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2020 Mathematical Methods Trial Exam 2 Solutions

SECTION A – Multiple-choice questions

1	2	3	4	5	6	7	8	9	10
B	B	D	C	E	B	A	B	A	C

11	12	13	14	15	16	17	18	19	20
D	A	C	D	B	A	E	C	E	A

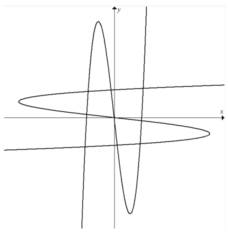
Q1 $y = ax^2 + \frac{b}{a}$, $ay = (ax)^2 + b$ B

Q2 $a_0 = 1$, $a_1 = -1$, $a_2 = 1$, $a_3 = -1$, …, $a_{n-1} = 1$
.: sum $= 1$ B

Q3 $\alpha = \beta$, .: $\alpha^{-1} = \beta^{-1}$ D

Q4 $5^{\log_a b} = 5^{\frac{\log_5 b}{\log_5 a}} = \left(5^{\log_5 b}\right)^{\frac{1}{\log_5 a}} = b^{\frac{1}{\log_5 a}} = b^{(\log_5 a)^{-1}}$ C

Q5 E



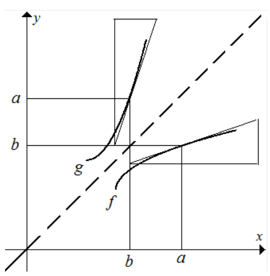
Q6 $\frac{2^n - 1}{1-0} = \frac{\int_0^1 (x+1)^n \, dx}{1-0}$, $n \approx 1.53$ B

Q7 $A = 4 \times \int_0^1 \left(x - \frac{1}{3} x\left(4x^2 - 1\right) \right) dx = \frac{4}{3}$ A

Q8 $y = \sin\left(\frac{x}{m}\right)$ is an odd function. The number of intersections it makes with $y = mx$ cannot be 2. B

Q9 $y = a\cos(nx)$ is an even function. When $b > 1$, $0 < \frac{1}{b} < 1$.
If p is a solution, then $-p$ is also a solution. .: Sum $= 0$ A

Q10 $g'(b) = \frac{1}{f'(a)} = \frac{1}{b} = b^{-1}$ C



Q11 $y = (1-x)\left(x^2 + bx + c\right)$ has three x-intercepts.
.: $\Delta = b^2 - 4c \ge 0$, $b \le -2\sqrt{c}$ or $b \ge 2\sqrt{c}$. D

Q12 $f(t+10) = f(t)$ has a period of 10.
Given $f(5+a) = -f(5-a)$ for $0 < a < 5$, .: $f(6) = -f(4)$
Hence $f(26) = -f(34)$. A

Q13 If the common tangent touches $y = f(x)$ at (a, b), then it touches $y = f(x+h) + k$ at $(a-h, b+k)$.
Gradient of the common tangent $= \frac{(b+k) - b}{(a-h) - a} = -\frac{k}{h}$ C

Q14 Let $P(x, y)$ on the curve be the point closest to O.
Gradient $OP = \frac{y}{x}$, gradient of tangent to curve at $P = \frac{1}{x}$.
At P, $\frac{y}{x} \times \frac{1}{x} = -1$, $x^2 + y = 0$, .: $x^2 + \log_e x = 0$ D

Q15 $a \times 1 + \int_2^4 \left(\left(\frac{b-a}{2} \right) x + 2a - b \right) dx = 1$, .: $2a + b = 1$ B

Q16 $\Pr(B') = \Pr(A \cap B') + \Pr(A' \cap B')$
$\Pr(A \cap B') = \Pr(B') - \Pr(A' \cap B') = \frac{2}{3} - \frac{7}{12} = \frac{1}{12}$
$\Pr\left(\left(A' \cup B\right)' \right) = \Pr(A \cap B') = \frac{1}{12}$ A

Q17 E

Q18 Sum of hidden numbers $= 63 - 58 = 5$
Possibilities: $(113), (131), (311), (122), (212), (221)$
$\Pr(58) = \frac{6}{6 \times 6 \times 6} = \frac{1}{36}$ C

