

VCE Mathematical Methods Units 3&4

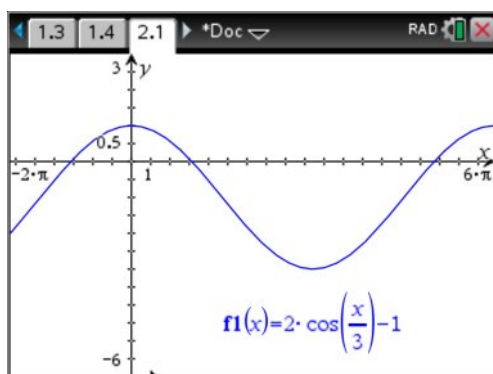
Written Examination 2

Suggested Solutions

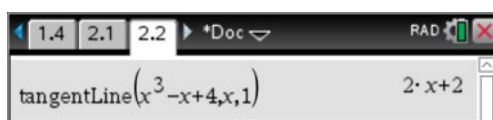
SECTION A – MULTIPLE-CHOICE QUESTIONS

1	A	B	C	D	E
2	A	B	C	D	E
3	A	B	C	D	E
4	A	B	C	D	E
5	A	B	C	D	E
6	A	B	C	D	E
7	A	B	C	D	E
8	A	B	C	D	E
9	A	B	C	D	E
10	A	B	C	D	E

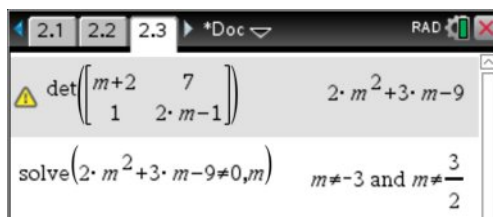
11	A	B	C	D	E
12	A	B	C	D	E
13	A	B	C	D	E
14	A	B	C	D	E
15	A	B	C	D	E
16	A	B	C	D	E
17	A	B	C	D	E
18	A	B	C	D	E
19	A	B	C	D	E
20	A	B	C	D	E

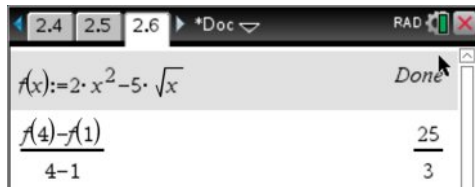
Question 1 E

$$\begin{aligned} \text{period} &= \frac{2\pi}{n} \\ &= \frac{2\pi}{\frac{1}{3}} \\ &= 6\pi \end{aligned}$$

Question 2 A**Question 3 B**

$$\begin{bmatrix} m+2 & 7 \\ 1 & 2m-1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} m+3 \\ 5 \end{bmatrix}$$



Question 4 **B****Question 5** **A****Question 6** **C**

$$kx^2 - kx + \frac{1}{4} = 0$$

$$\Delta = b^2 - 4ac$$

$$= k^2 - k$$

$$k^2 - k = 0 \text{ for one solution}$$

$$k(k - 1) = 0$$

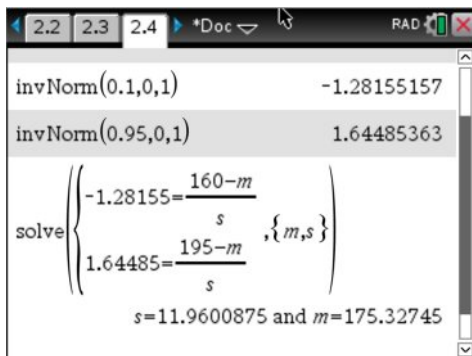
$$k = 0 \text{ or } k = 1$$

However, when $k = 0$, the quadratic reduces to $\frac{1}{4} = 0$ and therefore $k \neq 0$.

$$k = 1 \text{ only}$$

Question 7 **A**

$$z = \frac{x - \bar{x}}{\mu}$$



Question 8 **A**

$$g(x) = 3f(2x - 4) + 1$$

Transformations:

Dilation factor of 3 from x -axis: $(1, 3) \rightarrow (1, 9)$

Dilation factor of $\frac{1}{2}$ from y -axis: $(1, 9) \rightarrow \left(\frac{1}{2}, 9\right)$

