

2020 Mathematical Methods Trial Exam 1 Solutions
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Q1a $y = -x^2 + 4$

Q1b $y = b$, $x^2 = 4 - b$, $x = \pm\sqrt{4-b}$

Area $A = \frac{1}{2}b(4 + 2\sqrt{4-b}) = b(2 + \sqrt{4-b})$

Q1c Let $\frac{dA}{db} = (2 + \sqrt{4-b}) + b\left(\frac{-1}{2\sqrt{4-b}}\right) = 0$

$4\sqrt{4-b} + 2(4-b) - b = 0$, $4\sqrt{4-b} = 3b - 8$,

$16(4-b) = 9b^2 - 48b + 64$, $9b^2 - 32b = 0$, $b = \frac{32}{9}$

Q2ai

$$\begin{array}{r} x^4 - 4x^2 + 16 \\ x^2 + 4 \overline{) x^6 + 0x^4 + 0x^2 + 64} \\ \underline{x^6 + 4x^4} \\ -4x^4 + 0x^2 \\ \underline{-4x^4 - 16x^2} \\ 16x^2 + 64 \\ \underline{4x^2 + 64} \\ 0 \end{array}$$

$Q(x) = x^4 - x^2 + 4$, Remainder = 0

Q2aii $P(x) = (x^2 + 4)(x^4 - 4x^2 + 16)$
 $= (x^2 + 4)(x^4 + 8x^2 + 16 - 12x^2) = (x^2 + 4)((x^2 + 4)^2 - (2\sqrt{3}x)^2)$
 $= (x^2 + 4)(x^2 - 2\sqrt{3}x + 4)(x^2 + 2\sqrt{3}x + 4)$

Q2b The turning points are $(0, 4)$, $(\sqrt{3}, 1)$ and $(-\sqrt{3}, 1)$.

$\tan \theta = \pm \frac{4-1}{0-\sqrt{3}} = \mp\sqrt{3}$, $\theta = \frac{2\pi}{3}$ or $\frac{\pi}{3}$, $\therefore$ equilateral

Q3 $x' = -y$ and $y' = -x$

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \frac{1}{2} \left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -y \\ -x \end{bmatrix} \right) + \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} x-y \\ y-x \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{x-y}{2} - 1 \\ \frac{y-x}{2} + 1 \end{bmatrix} = \begin{bmatrix} \frac{x-y-2}{2} \\ \frac{y-x+2}{2} \end{bmatrix}$$

$\therefore X = \frac{x-y-2}{2}$, $Y = \frac{y-x+2}{2}$, $\therefore X+Y=0$

Q4a $5e^{-4x} + 2e^{-2x} - 3 = 0$, $(5e^{-2x} - 3)(5e^{-2x} + 1) = 0$

$\therefore 5e^{-2x} - 3 = 0$, $e^{2x} = \frac{5}{3}$, $x = \frac{1}{2} \log_e \frac{5}{3}$

Q4b $\log_4 x = \log_3 3 - \log_3 5 = \log_3 \frac{3}{5} = \frac{\log_{10} \frac{3}{5}}{\log_{10} 3}$

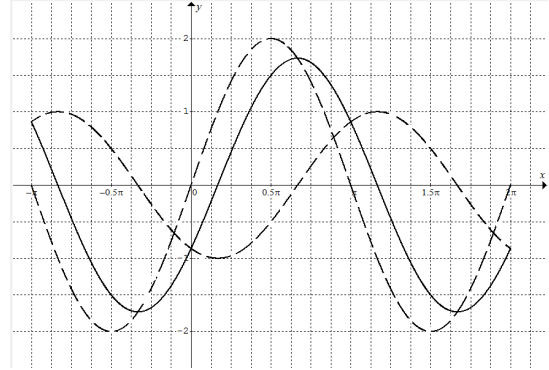
$\log_e x = \frac{\log_4 x}{\log_4 e} = \log_4 x \times \frac{\log_{10} 4}{\log_{10} e} = \frac{(\log_{10} \frac{3}{5})(\log_{10} 4)}{(\log_{10} 3)(\log_{10} e)}$

Q4c $y = Ae^{kx}$, $e = Ae^{ke}$ and $e^2 = Ae^{ke^2}$

$\frac{e^2}{e} = \frac{e^{ke^2}}{e^{ke}}$, $e = e^{ke^2-ke}$, $ke^2 - ke = 1$, $k = \frac{1}{e^2 - e}$

$A = \frac{e}{e^{ke}} = e^{1-ke} = e^{1-\frac{1}{e-1}} = e^{\frac{e-2}{e-1}}$

Q5a The two dotted curves



Q5b By addition of ordinates, sketch the solid curve

$y = 2 \sin x + \cos\left(x + \frac{5\pi}{6}\right)$.

