

Trial Examination 2020

MATHEMATICAL METHODS

Trial Written Examination 1 - SOLUTIONS

Question 1

$$h(x) = \frac{x^2}{\tan(2x)} \quad \text{Use the quotient rule.}$$

$$h'(x) = \frac{2x \tan(2x) - 2x^2 \sec^2(2x)}{\tan^2(2x)} \quad \mathbf{1A}$$

Other forms

OR

$$h'(x) = \frac{2x(\sin(2x)\cos(2x) - x)}{\sin^2(2x)} \quad \mathbf{1A}$$

Question 2

$$f(x) = \log_e(\sin(2x)) \quad \text{Use the chain rule.}$$

$$f'(x) = \frac{2 \cos(2x)}{\sin(2x)} \quad \mathbf{1A}$$

$$f'\left(\frac{\pi}{3}\right) = \frac{2 \cos\left(\frac{2\pi}{3}\right)}{\sin\left(\frac{2\pi}{3}\right)}$$

$$= \frac{2 \times -\frac{1}{2}}{\frac{\sqrt{3}}{2}}$$

$$= \frac{-2}{\sqrt{3}}$$

$$= \frac{-2}{\sqrt{3}}$$

1A

OR

$$f'\left(\frac{\pi}{3}\right)$$

$$= \frac{2}{\tan\left(\frac{2\pi}{3}\right)}$$

$$= \frac{-2}{\sqrt{3}}$$

1A

Question 3

$$\int_0^a \left(\frac{x}{x^2 + 4} \right) dx = 3.$$

$$\frac{1}{2} \int_0^a \left(\frac{2x}{x^2 + 4} \right) dx = 3$$

$$\frac{1}{2} \left[\log_e (x^2 + 4) \right]_0^a = 3 \quad \mathbf{1M}$$

$$\frac{1}{2} \left(\log_e (a^2 + 4) - \log_e (4) \right) = 3$$

$$\log_e \left(\frac{a^2 + 4}{4} \right) = 6 \quad \mathbf{1M}$$

$$\frac{a^2 + 4}{4} = e^6$$

$$a = \pm \sqrt{4e^6 - 4} = \pm 2\sqrt{e^6 - 1} \quad \mathbf{1A}$$

Question 4

$$f(x) = \frac{1}{9}(3x-1)e^{3x}$$

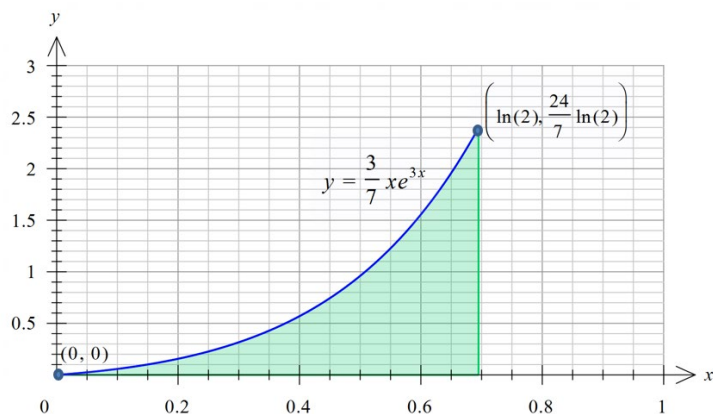
a. $f'(x) = \frac{3}{9}e^{3x} + \frac{3}{9}(3x-1)e^{3x}$ product rule

$$= \frac{1}{3}e^{3x} + xe^{3x} - \frac{1}{3}e^{3x}$$

$$= xe^{3x} \text{ as required} \quad \mathbf{1M \text{ Show that}}$$

b. $h: [0, \log_e(2)] \rightarrow R, h(x) = \frac{3}{7}e^{3x}$

$$y = xh(x)$$



$$\frac{3}{7} \int_0^{\log_e(2)} (xe^{3x}) dx \quad \mathbf{1M}$$

$$= \frac{3}{7} \times \frac{1}{9} \left[(3x-1)e^{3x} \right]_0^{\log_e(2)} \quad \mathbf{1M}$$

$$= \frac{1}{21} \left((3\log_e(2)-1)e^{3\log_e(2)} + 1 \right)$$

$$e^{3\log_e(2)} = e^{\log_e(2^3)} = e^{\log_e(8)} = 8$$

$$= \frac{1}{21} (24\log_e(2) - 8 + 1)$$

$$= \frac{8}{7} \log_e(2) - \frac{1}{3} \quad \mathbf{1A}$$

Question 5

$$f: [0, \infty) \rightarrow R, f(x) = e^{2x} \text{ and } g(x) = x^2 - 1$$

a. Test to see if $g(f(x))$ exists.

