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2020

Mathematical Methods

Trial Examination 2
(2 hours)

SECTION A Multiple-choice questions

Instructions for Section A

Answer **all** questions.
Choose the response that is **correct** for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Unless otherwise indicated, the diagrams in this examination are **not** drawn to scale.

Question 1 The graph of $y = x^2 + b$ is transformed to the graph of $y = ax^2 + \frac{b}{a}$.

The sequence of transformations is

A. Dilation from the y-axis by a factor of a, dilation from the x-axis by a factor of a

B. Dilation from the y-axis by a factor of a^{-1}, dilation from the x-axis by a factor of a^{-1}

C. Dilation from the y-axis by a factor of a, dilation from the x-axis by a factor of a^{-1}

D. Dilation from the y-axis by a factor of a^{-1}, dilation from the x-axis by a factor of a

E. Dilation from the y-axis by a factor of a, dilation from the x-axis by a factor of a, translation by ab in in the positive x-direction

Question 2 Given $1 + x^n = (1+x)(a_0 + a_1x + a_2x^2 + a_3x^3 + ... + a_{n-1}x^{n-1})$ where $x \in R \setminus \{0\}$, **odd** integer $n > 2$ and $a_0, a_1, a_2, a_3, ..., a_{n-1}$ are coefficients, the sum of the coefficients equals

A. 0 only

B. 1 only

C. 0 or $n-1$

D. 1 or n

E. 0 or $n+1$

Question 3 Consider $y = f(x)$. If the area bounded by $y = \alpha f(x)$ and the x-axis equals the area bounded by $y = f\left(\frac{x-b}{\beta}\right)$ and the x-axis, where $b, \alpha, \beta \in R \setminus \{0\}$, then

A. $\alpha + \beta^{-1} = 0$

B. $\alpha^{-1} - \beta = 0$

C. $\alpha^{-1} + \beta = 0$

D. $\alpha^{-1} - \beta^{-1} = 0$

E. $\alpha + b\beta^{-1} = 0$

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Question 4 $5^{\log_a b}$ can be written as

A. $b^{-\log_5 a}$

B. $\left(b^{-\log_5 a}\right)^{-1}$

C. $b^{(\log_5 a)^{-1}}$

D. $b^{(-\log_5 a)^{-1}}$

E. $b^{-\log_a 5}$

Question 5 The graph of $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ and the graph of $x = a_0 + a_1 y + a_2 y^2 + a_3 y^3$, where $a_0, a_1, a_2, a_3 \in R$, have

A. only one intersection

B. only two intersections

C. only three intersections

D. no more than five intersections

E. a maximum of nine intersections

Question 6 Consider $f(x) = (x+1)^n$ and $x \in [0, 1]$. The average value of $f(x)$ equals the average rate of change of $f(x)$ with respect to x over the interval $[0, 1]$ when n is closest to

A. 0.53

B. 1.53

C. 2.53

D. 3.81

E. 3.82

Question 7 The area of the regions bounded by the curve $y = \frac{1}{3}x(4x^2 - 1)$ and its inverse is closest to

A. $\frac{4}{3}$

B. $\frac{5}{4}$

C. $\frac{6}{5}$

D. $\frac{7}{6}$

E. $\frac{8}{7}$

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Question 8 Consider the equation $mx = \sin\left(\frac{x}{m}\right)$ for $m \in R \setminus \{0\}$.

The number of solutions for x **cannot** be

A. 1

B. 2

C. 3

D. 5

E. 15

Question 9 Consider the equation $a\cos(nx) = \frac{1}{b}$ for $a \geq b > 1$ and $n \in R \setminus \{0\}$.