Q19 $E(\hat{P}) = 0.20$, $\text{sd}(\hat{P}) = \sqrt{\frac{0.20 \times 0.80}{100}} = 0.04$
$\Pr(\hat{P} < 0.22) \approx 0.6915$
$\Pr(\text{at least one}) = 1 - \Pr(\text{none}) \approx 1 - 0.0028 = 0.9972$ E

Q20 $\Pr(X \le 1) < 0.1$
$\Pr(X = 0) + \Pr(X = 1) < 0.1$
$(0.2)^n + n\left(0.2^{n-1}\right)(0.8) < 1$, .: $n \ge 4$ A

SECTION B

Q1a $h=\frac{2}{9}\left(\frac{10}{t+1}-1\right)=0$, $t=9$, time taken $=9.000$ min.

Q1b $V(t)=\pi\times 1^2\times\frac{2}{9}\left(\frac{10}{t+1}-1\right)$, $\frac{dV}{dt}=-\frac{2\pi}{9}\times\frac{10}{(t+1)^2}$

$\left|\frac{dV}{dt}\right|=\frac{20\pi}{9(t+1)^2}$ m^3 min^{-1}

Q1c Max $=\frac{20\pi}{9}$ m^3 min^{-1}, min $=\frac{2\pi}{90}$ m^3 min^{-1}

Q1di At lowest level, $\frac{dV}{dt}=1-\frac{20\pi}{9(t+1)^2}=0$, $(t+1)^2=\frac{20\pi}{9}$

$t\approx 1.642$ min.

Q1dii $V(t)=1\times t+\frac{2\pi}{9}\left(\frac{10}{t+1}-1\right)$,

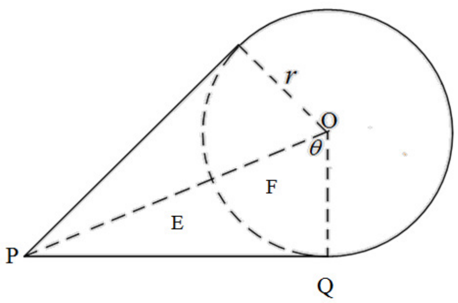
$V(1.642)\approx 1.642+\frac{2\pi}{9}\left(\frac{10}{1.642+1}-1\right)\approx 3.586\,\text{m}^3$

Q1diii $V(t)=1\times t+\frac{2\pi}{9}\left(\frac{10}{t+1}-1\right)=2\pi$, $t\approx 5.981$ min.

Q1div Let r m^3 min^{-1} be the constant rate,

$r\times 9+\frac{2\pi}{9}\left(\frac{10}{9+1}-1\right)=2\pi$, $r=\frac{2\pi}{9}\approx 0.698$

Q2a



$PQ=\sqrt{1-r^2}$,

Area E $=$ area $\Delta POQ-$ sector F $=\frac{1}{2}r\sqrt{1-r^2}-\frac{1}{2}r^2\theta$

$A=$ area of circle $+2\times$ area E $=\pi r^2+r\sqrt{1-r^2}-r^2\theta$

Q2b $0<r<1$

Q2ci $\lim_{r\to 1}\theta=0$

Q2cii $\lim_{r\to 1}A=\pi\times 1^2+1\sqrt{1-1^2}-1^2\times 0=\pi$

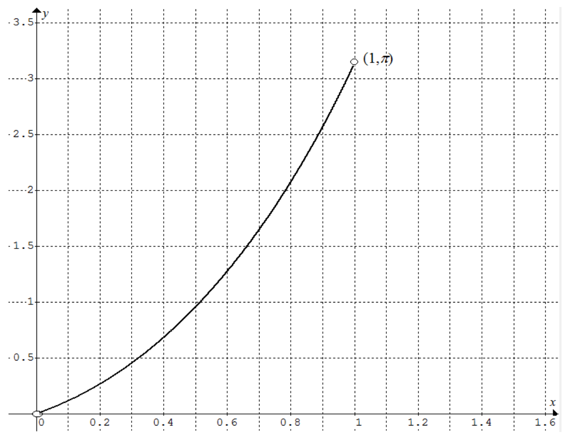
Q2di $\frac{dA}{dr}=2\pi r+\sqrt{1-r^2}-2r\cos^{-1}r$, $0<r<1$