If $g(f(x))$ exists then $r_f \subset d_g$.

$$\text{Range } f = [1, \infty) \quad \text{Domain } g = R$$

$$[1, \infty) \subset R$$

Therefore $g(f(x))$ exists. $\mathbf{1A}$

$g_1: D \rightarrow R, g_1(x) = x^2 - 1$, where D is the maximal domain of g_1 such that $h(x) = f(g_1(x))$ exists.

b. Test to see if $f(g_1(x))$ exists.

If $f(g_1(x))$ exists then $r_g \subset d_f$.

$$\text{Range } g = [-1, \infty), \text{ Domain } f = [0, \infty)$$

$$[-1, \infty) \not\subset [0, \infty)$$

$$\text{Restrict Range } g = [-1, \infty) \text{ to Range } g_1 = [0, \infty) \quad \mathbf{1M}$$

$$\text{Giving Domain } g_1 = (-\infty, -1] \cup [1, \infty)$$

$$D = (-\infty, -1] \cup [1, \infty) \text{ OR } R \setminus (-1, 1) \quad \mathbf{1A}$$

$$\mathbf{c.} \quad \text{Rule } h(x) = e^{2(x^2-1)} \quad \mathbf{1A}$$

$$h'(x) = 4xe^{2(x^2-1)}$$

$$h'(2) = 8e^6 \quad \mathbf{1A}$$

Question 6

$$f: [-1, \infty) \rightarrow \mathbb{R}, f(x) = a\sqrt{x+b} + c$$

a. $f(x) = a\sqrt{x+1} + 3$ **1A**

Substitute (0, 5) into $f(x)$.

$$a\sqrt{0+1} + 3 = 5, a = 2$$

$$f(x) = 2\sqrt{x+1} + 3$$
 1A

b. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} m & 0 \\ 0 & n \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} p \\ q \end{bmatrix}$

$$f(x) = 2\sqrt{x+1} + 3, g(x) = -\sqrt{2x-1} - 1 = -\sqrt{2\left(x - \frac{1}{2}\right)} - 1$$

1. Dilate by a factor of $\frac{1}{2}$ from the x -axis. $f_1(x) = \sqrt{x+1} + \frac{3}{2}$

2. Reflect in the x -axis. $f_2(x) = -\sqrt{x+1} - \frac{3}{2}$

3. Dilate by a factor of $\frac{1}{2}$ from the y -axis. $f_3(x) = -\sqrt{2x+1} - \frac{3}{2} = -\sqrt{2\left(x + \frac{1}{2}\right)} - \frac{3}{2}$

4. Translate 1 unit to the right and a $\frac{1}{2}$ unit up. $g(x) = -\sqrt{2\left(x - \frac{1}{2}\right)} - 1$

$$n = -\frac{1}{2} \text{ (from 1 and 2), } m = \frac{1}{2} \text{ (from 3), } p = 1 \text{ and } q = \frac{1}{2} \text{ (from 4)}$$

A correct method **1M**, 2 correct **2H**, all correct **3A**

OR

$$x' = mx + p, x = \frac{x' - p}{m}$$

$$y' = ny + q, y = \frac{y' - q}{n}$$

$$y' = 2n\sqrt{\frac{x' - p}{m}} + 1 + 3n + q$$

$$y' = 2n\sqrt{\frac{x'}{m} + 1 - \frac{p}{m}} + 3n + q$$

$$= -\sqrt{2x-1} - 1$$

$$2n = -1, n = -\frac{1}{2}, \frac{1}{m} = 2, m = \frac{1}{2}, 1 - 2p = -1, p = 1, -\frac{3}{2} + q = -1, q = \frac{1}{2}$$