The **sum** of the values of x satisfying the equation is

A. 0

B. 1

C. ab

D. bn

E. ∞ or undefined

Question 10 Given $f(a) = b$, $f'(a) = b$ for $a, b \in R \setminus \{0\}$ and $g(x) = f^{-1}(x)$, $g'(b) =$

A. a

B. a^{-1}

C. b^{-1}

D. b

E. $a^{-1}b$

Question 11 Given $b \in R$ and $c \in R^+$, the area of the region(s) bounded by $y = (1-x)(x^2 + bx + c)$ and the x-axis is defined if

A. $b > c$

B. $c > b$

C. $4c > b^2$

D. $b \leq -2\sqrt{c}$ or $b \geq 2\sqrt{c}$

E. $-2\sqrt{c} \leq b \leq 2\sqrt{c}$

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Question 12 Given $f(t+10)=f(t)$ and $f(5+a)=-f(5-a)$ for $t\in R$ and $0<a<5$, then $f(26)=$

A. $-f(34)$

B. $f(-26)$

C. $-f(6)$

D. $f(14)$

E. $-f(-4)$

Question 13 Consider a quadratic function with equation $y=f(x)$ and its transformation with equation $y=f(x+h)+k$ where $h,k\in R\setminus\{0\}$.
The gradient of the common tangent to the graphs of the two functions is

A. $-\dfrac{h}{k}$

B. $\dfrac{h}{k}$

C. $-\dfrac{k}{h}$

D. $\dfrac{k}{h}$

E. not determinable without more information

Question 14 The distance from the origin O to the curve $y=\log_e x$ is shortest when

A. $0.65x+\log_e x=0$

B. $x+1.54\log_e x=0$

C. $0.95x^2+\log_e x=0$

D. $x^2+\log_e x=0$

E. $x^2+1.05\log_e x=0$

Question 15 The probability density function of random variable X is given by

$$f(x)=\begin{cases} a & \text{for } 1 \le x < 2 \\ \left(\frac{b-a}{2}\right)x + 2a - b & \text{for } 2 \le x < 4 \\ 0 & \text{elsewhere} \end{cases}$$

The relationship between a and b is

A. $a+2b=1$

B. $2a+b=1$

C. $2b-a=0$

D. $a-2b=0$

E. $a+b=1$

Question 16 Given $\Pr(A)=\frac{1}{4}$, $\Pr(B)=\frac{1}{3}$ and $\Pr(A' \cap B')=\frac{7}{12}$, then $\Pr\left((A' \cup B)'\right)=$

A. $\frac{1}{12}$

B. $\frac{2}{3}$

C. $\frac{5}{6}$

D. $\frac{3}{4}$

E. $\frac{5}{18}$

Question 17 The probability distribution of random variable X is given by the table below.

X	1	2	3	4
$\Pr(X=x)$	0.48	ba^2	0.20	a^4

A possible value of a is

A. $-\sqrt{0.8}$

B. $\sqrt{0.8}$

C. -0.8

D. 0.8

E. -0.4

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Question 18 Each face of a die is marked with a different number out of 1, 2, 3, 4, 5 and 6 for three fair dice. The three dice are rolled on a table. When the dice come to rest, the **sum** of the numbers that appear on the tops and the sides of the dice is determined.
$\Pr(\textit{the sum is } 58) =$

A. $\frac{1}{12}$

B. $\frac{1}{24}$

C. $\frac{1}{36}$

D. $\frac{1}{108}$

E. $\frac{1}{216}$

Question 19 Given population proportion $p = 0.20$, and 5 random samples of size 100 are taken from this large population, the probability that at least 1 of the 5 samples have $\hat{p} < 0.22$ is closest to

A. 0.033

B. 0.328

C. 0.609

D. 0.891

E. 0.997

Question 20 A random sample of size n is taken from a large population with a population proportion of certain attribute given by $p = 0.80$.
Let random variable X be the number in the sample having the certain attribute and $\Pr(X \le 1) < 0.1$.
The minimum value of n is

A. 4

B. 12

C. 24

D. 36

E. 48

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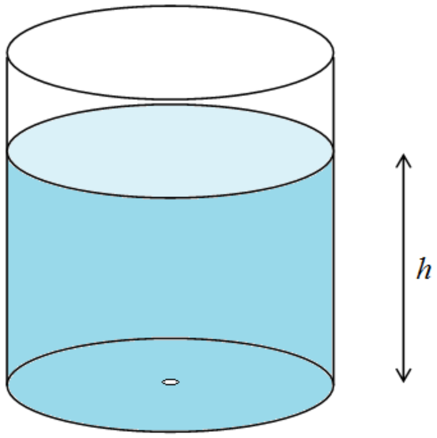
SECTION B

Instructions for Section B

Answer **all** questions in the spaces provided.
In all questions where a numerical answer is required, an exact value must be given unless otherwise stated.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this examination are **not** drawn to scale.