Translate 2 units right: $(1, 9) \rightarrow \left(\frac{5}{2}, 9\right)$

Translate 1 unit down: $\left(\frac{5}{2}, 9\right) \rightarrow \left(\frac{5}{2}, 10\right)$

Question 9 **A**

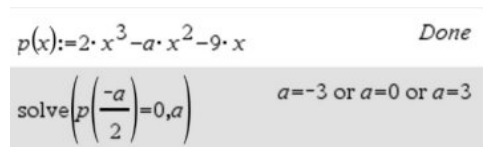
Let f and g be two functions such that $f(x + 1) = x$ and $g(x + 2) = f(x)$.

$$f(x + 1) = x \rightarrow f(x) = x - 1$$

$$g(x + 2) = f(x) = x - 1 \rightarrow g(x) = x - 3$$

$$\begin{aligned} f(g(x)) &= (x - 3) - 1 \\ &= x - 4 \end{aligned}$$

Question 10 **E**



$p(x) := 2 \cdot x^3 - a \cdot x^2 - 9 \cdot x$ Done
 solve $\left(p\left(\frac{-a}{2}\right) = 0, a\right)$ $a = -3$ or $a = 0$ or $a = 3$

Question 11 **D**

Let S = success, M = miss and A = any (success or miss).

$$\text{Pr(at least 3 consecutive)} = \text{Pr}(SSSAA) + \text{Pr}(MSSSA) + \text{Pr}(AMSSS)$$

$$\text{Pr(at least 3 consecutive)} = \left(\frac{4}{5}\right)^3 \times 1 \times 1 + \left(\frac{1}{5}\right) \times \left(\frac{4}{5}\right)^3 \times 1 + 1 \times \frac{1}{5} \times \left(\frac{4}{5}\right)^3 = 0.7168$$

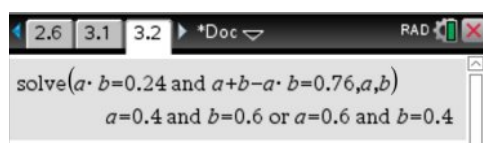
Question 12 **E**

Let $a = \text{Pr}(A)$ and $b = \text{Pr}(B)$.

$$\text{Pr}(A \cup B) = \text{Pr}(A) + \text{Pr}(B) - \text{Pr}(A \cap B)$$

$$0.76 = a + b - ab$$

$$\text{Pr}(A \cap B) = ab = 0.24$$



solve $(a \cdot b = 0.24 \text{ and } a + b - a \cdot b = 0.76, a, b)$
 $a = 0.4 \text{ and } b = 0.6 \text{ or } a = 0.6 \text{ and } b = 0.4$

Question 13 B

The derivative function f' is a positive cubic and matches the positive quartic in **B**.

Question 14 C

$$2 - x > 0 \rightarrow x < 2$$

$$\log_e(2 - x) \neq 0 \rightarrow x \neq 1$$

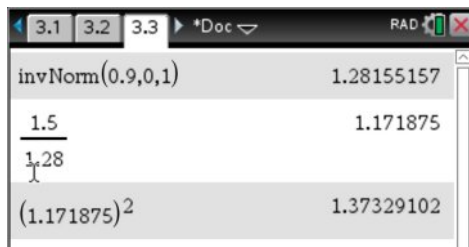
$$\therefore x \in (-\infty, 2) \setminus \{1\}$$

Question 15 C

$$z = \frac{x - \mu}{\sigma}$$

$$\mu = 0 \rightarrow \sigma = \frac{x}{z}$$

$$\text{Var}(X) = \sigma^2$$



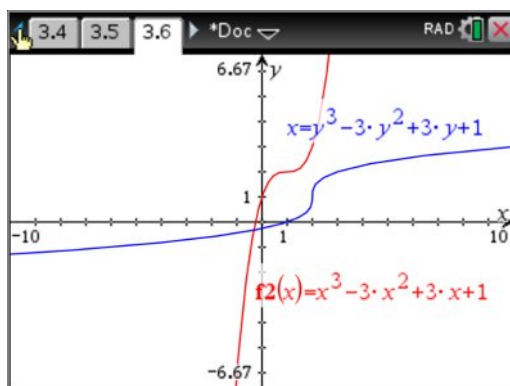
invNorm(0.9,0,1)	1.28155157
1.5	1.171875
1.28	
(1.171875) ²	1.37329102

Question 16 C

$$(x - 1)^3 = x^3 - 3x^2 + 3x - 1$$

$$\therefore g(x) = (x - 1)^3 + 2$$

Thus the inverse will exist.