A correct method **1M**, 2 correct **2H**, all correct **3A**

Question 7

$$X \sim \text{Bi}\left(10, \frac{1}{10}\right)$$

$$\Pr(X \leq 2) = \Pr(X = 0) + \Pr(X = 1) + \Pr(X = 2) \quad \mathbf{1M}$$

$$= \binom{10}{0} \left(\frac{1}{10}\right)^0 \left(\frac{9}{10}\right)^{10} + \binom{10}{1} \left(\frac{1}{10}\right)^1 \left(\frac{9}{10}\right)^9 + \binom{10}{2} \left(\frac{1}{10}\right)^2 \left(\frac{9}{10}\right)^8 \quad \mathbf{1M}$$

$$= \left(\frac{9}{10}\right)^8 \left(\left(\frac{9}{10}\right)^2 + 10 \left(\frac{1}{10}\right) \left(\frac{9}{10}\right) + 45 \left(\frac{1}{10}\right)^2 \right)$$

$$= \left(\frac{9}{10}\right)^8 \left(\frac{81}{100} + \frac{9}{10} + \frac{45}{100} \right)$$

$$= \left(\frac{216}{100}\right) \left(\frac{9}{10}\right)^8$$

$$= \left(\frac{54}{25}\right) \left(\frac{9}{10}\right)^8 \quad \mathbf{1A}$$

OR

$$\Pr(X \leq 2) = \Pr(X = 0) + \Pr(X = 1) + \Pr(X = 2) \quad \mathbf{1M}$$

$$= \binom{10}{0} \left(\frac{1}{10}\right)^0 \left(\frac{9}{10}\right)^{10} + \binom{10}{1} \left(\frac{1}{10}\right)^1 \left(\frac{9}{10}\right)^9 + \binom{10}{2} \left(\frac{1}{10}\right)^2 \left(\frac{9}{10}\right)^8 \quad \mathbf{1M}$$

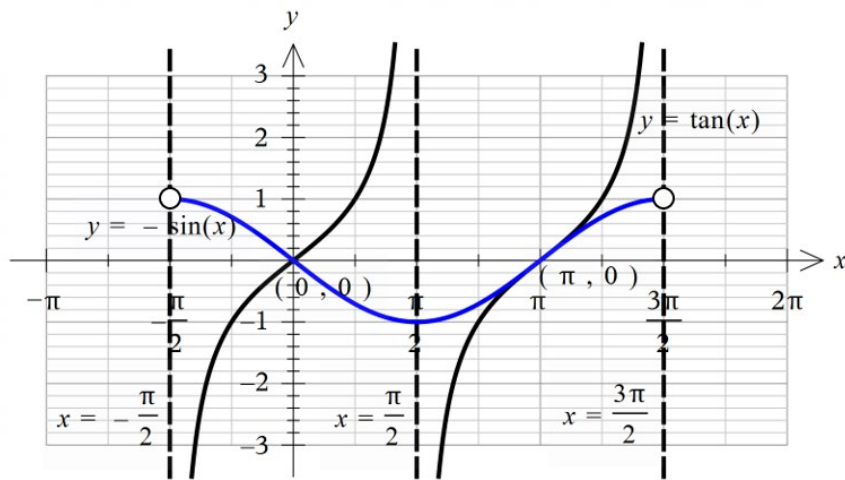
$$= \left(\frac{9}{10}\right)^8 \left(\left(\frac{9}{10}\right)^2 + 10 \left(\frac{1}{10}\right) \left(\frac{9}{10}\right) + 45 \left(\frac{1}{10}\right)^2 \right)$$

$$= \left(\frac{9}{10}\right)^{10} \left(1 + \frac{10}{9} + \frac{5}{9} \right)$$