Question 1 (11 marks)

Water is drained from a vertical cylindrical tank of radius 1 m and height 2 m.
Let V (m^3) be the volume of water and h (m) the depth of water at time t (min) after the start of draining.
The tank is full at $t = 0$.



The depth of water is given by $h(t) = \frac{2}{9}\left(\frac{10}{t+1} - 1\right)$.

Correct all numerical answers to 3 decimal places unless stated otherwise.

a. Determine the time taken to empty the tank. 1 mark

b. Find the magnitude of the draining rate $\frac{dV}{dt}$ at time t. 2 marks

c. Determine the exact magnitude of the maximum and minimum rates of draining. 2 marks

d i. If water is added at the top at a constant rate of 1 m^3 per minute during draining (assume the same draining rate as in part b), determine the time t when the water level is at the lowest.

2 marks

d ii. Determine the volume of water in the tank at the lowest level. 2 marks

d iii. Find the time t when the tank is full again and begins to overflow. 1 mark

d iv. Find the constant rate of adding water to the tank such that it is full again at $t = 9$. 1 mark

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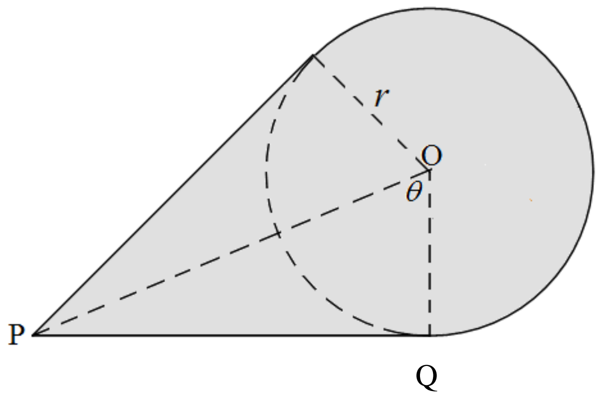
Question 2 (11 marks)

Two tangents are drawn from a point P to a circle centred at point O such that line segment $\mathrm{OP} = 1\,\mathrm{m}$.



The region enclosed by an arc of the circle and the two tangents is shaded.
Let $\angle\mathrm{POQ}$ be θ.

Area of a sector $= \frac{1}{2} \times \text{angle} \times r^2$ where angle is measured in radians

a. Show that the area of the shaded region is given by $A = \pi r^2 + r\sqrt{1-r^2} - r^2\theta$ with clear explanation.

2 marks

b. Find the values of r such that the region *enclosed by an arc of the circle and two tangents from point* P is defined.

1 mark

c i. From the diagram $r = \cos\theta$ or $\theta = \cos^{-1} r$. Find $\lim\limits_{r\to 1} \theta$. 1 mark

c ii. Hence find $\lim\limits_{r\to 1} A$. 2 marks

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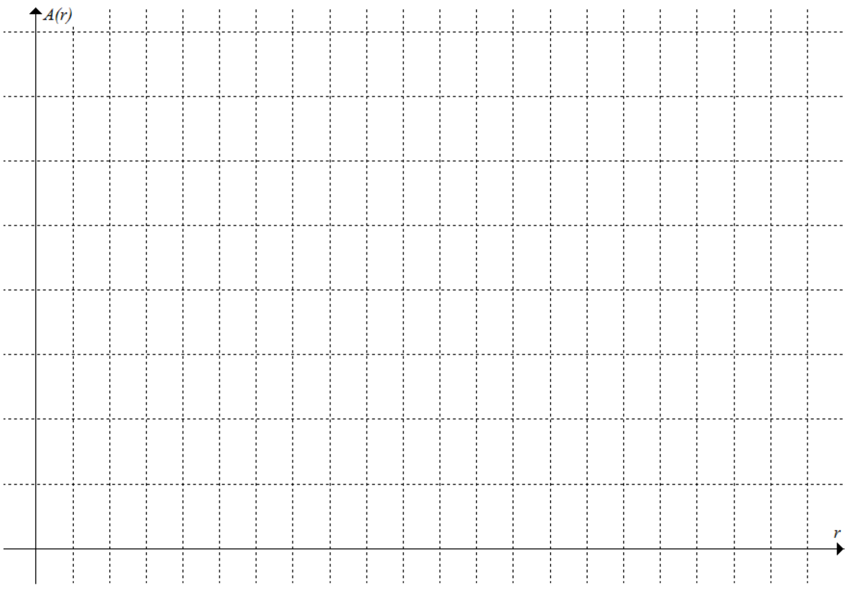
d i. Use CAS to find the derivative of $A(r)$ with respect to r. 1 mark