Question 17 **D**

If $f(x) = \frac{1}{x}$:

$$\text{LHS} = f(x) + 2f(y)$$

$$= \frac{1}{x} + 2 \times \frac{1}{y}$$

$$= \frac{1}{x} + \frac{2}{y}$$

$$= \frac{2x + y}{xy}$$

$$\text{RHS} = (2x + y)f(xy)$$

$$= (2x + y) \frac{1}{xy}$$

$$= \frac{2x + y}{xy}$$

$$\therefore \text{LHS} = \text{RHS}$$

$$\therefore f(x) + 2f(y) = (2x + y)f(xy)$$

TI-Nspire calculator screenshot showing the verification of the identity $f(x) + 2f(y) = (2x + y)f(xy)$ for $f(x) = \frac{1}{x}$. The first line defines $f(x) := \frac{1}{x}$ and the second line shows the identity being verified, resulting in 'true'.

Question 18 **C**

$$\begin{aligned} \int_3^{10} f(x) + 1 \, dx &= \int_3^8 f(x) \, dx + \int_8^{10} f(x) \, dx + \int_3^{10} 1 \, dx \\ &= 10 - 4 + 7 \\ &= 13 \end{aligned}$$

Question 19 **C**

TI-Nspire calculator screenshot showing the solution to the differential equation $\frac{d}{dx}(\sqrt{x^2 + (x^2 - 4)^2}) = 0$. The first line defines $d(x) := \sqrt{x^2 + (x^2 - 4)^2}$. The second line shows the equation being solved: $\text{solve}\left(\frac{d}{dx}(d(x)) = 0, x\right)$. The third line shows the solutions: $x = \frac{-\sqrt{14}}{2}$ or $x = 0$ or $x = \frac{\sqrt{14}}{2}$. The fourth line shows the value of $d\left(\frac{\sqrt{14}}{2}\right)$ as $\frac{\sqrt{15}}{2}$.

Question 20 B

Calculator screen showing the function $f(x) := n^{n-1} \cdot x^n \cdot \ln(n \cdot x)$ and the command $\text{solve}\left(\frac{d}{dx}(f(x))=0, x\right)$. The result displayed is $x=0 \text{ and } n>0 \text{ or } x = \frac{e^{-1/n}}{n}$.

However $f(x)$ is undefined for $x = 0$ and therefore there is only one possible stationary point.

SECTION B**Question 1** (7 marks)**a.**

$$f(x) = x^3 - 6x^2 + 8x$$

$$\text{solve}(f(x)=0, x) \quad x=0 \text{ or } x=2 \text{ or } x=4$$

$$(0, 0), (2, 0), (4, 0)$$

A1

b.

$$\text{Let } f(x) = g(x).$$

$$x^3 - 6x^2 + 8x = -ax$$

$$x^2 - 6x + 8 = -a \quad (1)$$

$$\text{Let } f'(x) = g'(x).$$

$$3x^2 - 12x + 8 = -a \quad (2)$$

M1

Equate (1) and (2).

$$3x^2 - 12x + 8 = x^2 - 6x + 8$$

$$2x^2 - 6x = 0$$

$$2x(x - 3) = 0$$

$$x = 0 \text{ or } x = 3$$

M1

$$\therefore p = 3, \text{ as } p > 0.$$

Substitute $p = 3$ into (1):

$$(3)^2 - 6(3) + 8 = -a$$

M1

$$a = -1$$

- c. Let $d(x)$ = vertical distance between $f(x)$ and $g(x)$.

