part-time job.
For random samples of 24 school leavers, $\hat{P}$ is the random variable that represents the proportion of school leavers who have had a part-time job.
The mean and standard deviation of $\hat{P}$ are given respectively by

A. 0.4 and 0.1
B. 0.4 and 0.5
C. 2.4 and 0.01
D. 9.6 and 0.5
E. 9.6 and 0.01

Question 16

The continuous random variable X has a probability density function given by

$$f(x)=\begin{cases}\frac{3}{28}\left(1+\sqrt{x}\right)dx, & 0\le x\le 4\\ 0 & \text{elsewhere}\end{cases}$$

The median of this function, m, satisfies the equation

A. $m+\frac{1}{2\sqrt{m}}-\frac{14}{3}=0$

B. $2\sqrt{m^3}-11=0$

C. $3m+2\sqrt{m^3}-14=0$

D. $\sqrt[3]{m^2}+14=0$

E. $6m+9\sqrt{m^3}-28=0$

Question 17

Let $f:[-\pi,\pi]\rightarrow R,\ f(x)=-\cos\left(\frac{x}{2}\right)$.

A tangent is drawn to the graph of f.
The y-intercept of this tangent occurs at the point $(0, c)$.
The maximum possible value of c is

A. $-\frac{\pi}{2}$

B. -1

C. 0

D. $\frac{\pi}{2}$

E. π

Question 18

Let $f:[0,\infty)\rightarrow R,\ f(x)=\sqrt{x}$ and $g:(-\infty,0]\rightarrow R,\ g(x)=x^2-2x$.
The domain and range of $f(g(x))$ are respectively

A. $(-\infty,0]$ and R

B. $(-\infty,0]$ and $[0,\infty)$

C. R and $(-\infty,0]$

D. $[0,\infty)$ and $(-\infty,0]$

E. $[0,\infty)$ and $[0,\infty)$

Question 19

Let $g:[-a, a)\cup(a, 2a]\to R,\ g(x)=\dfrac{1}{x-a}+b$ where a and b are positive real numbers.
The range of g is

A. $R\backslash\{b\}$

B. $\left(b-\dfrac{1}{2a}, b+\dfrac{1}{a}\right)$

C. $[-b, b)\cup(b, 2b]$

D. $\left(-\infty, b-\dfrac{1}{2a}\right)\cup\left(b+\dfrac{1}{a}, \infty\right)$

E. $\left(-\infty, b-\dfrac{1}{2a}\right]\cup\left[b+\dfrac{1}{a}, \infty\right)$

Question 20

The graph of the cubic function f has a minimum turning point at the point $(1,0)$.
The domain of f is $x\in(0,\infty)$.

The transformation given by

$$T: R^2\to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)=\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}+\begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

maps the graph of f onto the graph of g.

The graph of g could have a **minimum** turning point at a point where

A. $x<-2$
B. $0<x<1$
C. $x<1$
D. $x=3$
E. $x>3$

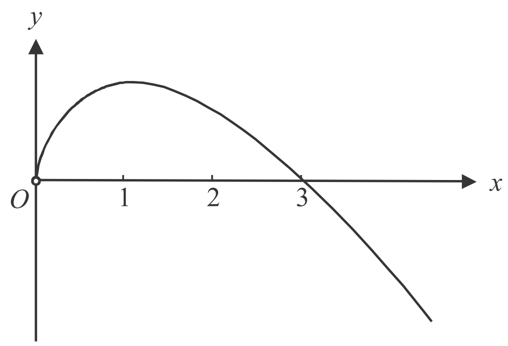
SECTION B

Answer all questions in this section.

Question 1 (10 marks)

Let $f:(0,\infty)\to R,\ f(x)=-x\log_e\left(\frac{x}{3}\right)$.

Part of the graph of *f* is shown below.



a. Find the interval for which the graph of *f* is strictly decreasing. 2 marks

b. Find the slope of the tangent to *f* at $x=\frac{3}{\sqrt{e}}$. 1 mark

c. Find the obtuse angle that the tangent to *f* at $x=\frac{3}{\sqrt{e}}$ makes with the positive direction of the *x*-axis. Give your answer correct to the nearest degree. 1 mark

d. The tangent to the graph of *f* at $x = \frac{3}{\sqrt{e}}$ is perpendicular to the tangent of *f* at $x = a$.

Find the value of *a*. 2 marks

e. The point $P(p, f(p))$ lies on the graph of *f*.
Find the *y*-intercept of the tangent to the graph of *f* at $x = p$. Give your answer in terms of *p*. 2 marks

f. Hence find the distance between the *y*-intercepts of the tangent to *f* at $x = \frac{3}{\sqrt{e}}$, and the tangent to *f* that is perpendicular to this tangent.