d ii. Hence show/explain that $A(r)$ is a strictly increasing function of r. 2 marks

e. Sketch accurately the graph of $A(r)$. 2 marks



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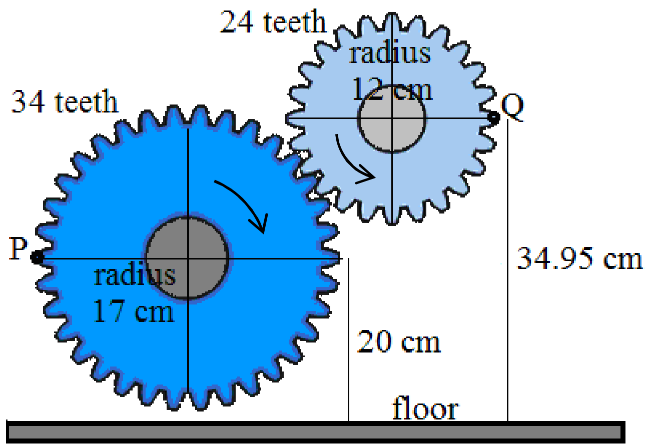
Question 3 (12 marks)

Two gear wheels are shown in the diagram below and two marked points are labeled as P and Q.
The small wheel has 24 teeth and a radius of 12 cm. The large wheel has 34 teeth and a radius of 17 cm.
The small wheel rotates at constant speed in the anticlockwise direction and the large wheel rotates in the clockwise direction.
At time t (seconds) the heights of P and Q above the floor are given by $h_{\mathrm{P}}(t)$ and $h_{\mathrm{Q}}(t)$ cm respectively.
$h_{\mathrm{P}}(0)=20$, $h_{\mathrm{Q}}(0)=34.95$, and point Q complete one revolution in 12 s, i.e. $h_{\mathrm{Q}}(12)=34.95$



a. Given $h_{\mathrm{Q}}(t)=12\sin(mt)+34.95$, show that $m\approx 0.5236$. 1 mark

b. Given $h_{\mathrm{P}}(t)=17\sin(nt)+20$, show that $n\approx 0.3696$ 2 marks

c. Expressed h_{P} in the form $h_{\mathrm{P}}(t)=a\cos(k(b-t))+c$ where $a, k, b, c\in R\setminus\{0\}$. 2 marks

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d. Determine $h_Q(5)$, correct to 2 decimal places. 1 mark

e. Determine the time t, correct to 2 decimal places, when $h_P = 30$ for the first time. 1 mark

f. Determine the time t, correct to 2 decimal places, when points P and Q are together for the first time.

3 marks

g. Determine the exact length of the time interval between two consecutive occasions when points P and Q are together.

2 marks

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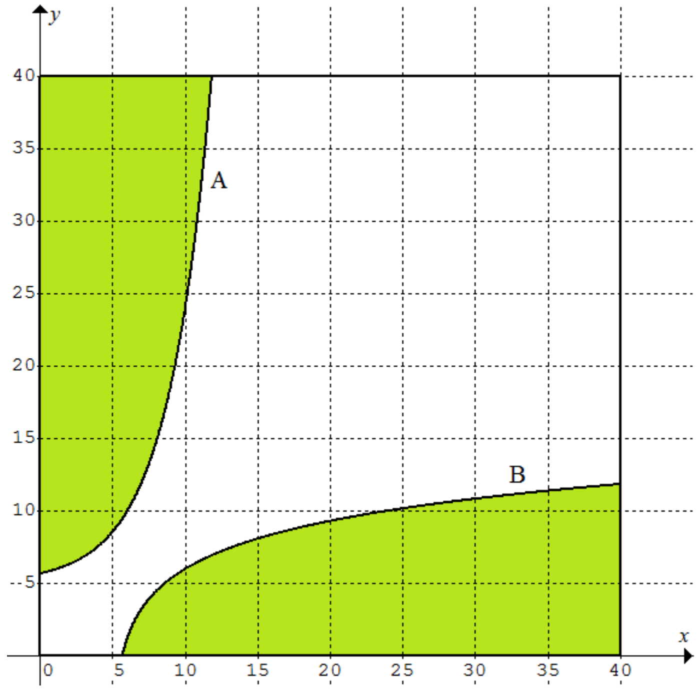
Question 4 (13 marks)