$$d(x) = f(x) - g(x)$$

$$= x^3 - (6x^2 + 9x)$$

M1

$$d'(x) = 3x^2 - 12x + 9$$

$$\text{Let } d'(x) = 0.$$

$$x = 1 \text{ or } x = 3$$

M1

$$d(1) = 4$$

A1

maximum vertical distance = 4

$$\frac{d}{dx}(d(x)) \quad 3 \cdot x^2 - 12 \cdot x + 9$$

$$\text{solve}(3 \cdot x^2 - 12 \cdot x + 9 = 0, x) \quad x = 1 \text{ or } x = 3$$

$$d(x) \quad x^3 - 6 \cdot x^2 + 9 \cdot x$$

$$d(1) \quad 4$$

Question 2 (18 marks)

- a. i.

$$\left(-\frac{4}{3}, -\frac{1}{3}\right)$$

A1

- ii.

$$\sqrt{\left(-\frac{4}{3}\right)^2 + \left(-\frac{1}{3}\right)^2} \quad \frac{\sqrt{17}}{3}$$

$$\frac{\sqrt{17}}{3}$$

A1

- iii.

$$\tan^{-1}(4) - \tan^{-1}(1) \quad 30.9637565$$

$$31^\circ$$

A1

- b. i. There is a dilation factor of a from the y -axis and a dilation factor of $\frac{1}{a}$ from the x -axis.

A1

ii. $x' = ax \rightarrow x = \frac{x'}{a}$

$$y' = \frac{1}{a}y \rightarrow y = ay'$$

$$y = x + 1$$

$$ay' = \frac{x'}{a} + 1$$

$$y' = \frac{1}{a^2}x + \frac{1}{a}$$

$$h(x) = \frac{1}{a^2}x + \frac{1}{a}$$

M1

A1

c. $f(x) = h(x)$

$$4x + 5 = \frac{1}{a^2}x + \frac{1}{a}$$

Unique solutions occur where $m_1 \neq m_2$.

$$\frac{1}{a^2} \neq 4$$

$$a \neq \pm \frac{1}{2}$$

M1

But $a \in \mathbb{R}^+$ and therefore unique solutions occur for $a \in (0, \infty) / \left\{ \frac{1}{2} \right\}$.

A1

d.

Handwritten solution for part d:

$$\text{solve}(f(x)=h(x),x) \quad x = \frac{-a \cdot (5a-1)}{4a^2-1}$$

$$f\left(\frac{-a \cdot (5a-1)}{4a^2-1}\right) = \frac{4a-5}{4a^2-1}$$

$$\left(\frac{-a(5a-1)}{4a^2-1}, \frac{4a-5}{4a^2-1} \right)$$

x-coordinate A1

y-coordinate A1

- e. Let distance = $d(a)$.

$$d(a) = \sqrt{\left(\frac{-a(5a-1)}{4a^2-1}\right)^2 + \left(\frac{4a-5}{4a^2-1}\right)^2} \quad \text{M1}$$

For min/max, let $d'(a) = 0$. M1

$$\begin{aligned} a &= \frac{\pm\sqrt{21} + 5}{4} \\ &= \frac{\sqrt{21} + 5}{4} \quad \text{A1} \end{aligned}$$

$$d(a) := \sqrt{\left(\frac{-a \cdot (5 \cdot a - 1)}{4 \cdot a^2 - 1}\right)^2 + \left(\frac{4 \cdot a - 5}{4 \cdot a^2 - 1}\right)^2}$$

$$\begin{aligned} &\text{solve}\left(\frac{d}{da}(d(a))=0, a\right) \\ &a = \frac{-(\sqrt{21}-5)}{4} \text{ or } a = \frac{\sqrt{21}+5}{4} \end{aligned}$$

$$\text{fMin}(d(a), a) \quad a = \frac{\sqrt{21}+5}{4}$$

Note: Students may also determine the correct value of a from a graph of the distance function.

- f. i. $\text{dom}_p = [-1, 1]$

Transformation T_2 represents a dilation factor of a from the y -axis and, as $a < 0$, there is also a reflection in the y -axis.

$$\therefore \text{dom}_q = [a, -a] \quad \text{A1}$$

ii. $q : [a, -a] \rightarrow \mathbb{R}, q(x) = \frac{1}{a^2}x + \frac{1}{a}$

$$f(x) = q(x)$$

$$4x + 5 = \frac{1}{a^2}x + \frac{1}{a}$$

Solution is $\left(\frac{-a(5a-1)}{4a^2-1}, \frac{4a-5}{4a^2-1} \right)$ from **part d**.