$$= \left(\frac{8}{3}\right) \left(\frac{9}{10}\right)^{10} \quad \mathbf{1A}$$

Question 8

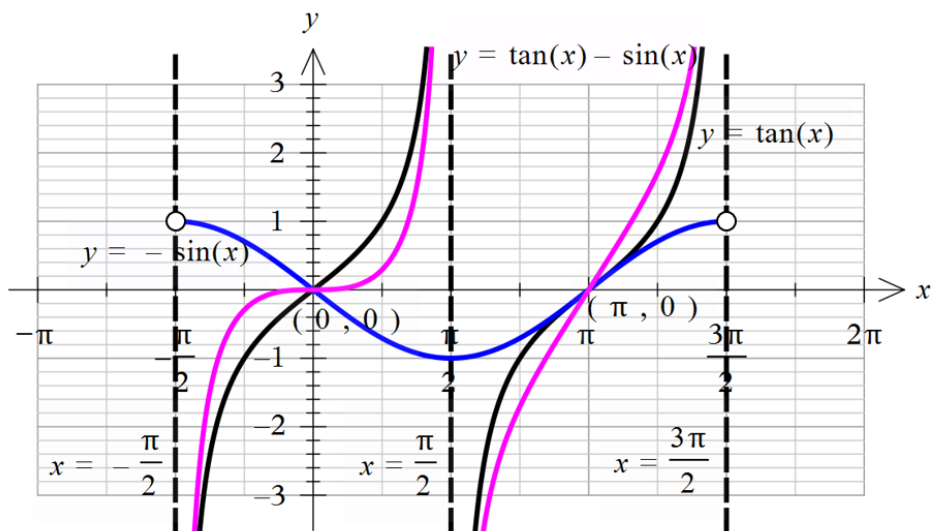
a. $y = \tan(x)$ for $x \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right) \setminus \left\{\frac{\pi}{2}\right\}$



Shape 1A

Asymptotes and intercepts labelled 1A

b.



Evidence of addition of ordinates either graphically or a table of values. 1M

x	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$
$\tan(x)$	undefined	-1	0	1	undefined	-1	0	1	undefined
$-\sin(x)$	undefined	$\frac{\sqrt{2}}{2}$	0	$-\frac{\sqrt{2}}{2}$	-1	$-\frac{\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	undefined
$\tan(x) - \sin(x)$	undefined	$-1 + \frac{\sqrt{2}}{2}$	0	$1 - \frac{\sqrt{2}}{2}$	undefined	$-1 - \frac{\sqrt{2}}{2}$	0	$1 + \frac{\sqrt{2}}{2}$	undefined

Correct graph 1A

c. $x = 0, x = \pi$ 1A

Question 9

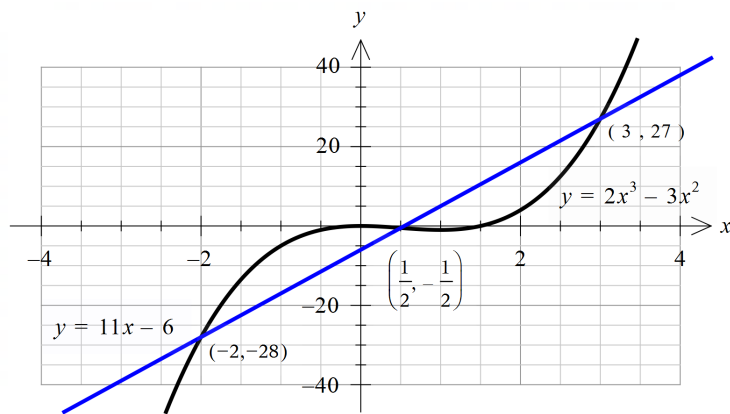
a. $x^2(2x-3) = 11x-6$

$$2x^3 - 3x^2 - 11x + 6 = 0 \quad \mathbf{1M}$$

$$(x+2)\left(x-\frac{1}{2}\right)(2x-6) = 0$$

$$x = 3 \quad \mathbf{1A}$$

b. Area enclosed by the graphs of $y = 2x^3 - 3x^2$ and $y = 11x - 6$



The bounded areas are equal.

$$\text{Area} = 2 \int_{-2}^{\frac{1}{2}} \left((2x^3 - 3x^2) - (11x - 6) \right) dx \quad \mathbf{1M}$$

$$= 2 \left[\frac{1}{2}x^4 - x^3 - \frac{11}{2}x^2 + 6x \right]_{-2}^{\frac{1}{2}}$$

$$= 2 \left(\left(\frac{1}{32} - \frac{1}{8} - \frac{11}{8} + 3 \right) - (8 + 8 - 22 - 12) \right) \quad \mathbf{1M}$$

$$= 2 \left(\left(-\frac{47}{32} + 3 \right) + 18 \right)$$

$$= 2 \left(-\frac{47}{32} + \frac{672}{32} \right)$$

$$= \frac{625}{16} = 39.0625 = 39\frac{1}{16} \quad (\text{any correct form})$$

1A

Question 10

$$f : (-\infty, 1) \rightarrow \mathbb{R}, f(x) = \frac{2}{(x-1)^2} - \frac{20}{9}$$

a. Let $y = \frac{2}{(x-1)^2} - \frac{20}{9}$.