Q2dii $0<r<1$, $\therefore\ 0<\theta<\frac{\pi}{2}$,

$\therefore\ \pi r+\sqrt{1-r^2}<\frac{dA}{dr}<2\pi r+\sqrt{1-r^2}$, $\therefore\ \frac{dA}{dr}>0$,

$\therefore\ A(r)$ is a strictly increasing function in $0<r<1$

Q2e



Q3a $\frac{2\pi}{m}=12$, $m\approx 0.5236$

Q3b $T_P=12\times\frac{34}{24}=17$, $\frac{2\pi}{n}=17$, $n\approx 0.3696$

Q3c $h_P(t)=17\sin(0.3696t)+20=17\cos\left(\frac{\pi}{2}-0.3696t\right)+20$

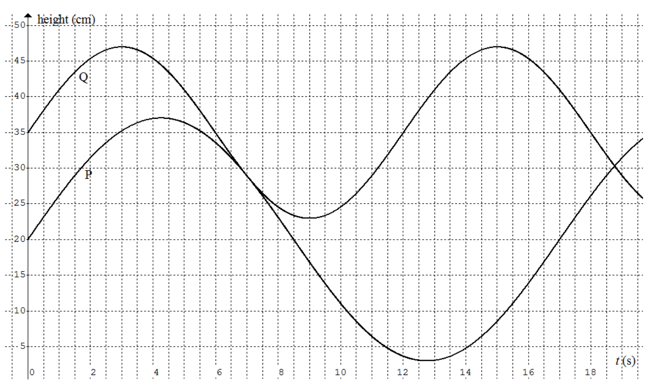
$=17\cos(0.3696(4.25-t))+20$

Q3d $h_Q(5)=12\sin(0.5236\times 5)+34.95\approx 40.95$

Q3e $h_P(t)=17\sin(0.3696t)+20=30$

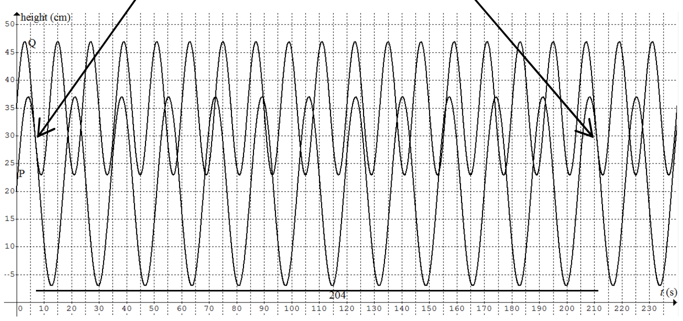
$t=\frac{1}{0.3696}\sin^{-1}\left(\frac{10}{17}\right)\approx 1.70$

Q3f $t\approx 7.03$ (± 0.02)



Q3g $LCM(T_P, T_Q) = LCM(12, 17) = 204$ s or by graphs (see below)

First time Second time



Q4a $y = 5e^{\frac{x}{3}-2} + 5$, $\frac{y}{5} - 1 = e^{\frac{x}{3}-2}$

Sequence of transformations: Right 2, up 1, dilate from y-axis by 3, dilate from x-axis by 5.

$c = -2$, $d = -1$, $a = 3$, $b = 5$

Q4b Equation of curve B: $\frac{x}{5} - 1 = e^{\frac{y}{3}-2}$, $\frac{y}{3} - 2 = \log_e\left(\frac{x}{5} - 1\right)$

$y = 3\log_e\left(\frac{x}{5} - 1\right) + 6$

Q4c Range of A: When $x = 0$, $y = 5e^{-2} + 5$

Domain of B is the range of A, i.e. $\left[5e^{-2} + 5, 40\right]$

Curve B: When $x = 40$, $y = 3\log_e\left(\frac{40}{5} - 1\right) + 6 = 3\log_e 7 + 6$

Range of B is $\left[0, 3\log_e 7 + 6\right]$.