M1

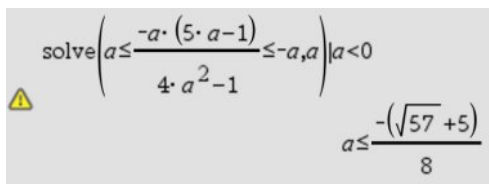
However, the restricted domain of $\text{dom}_q = [a, -a]$.

$$\therefore a \leq \frac{-a(5a-1)}{4a^2-1} \leq -a \text{ for } a < 0 \text{ and } 4a^2-1 \neq 0.$$

M1

$$a \in \left(-\infty, \frac{-5-\sqrt{57}}{8} \right]$$

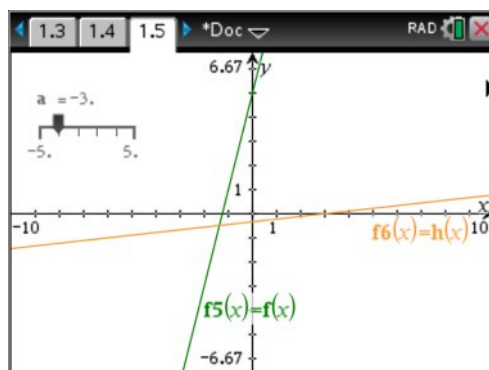
A1



solve $\left(a \leq \frac{-a(5a-1)}{4a^2-1} \leq -a, a < 0 \right)$

$a \leq \frac{-(\sqrt{57}+5)}{8}$

Note: A graphical approach with sliders can be also used, as shown below.



iii. $q(x) = \frac{1}{a^2}x + \frac{1}{a}$

As $a \rightarrow -\infty, q(x) \rightarrow 0$.

$$f(x) = q(x)$$

$$4x + 5 = 0$$

$$x = -\frac{5}{4}$$

$$\therefore m_1 = -\frac{5}{4}$$

A1

Question 3 (13 marks)

a. i. average rate of change $= \frac{f(a) - 1}{a - 1}$

$$= \frac{\frac{1}{a} - 1}{a - 1}$$

$$= \frac{1 - a}{a(a - 1)}$$

$$= -\frac{1}{a}$$

A1

ii. $f'(a) = -\frac{1}{x^2}$

A1

$$-\frac{1}{x^2} = -\frac{1}{a}$$

$$x^2 = a$$

As $x > 0$, $x = \sqrt{a}$.

$$\therefore \text{point } Q = \left(\sqrt{a}, \frac{1}{\sqrt{a}} \right)$$

A1

b. i.

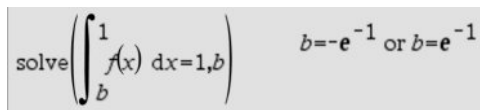


The image shows a TI-84 Plus calculator screen. The top line displays 'f(x):=1/x' and 'Done'. The second line shows the integral calculation: $\int_1^e f(x) dx = 1$.

$$\int_1^e f(x) dx = 1$$

A1

ii.



The image shows a TI-84 Plus calculator screen. The top line displays 'solve(integral from b to 1 of f(x) dx = 1, b)' and the bottom line shows the result: 'b = -e^-1 or b = e^-1'.

$$b = \frac{1}{e} \text{ as } 0 < b < 1$$

A1

- c. i. The area is a trapezium.

$$\frac{1}{2} \cdot \left(1 + \frac{1}{a}\right) \cdot (a-1) \qquad \frac{(a-1) \cdot (a+1)}{2 \cdot a}$$

$$A = \frac{1}{2} \left(1 + \frac{1}{a}\right) \times (a-1) \qquad \text{M1}$$

$$= \frac{(a-1)(a+1)}{2a} \qquad \text{A1}$$

- ii.