Inverse swap x and y .

$$x = \frac{2}{(y-1)^2} - \frac{20}{9} \quad \mathbf{1M}$$

$$x + \frac{20}{9} = \frac{2}{(y-1)^2}$$

$$\left(x + \frac{20}{9}\right)(y-1)^2 = 2$$

$$(y-1)^2 = \frac{2}{\left(x + \frac{20}{9}\right)}$$

$$y = \pm \sqrt{\frac{2}{\left(x + \frac{20}{9}\right)}} + 1$$

$$y = -\sqrt{\frac{2}{\left(x + \frac{20}{9}\right)}} + 1 \text{ as range is } (-\infty, 1).$$

$$f^{-1}(x) = -\sqrt{\frac{2}{\left(x + \frac{20}{9}\right)}} + 1 = -\sqrt{\frac{18}{9x+20}} + 1 \quad \mathbf{1A}$$

$$\text{Domain } \left(-\frac{20}{9}, \infty\right) \quad \mathbf{1A}$$

b. Solve $\frac{2}{(x-1)^2} - \frac{20}{9} = x$ for x . **1M**

$$\frac{2}{(x-1)^2} = x + \frac{20}{9}$$

$$2 = \left(x + \frac{20}{9}\right)(x-1)^2$$

$$x^3 + \frac{20}{9}x^2 - 2x^2 - \frac{40}{9}x + x + \frac{20}{9} - 2 = 0$$

$$x^3 + \frac{20}{9}x^2 - 2x^2 - \frac{40}{9}x + x + \frac{20}{9} - 2 = 0$$

$$x^3 + \frac{2}{9}x^2 - \frac{31}{9}x + \frac{2}{9} = 0 \quad \mathbf{1M}$$

$$(x+2)\left(x^2 - \frac{16}{9}x + \frac{1}{9}\right) = 0$$

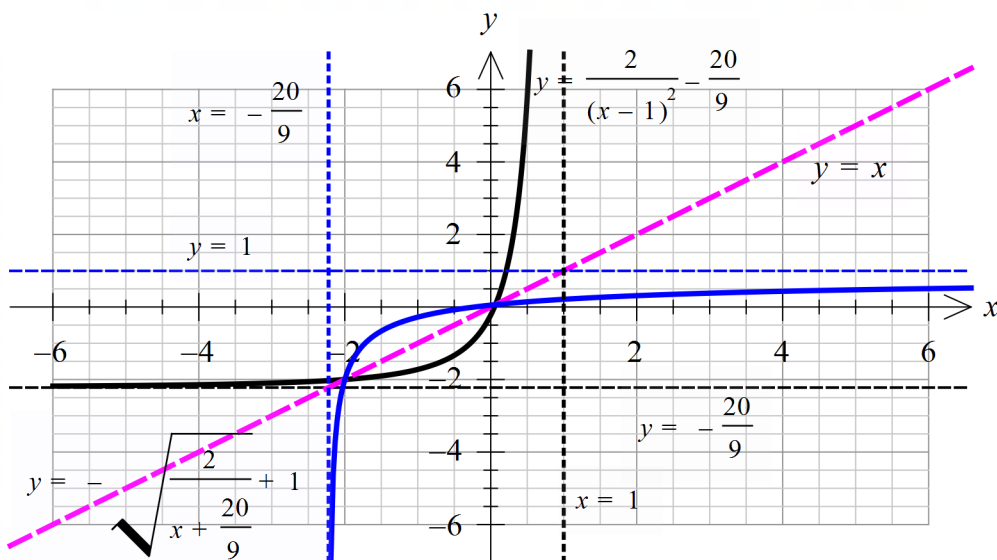
$$(x+2)(9x^2 - 16x + 1) = 0$$

$$x = -2, (-2, -2) \quad \mathbf{1A}$$

$$x = \frac{16 \pm \sqrt{256 - 36}}{18} = \frac{16 \pm \sqrt{220}}{18} = \frac{8 \pm \sqrt{55}}{9}$$

Select the negative branch as the range is $(-\infty, 1)$.

$$x = \frac{8 - \sqrt{55}}{9}, \left(\frac{8 - \sqrt{55}}{9}, \frac{8 - \sqrt{55}}{9}\right) \quad \mathbf{1A}$$



END OF SOLUTIONS