A sketch of a square block of land of side 40 m is shown below. Curve A is the transformation of $y = e^{x}$ and curve B is the inverse of curve A.



a. Let the equation of curve A be $y = 5e^{\left(\frac{x}{3}-2\right)} + 5$ after applying the transformation T to the equation $y = e^{x}$, where

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} c \\ d \end{bmatrix}\right) \text{ and } a, b, c, d \in R$$

Find the values of a, b, c and d. 2 marks

b. Write down the equation of curve B in the form $y = f(x)$. 1 mark

c. State the domain and range of curve B. 2 marks

d. Determine exactly the shortest distance between curve A and curve B. 3 marks

e. Determine the area of the ground in the square block of land between curve A and curve B.
Correct your answer to 2 decimal places. 2 marks

The largest circle that can be drawn on the ground in the square block of land between curve A and curve B has its centre at point (p, p) and touches curve A at point (a, b) where $p, a, b \in R^{+}$ satisfying

$$b = 5e^{\left(\frac{a}{3}-2\right)} + 5 \qquad (1)$$

$$(p-a)^2 + (p-b)^2 = (40-p)^2 \qquad (2)$$

$$\frac{p-b}{p-a} = -\frac{3}{5}e^{\left(2-\frac{a}{3}\right)} \qquad (3)$$

f. Show that equation (3) above is true. 2 marks

g. Solve the above simultaneous equations by CAS to determine the value of p.
Correct your answer to 2 decimal places. 1 mark

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Question 5 (13 marks)

The following graph of $y = f(t)$ is to be dilated from the horizontal axis by a factor k so that it can be used as a probability density function. The image after the dilation has equation $y = k\,f(t)$.

a. Show that $k = \dfrac{4}{41}$. 2 marks

The probability density function $y = k\,f(t)$ is used to find the probability of a 7:30 am (Monday to Friday) bus from Company A arriving late at the bus stop by t minutes. There is also a 7:35 am bus from Company B, which has the same probability density function $y = k\,f(t)$ for lateness.

b. If a student arrives at the bus stop at 7:35 am, what is the probability (correct to 4 decimal places) she catches the 7:30 am bus from Company A?

2 marks

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c. If the student arrives at the bus stop at 7:37 am, what is the probability (correct to 4 decimal places) she will miss either bus?

2 marks

d. Find the probability (correct to 4 decimal places) that the 7:30 am bus from Company A is late by more than 5 minutes at least two days in a week from Monday to Friday.

1 mark

e. Show that the mean late time of the 7:30 am bus from Company A is $\frac{575}{123}$ min. 2 marks

f. Calculate the probability (correct to 4 decimal places) the 7:30 am bus from Company A is late for more than $\frac{575}{123}$ min. 1 mark

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g. A survey of the lateness of the 7:30 am bus (Monday to Friday) over 4 weeks is to be carried out by Company A.
Find the probability (correct to 4 decimal places) that 12 out of the 20 days the 7:30 am bus from Company A is late for more than $\frac{575}{123}$ min. 1 mark

h. Company B did a similar survey of the 7:35 am bus (Monday to Friday) over 4 weeks and found that on 11 out of 20 days the 7:35 am bus was late for more than $\frac{575}{123}$ min.
Determine the approximate 95% confidence interval for the proportion of week days the 7:35 am bus was late for more than $\frac{575}{123}$ min, correct to 4 decimal places.
Interpret the approximate 95% confidence interval in this context. 2 marks

End of Examination 2

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