Give your answer in the form $\frac{r\left(e^{\frac{s}{t}} - u\right)}{e^r}$ where *r, s*, *t* and *u* are positive integers. 2 marks

Question 2 (11 marks)

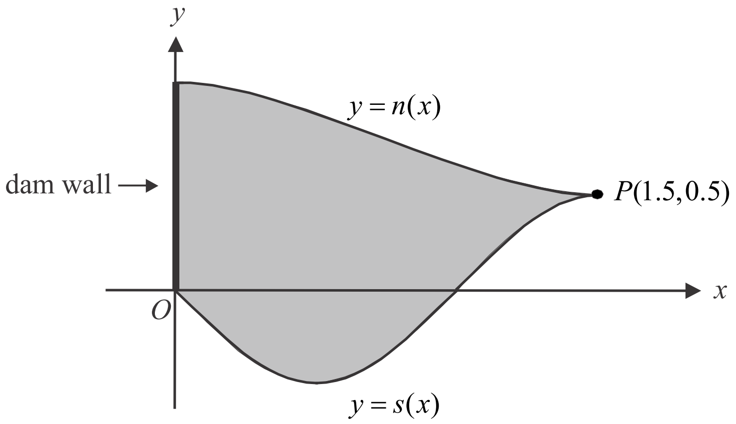
A reservoir has a dam wall on its west boundary.
The northern shoreline of the reservoir is modelled by the function

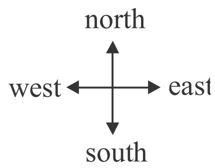
$$n:[0,a] \to R,\ n(x) = 0.3\cos\left(\frac{2\pi x}{3}\right) + 0.8 .$$

The southern shoreline of the reservoir is modelled by the function

$$s:[0,a] \to R,\ s(x) = -0.5\sin(\pi x) .$$

The dam wall runs in a straight line in a north–south direction.
The horizontal axis points east, the vertical axis points north and the surface of the water in the reservoir is shaded in the diagram below.





The eastern tip of the reservoir is located at the point $P(1.5, 0.5)$. The point $O(0,0)$ lies at the southern end of the dam wall.
All distances are measured in kilometres.

a. State the value of a. 1 mark

b. Find the length of the dam wall, in kilometres. 1 mark

c. Find the surface area of the water in the reservoir, in square kilometres, correct to two decimal places. 1 mark

A portable, straight barrier, of adjustable length, is used at the reservoir to trap surface debris.

In March, the barrier is installed so that one end is anchored at point O and the other at point P.

d. Find the length of the barrier required for this installation, in kilometres, correct to two decimal places. 1 mark

In June, the barrier is moved and is installed in a new position. It now runs in an east-west direction with its ends anchored at two points that lie on the southern shoreline. One end of the barrier is anchored at the point where $x = 0.2$.

e. Find the length, in kilometres, of the barrier required for this installation. 1 mark

In August, the barrier is moved again. This time the barrier runs parallel to the dam wall from a point on the northern shoreline to a point due south on the southern shoreline.

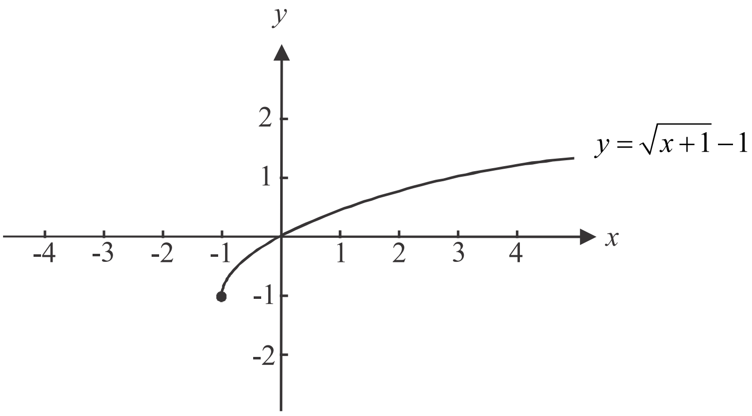
f. Find the length of the barrier required if half the surface area of the water in the reservoir lies to the east of the barrier and half lies to the west.
Give your answer in kilometres correct to two decimal places. 3 marks

g. For this installation in August, what is the maximum possible length of the barrier, in kilometres, correct to two decimal places? 3 marks

Question 3 (10 marks)

Let $f:[-1,\infty)\to R,\ f(x)=\sqrt{x+1}-1$.