Q4d Curve A: $y = 5e^{\frac{x}{3}-2} + 5$. Let $\frac{dy}{dx} = \frac{5}{3}e^{\frac{x}{3}-2} = 1$, $\frac{x}{3} - 2 = \log_e\frac{3}{5}$

$x = 3\log_e\frac{3}{5} + 6$, $y = 3 + 5 = 8$, $\left(3\log_e\frac{3}{5} + 6,\ 8\right)$

Curve B: $\left(8,\ 3\log_e\frac{3}{5} + 6\right)$

Shortest distance

$= \sqrt{\left(8 - 3\log_e\frac{3}{5} - 6\right)^2 + \left(3\log_e\frac{3}{5} + 6 - 8\right)^2} = \sqrt{2}\left(2 - 3\log_e\frac{3}{5}\right)$

Q4e Area between curve A and curve B

$= 40^2 - 2\int_{5e^{-2}+5}^{40}\left(3\log_e\left(\frac{x}{5} - 1\right) + 6\right)dx \approx 977.30\ \text{m}^2$

Q4f At (a, b), $\frac{dy}{dx} = \frac{5}{3}e^{\frac{x}{3}-2} = \frac{5}{3}e^{\frac{a}{3}-2}$, gradient from (p, p) to

$(a, b) = \frac{p-b}{p-a}$, $\therefore$ $\frac{5}{3}e^{\frac{a}{3}-2} \times \frac{p-b}{p-a} = -1$, $\frac{p-b}{p-a} = -\frac{3}{5}e^{2-\frac{a}{3}}$

Q4g $p = 25.17$

Q5a $k \times$ area under graph $= 1$

$k\left(\frac{1}{2}(2\times 0.5 + 3\times 3) + \frac{1}{2}(0.5 + 3)3\right) = 1$, $k = \frac{4}{41}$

Q5b $\Pr(\text{bus A late by} > 5) = \frac{4}{41}\left(\frac{1}{2}\times 3\times 3\right) \approx 0.4390$

Q5c $\Pr(\text{miss either bus})$

$= \Pr(\text{bus A late by} < 7) + \Pr(\text{bus B late by} < 2)$

$- \Pr(\text{bus A late by} < 7 \text{ and bus B late by} < 2)$

$= \left(1 - \frac{4}{41}\left(\frac{1}{2}\times 1\times 1\right)\right) + \frac{4}{41}\left(\frac{1}{2}\times 0.5\times 2\right)$

$- \left(1 - \frac{4}{41}\left(\frac{1}{2}\times 1\times 1\right)\right) \times \frac{4}{41}\left(\frac{1}{2}\times 0.5\times 2\right) \approx 0.9536$

Q5d Binomial: $p \approx 0.4390244$, $n = 5$

$\Pr(X \geq 2) \approx 0.7271$

Q5e

$$f(t) = \begin{cases} \frac{t}{4} & 0 \leq t \leq 2 \\ \frac{5t-7}{6} & 2 < t \leq 5 \\ 8 - t & 5 < t \leq 8 \\ 0 & \text{elsewhere} \end{cases}$$

Mean

$= \int_0^2 t \times kf(t)\,dt + \int_2^5 t \times kf(t)\,dt + \int_5^8 t \times kf(t)\,dt$

$= \frac{8}{123} + \frac{81}{41} + \frac{108}{41} = \frac{575}{123} \approx 4.674797$

Q5f $\int_{4.674797}^{8} kf(t)\,dt \approx 0.5299$

Q5g Binomial: $n = 20$, $p \approx 0.529907$

$\Pr(X = 12) \approx 0.1473$

Q5h $\left(\frac{11}{20} - 1.96\sqrt{\frac{\frac{11}{20}\times\frac{9}{20}}{20}},\ \frac{11}{20} + 1.96\sqrt{\frac{\frac{11}{20}\times\frac{9}{20}}{20}}\right)$

$\approx (0.3320, 0.7680)$

Let p be the long term proportion of week days the 7:35 am bus of Company B is late for more than $\frac{575}{123}$ min.

If many similar surveys were carried out and the 95% confidence interval calculated in each survey, 95% of them would have p in the interval.

Please inform mathline@itute.com re conceptual and/or mathematical errors