$$\text{solve} \left(\frac{(a-1) \cdot (a+1)}{2 \cdot a} = 1, a \right) \\ a = -(\sqrt{2} - 1) \text{ or } a = \sqrt{2} + 1$$

$$\frac{(a-1)(a+1)}{2a} = 1 \qquad \text{M1}$$

$$a = \sqrt{2} + 1 \qquad \text{A1}$$

- iii. The line PA is above the graph of f . The area below the line (the trapezium) has

an area of 1 square unit so that $\int_1^a f(x)dx$ must be less than this. The integral

$$\int_1^e f(x)dx = 1 \text{ so that } \int_1^a f(x)dx \text{ must be less than 1 and hence } a < e. \qquad \text{A1}$$

d. If $\int_k^m f(kx)dx = \frac{1}{k}$, then $\int_1^{\frac{m}{k}} f(x)dx = 1$, as k represents a dilation factor of $\frac{1}{k}$ units

from the y-axis.

If $\int_{m-1}^k f(kx)dx = \frac{1}{k}$, then by the same logic $\int_{\frac{m-1}{k}}^1 f(x)dx = 1$. M1

In **part b.** it was found that and $\int_1^e f(x)dx = 1$ and $\int_{\frac{1}{e}}^1 f(x)dx = 1$.

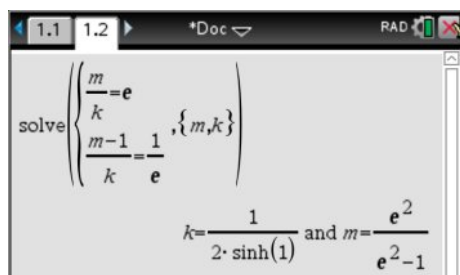
Therefore $\frac{m}{k} = e$ and $\frac{m-1}{k} = \frac{1}{e}$. M1

Solve simultaneously to give $m = \frac{e^2}{e^2 - 1}$ and $k = \frac{e}{e^2 - 1}$. A1

Note: CAS gives k using hyperbolic sine, which is not permitted in the VCAA exams.

Students can use the CAS but then should be able to substitute m back into the equation

$\frac{m}{k} = e$ to find the acceptable version for k .



solve $\left(\begin{array}{l} \frac{m}{k} = e \\ \frac{m-1}{k} = \frac{1}{e} \end{array} \right), \{m, k\}$

$k = \frac{1}{2 \cdot \sinh(1)}$ and $m = \frac{e^2}{e^2 - 1}$

Question 4 (10 marks)

a. i. $X \sim Bi\left(5, \frac{1}{5}\right)$

$$\begin{aligned}\Pr(X = 0) &= \left(\frac{4}{5}\right)^5 \\ &= \frac{1024}{3125}\end{aligned}\quad \text{A1}$$

ii. $\Pr(X \geq 3 | X \geq 1) = \frac{\Pr(X \geq 3 \cap X \geq 1)}{\Pr(X \geq 1)}$

$$= \frac{\Pr(X \geq 3)}{1 - \Pr(X = 0)} \quad \text{M1}$$

$$\text{binomCdf}\left(5, \frac{1}{5}, 3, 5\right) \quad 0.05792$$

$$\begin{aligned}\Pr(X \geq 3 | X \geq 1) &= \frac{0.05792}{\left(1 - \frac{1024}{3125}\right)} \\ &= 0.0861\end{aligned}\quad \text{A1}$$

b. $\Pr(Y = 4) + \Pr(Y = 5) = 11\Pr(Y = 5)$

$$\Pr(Y = 4) = 10\Pr(Y = 5)$$

$$\begin{aligned}{}^4C_4 p^4 (1-p)^1 &= 10 {}^5C_4 p^5 (1-p)^0 \\ 5p^4 (1-p) &= 10p^5 \\ p^4 (1-p) &= 2p^5\end{aligned}\quad \text{M1}$$

$$1 - p = 2p \text{ as } p \neq 0$$

$$3p = 1$$

$$p = \frac{1}{3} \quad \text{A1}$$

- c. $\Pr((X + Y) < 2) = \Pr((X + Y) = 0) + \Pr((X + Y) = 1)$
 $= \Pr(X = 0) \times \Pr(Y = 0) + \Pr(X = 0) \times \Pr(Y = 1) + \Pr(X = 1) \times \Pr(Y = 0)$ M1
 $= 0.2050$ A1