Part of the graph of *f* is shown below.



a. The transformation $T:R^2\to R^2,\ T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}c\\d\end{bmatrix}$ maps the graph of *f* onto the graph of $y=\sqrt{x}$.

State the values of *c* and *d*. 1 mark

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b. The graph of *f* is to undergo a dilation from the *y*-axis so that the endpoint of the image graph is $(-2,-1)$. This transformation is given by

$$T_1:R^2\to R^2,\ T_1\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}a&0\\0&b\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}.$$

State the values of *a* and *b*. 1 mark

Let $h:[-n,\infty)\to R,\ h(x)=\sqrt{x+n}-n$ where n is a real number.

c. Find the values of n for which $h(x)>0$. 1 mark

d. Find the rule and domain for h^{-1}, the inverse function of h. 2 marks

e. Find the coordinates of the points of intersection of the graphs of h and of h^{-1}. 2 marks

f. Show that the area enclosed by the graphs of h and h^{-1} is constant for all values of n. 1 mark

g. If $n=0$, find the value of q where $q>0$, for which the average value of h^{-1} between $x=0$ and $x=q$ is equal to the average value of h between $x=0$ and $x=1$. 2 marks

Question 4 (17 marks)

A fishing licence is required by anglers involved in recreational fishing. Anglers can purchase licences that last for different periods of time. The random variable X represents the amount, in dollars, anglers must pay for a licence.
The distribution of X is given in the table below.

x	10	20	40	100
$\Pr(X = x)$	0.2	k	0.1	0.4

a. **i.** Find k. 1 mark

ii. Calculate the expected value of the amount anglers must pay for a licence. 1 mark

iii. Calculate the standard deviation of X, correct to the nearest dollar. 1 mark

iv. Fishing licences are purchased online. The amount paid for one licence being purchased is independent of the amount paid for the next licence being purchased. Find the probability that for the next four fishing licences purchased, exactly three will be for more than \$20. 1 mark

Fisheries officers routinely inspect fish that have been caught and kept by recreational anglers.
A commonly inspected fish is the Golden Perch.
The weight, in kg, of Golden Perch fish that have been inspected over the years is normally distributed with a mean of 5.2 kg and a standard deviation of 0.4 kg.

b. **i.** What proportion of these Golden Perch fish have a weight of more than 5.5 kg, correct to four decimal places? 1 mark

ii. What is the probability that the next five Golden Perch fish that are inspected, weigh more than 5.5 kg, correct to four decimal places? 1 mark

iii. Of the Golden Perch fish that have been inspected over the years, the 20% which have the least weight are supposed to have the best flavour.
What is the maximum weight, in kg, that these fish can have, correct to four decimal places? 1 mark

The probability density function that describes the length, in cm, of Golden Perch fish that have been inspected over the years is given by

$$f(x)=\begin{cases}\dfrac{-2(x-60)\log_e(x-25)}{1225\log_e(35)-1768}, & 26\le x\le 60\\ 0 & \text{elsewhere}\end{cases}$$

c. **i.** Find the mean length, in cm, of these Golden Perch fish, correct to two decimal places. 2 marks

ii. It is legal to keep Golden Perch fish that are more than 30 cm in length.
What proportion of Golden Perch fish that were inspected, had been kept legally, correct to two decimal places? 1 mark

iii. Find the probability that a Golden Perch fish measured more than 50 cm in length, given that it had been kept legally, correct to four decimal places. 2 marks

Fishing authorities have determined over time that 8% of recreational anglers in a particular state don't have a fishing licence.
One particular day, Fisheries officers randomly select and question 25 anglers in this state.

d. **i.** Find the probability that at least two of these anglers don't have a licence, correct to four decimal places. 1 mark

ii. For random samples of 25 anglers, $\hat{P}$ is the random variable that represents the proportion of anglers who don't have a license. Find the probability that $\hat{P}$ is less than 10%, correct to four decimal places. Do not use a normal approximation. 2 marks

Fishing authorities in **another state** randomly select a very large sample of recreational anglers. They calculate an approximate 95% confidence interval of $(0.1092,\ 0.1504)$ for the proportion of anglers in that state who don't have a fishing license.

e. **i.** Find the sample proportion used in the calculation of this confidence interval. 1 mark

ii. Explain why this confidence interval suggests that the proportion of anglers who don't have a licence could be different in the two states. 1 mark

Question 5 (12 marks)