TI-Nspire calculator screenshot showing the following calculations:

- $\text{binomPdf}\left(5, \frac{1}{5}, 0\right) \cdot \text{binomPdf}\left(5, \frac{1}{3}, 0\right) = 0.043151276$
- $\text{binomPdf}\left(5, \frac{1}{5}, 0\right) \cdot \text{binomPdf}\left(5, \frac{1}{3}, 1\right) = 0.107878189$
- $\text{binomPdf}\left(5, \frac{1}{5}, 1\right) \cdot \text{binomPdf}\left(5, \frac{1}{3}, 0\right) = 0.053939095$
- Sum: $0.043151275720167 + 0.10787818930041 + 0.053939095 = 0.20496856$

- d. $\Pr(Y \geq 1) = 0.8683$ M1

TI-Nspire calculator screenshot showing the calculation:

- $\text{binomCdf}\left(5, \frac{1}{3}, 1, 5\right) = 0.868312757$

$$\Pr(T > 91) = 0.8683 \dots$$

$$\Pr(T < 91) = 1 - 0.8683 \dots$$

$$= 0.1316 \dots$$

$$z = \frac{x - \bar{x}}{\sigma}$$

M1

TI-Nspire calculator screenshot showing the calculation:

- $\text{invNorm}(0.13168724279836, 0, 1) = -1.11845085$

TI-Nspire calculator screenshot showing the solve command:

- $\text{solve}\left(-1.1184508519533 = \frac{91 - 100}{s}, s\right)$
- $s = 8.04684442$

$$\sigma = 8.0468$$

A1

Question 5 (12 marks)

a. i. $x \in (-1, \infty)$ A1

ii. $y \in [-1, 1]$ A1

b. Let $\sin(\log_e(x+1)) = 0$.

$$\log_e(x+1) = n\pi, \quad n \in \mathbb{Z} \quad \text{M1}$$

$$n = 0 \rightarrow \log_e(x+1) = 0 \rightarrow x = 0$$

$$n = 1 \rightarrow \log_e(x+1) = \pi \rightarrow x+1 = e^\pi \rightarrow x = e^\pi - 1 \quad \text{A1}$$

Therefore $(e^\pi - 1, 0)$ is the first positive x -intercept.

c. i. $g(x) = f(x-1)$

$$f(x) = \sin(\log_e(x+1))$$

$$g(x) = \sin(\log_e(x-1+1)) \quad \text{M1}$$

$$\therefore g(x) = \sin(\log_e(x)) \text{ as required}$$

ii. $g(x) = f(x-1)$

The graph of $y = g(x)$ is a translation of 1 unit right from the graph of $y = f(x)$.

Therefore $(1, 0)$ and $(e^\pi, 0)$ are the required x -intercepts of $y = g(x)$. A1

d. $\frac{d}{dx}x(\sin(\log_e(x)) - \cos(\log_e(x)))$

$$= 1 \times (\sin(\log_e(x)) - \cos(\log_e(x))) + x \times \left(\cos(\log_e(x)) \times \frac{1}{x} + \sin(\log_e(x)) \times \frac{1}{x} \right)$$

$$= 2 \sin(\log_e(x)) \quad \text{A1}$$

$$\rightarrow \int 2 \sin(\log_e(x)) dx = x(\sin(\log_e(x)) - \cos(\log_e(x))) \quad \text{M1}$$

$$\rightarrow \int \sin(\log_e(x)) dx = \frac{x}{2}(\sin(\log_e(x)) - \cos(\log_e(x))) \quad \text{A1}$$

e. $\int_0^{e^\pi - 1} f(x) dx = \int_1^{e^\pi} g(x) dx \quad \text{M1}$

$$\int_1^{e^\pi} g(x) dx = \left[\frac{x}{2}(\sin(\log_e(x)) - \cos(\log_e(x))) \right]_1^{e^\pi} \quad \text{M1}$$

$$\int_0^{e^\pi - 1} f(x) dx = \frac{1}{2}(e^\pi + 1) \quad \text{A1}$$

$$\int_1^{e^\pi} g(x) dx = \frac{e^\pi}{2} + \frac{1}{2}$$