Let $f : R \to R,\ f(x) = -x^3 + ax^2$ and $g : R \to R,\ g(x) = ax$ where a is a positive real constant.

a. Find the coordinates of the x-intercepts of the graph of f, in terms of a where appropriate. 1 mark

b. Find the coordinates of the local maximum of the graph of f in terms of a. 2 marks

c. **i.** Find the values of a for which the graphs of f and g have just one point of intersection. 2 marks

ii. Show that the graphs of f and g will have exactly two points of intersection when $a = 4$. 2 marks

d. Find the coordinates of the point of tangency when the graph of g is a tangent to the graph of f. 1 mark

The graphs of f and g have three points of intersection when $a > 4$.
Let the x-coordinates of these three points of intersection be r, s and t where $r < s < t$.

e. **i.** Find the values of r, s and t in terms of a where appropriate. 2 marks

ii. Find the value of a, where $a > 4$, if there are equal areas enclosed by the functions f and g between $x = r$ and $x = s$ and between $x = s$ and $x = t$. 2 marks

Mathematical Methods formulas

Mensuration

area of a trapezium	$\frac{1}{2}(a+b)h$	volume of a pyramid	$\frac{1}{3}Ah$
curved surface area of a cylinder	$2\pi rh$	volume of a sphere	$\frac{4}{3}\pi r^3$
volume of a cylinder	$\pi r^2 h$	area of a triangle	$\frac{1}{2}bc\sin(A)$
volume of a cone	$\frac{1}{3}\pi r^2 h$		

Calculus

$\frac{d}{dx}\left(x^n\right)=nx^{n-1}$		$\int x^n dx=\frac{1}{n+1}x^{n+1}+c,\ n\neq-1$	
$\frac{d}{dx}\left((ax+b)^n\right)=an(ax+b)^{n-1}$		$\int(ax+b)^n dx=\frac{1}{a(n+1)}(ax+b)^{n+1}+c,\ n\neq-1$	
$\frac{d}{dx}\left(e^{ax}\right)=ae^{ax}$		$\int e^{ax}dx=\frac{1}{a}e^{ax}+c$	
$\frac{d}{dx}\left(\log_e(x)\right)=\frac{1}{x}$		$\int\frac{1}{x}dx=\log_e(x)+c,\ x>0$	
$\frac{d}{dx}\left(\sin(ax)\right)=a\cos(ax)$		$\int\sin(ax)dx=-\frac{1}{a}\cos(ax)+c$	
$\frac{d}{dx}\left(\cos(ax)\right)=-a\sin(ax)$		$\int\cos\ (ax)\,dx=\frac{1}{a}\sin(ax)+c$	
$\frac{d}{dx}\left(\tan(ax)\right)=\frac{a}{\cos^2(ax)}=a\sec^2(ax)$			
product rule	$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$	quotient rule	$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$
chain rule	$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$		

Probability

$\Pr(A) = 1 - \Pr(A')$		$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$	
$\Pr(A \mid B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$			
mean	$\mu = \mathrm{E}(X)$	variance	$\mathrm{var}(X) = \sigma^2 = \mathrm{E}((X-\mu)^2) = \mathrm{E}(X^2) - \mu^2$

Probability distribution		**Mean**	**Variance**
discrete	$\Pr(X = x) = p(x)$	$\mu = \Sigma\, x\, p(x)$	$\sigma^2 = \Sigma\,(x-\mu)^2\, p(x)$
continuous	$\Pr(a < X < b) = \int_a^b f(x)dx$	$\mu = \int_{-\infty}^{\infty} x\, f(x)dx$	$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$

Sample proportions

$\hat{P} = \dfrac{X}{n}$		mean	$\mathrm{E}(\hat{P}) = p$
standard deviation	$\mathrm{sd}(\hat{P}) = \sqrt{\dfrac{p(1-p)}{n}}$	approximate confidence interval	$\left(\hat{p} - z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}\right)$

MATHEMATICAL METHODS

TRIAL EXAMINATION 2

MULTIPLE - CHOICE ANSWER SHEET

STUDENT NAME:..

INSTRUCTIONS

Fill in the letter that corresponds to your choice. Example: A ■ C D E

The answer selected is B. Only one answer should be selected.

1. A B C D E	11. A B C D E
2. A B C D E	12. A B C D E
3. A B C D E	13. A B C D E
4. A B C D E	14. A B C D E
5. A B C D E	15. A B C D E
6. A B C D E	16. A B C D E
7. A B C D E	17. A B C D E
8. A B C D E	18. A B C D E
9. A B C D E	19. A B C D E
10. A B C D E	20. A B C D E