From graph, $2 \sin x + \cos\left(x + \frac{5\pi}{6}\right) = 0$ at $x = -\frac{5\pi}{6}, \frac{\pi}{6}, \frac{7\pi}{6}$

Q5c $x = \frac{\pi}{6} + n\pi$ where n is an integer.

Q6a $f'(x) = \frac{\sin x(-\sin x) - \cos x(\cos x)}{\sin^2 x} = -\frac{1}{\sin^2 x}$

Q6b $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos^2 x}{\sin^2 x} dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1 - \sin^2 x}{\sin^2 x} dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\frac{1}{\sin^2 x} - 1 \right) dx$
 $= \left[-\frac{\cos x}{\sin x} - x \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \left(-\frac{\pi}{2} \right) - \left(-1 - \frac{\pi}{4} \right) = 1 - \frac{\pi}{4}$

Q7a $\Pr(A \cap B) = \Pr(A) - \Pr(A \cap B') = \frac{3}{5} - \frac{1}{4} = \frac{7}{20}$

Q7b $\Pr(A|B) = \frac{3}{5}$, $\frac{\Pr(A \cap B)}{\Pr(B)} = \frac{3}{5}$, $\therefore \Pr(B) = \frac{7}{12}$

$\therefore \Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B) = \frac{5}{6}$

Q7c $\Pr(A' \cap B') = \Pr(B') - \Pr(A \cap B') = \frac{5}{12} - \frac{1}{4} = \frac{1}{6}$

$\Pr(A')\Pr(B') = \frac{2}{5} \times \frac{5}{12} = \frac{1}{6}$, $\therefore A'$ and B' are independent.

Q8a A: $E(\hat{p}) \approx 0.36$, $sd(\hat{p}) \approx \sqrt{\frac{0.36 \times 0.64}{4}} = 0.24$,

Approx. 95% confidence interval $(0, 0.84)$

B: $E(\hat{p}) \approx 0.64$, $sd(\hat{p}) \approx \sqrt{\frac{0.64 \times 0.36}{144}} = 0.04$

Approx. 95% confidence interval $(0.56, 0.72)$

Better to choose A, because the chance of waiting longer than 5 minutes can be close to zero.

Q8b $\Pr(\text{waiting} > 5 \text{ min}) \approx \frac{1}{2} \times 0.36 + \frac{1}{3} \times 0.64 + \frac{1}{6} \times 0 \approx 0.4$

Q9a Given n is a positive odd integer, $\therefore n+2$ is also a positive

odd integer, $\therefore$ both $x^{\frac{1}{n}}$ and $x^{\frac{1}{n+2}}$ are odd functions

Given m is a positive even integer, $\therefore m+2$ is also a positive

even integer, $x^{\frac{m}{n}} = \left(x^{\frac{1}{n}}\right)^m$, $x^{\frac{m+2}{n+2}} = \left(x^{\frac{1}{n+2}}\right)^{m+2}$,

$\therefore$ both $x^{\frac{m}{n}}$ and $x^{\frac{m+2}{n+2}}$ are even functions $f(-x) = f(x)$

Given $m > n$, $2m > 2n$, $mn + 2m > mn + 2n$,

$$m(n+2) > n(m+2), \therefore \frac{m}{n} > \frac{m+2}{n+2}$$

$\therefore$ for $x \in (-1, 0)$ or $(0, 1)$, $x^{\frac{m}{n}} < x^{\frac{m+2}{n+2}}$, $\therefore x^{\frac{m+2}{n+2}} > x^{\frac{m}{n}}$

Q9b $y = x^{\frac{m}{n}}$ and $y = x^{\frac{m+2}{n+2}}$ intersect at $x = -1, 0, 1$

$$\begin{aligned} A &= \int_{-1}^0 \left(x^{\frac{m+2}{n+2}} - x^{\frac{m}{n}}\right) dx + \int_0^1 \left(x^{\frac{m+2}{n+2}} - x^{\frac{m}{n}}\right) dx = 2 \int_0^1 \left(x^{\frac{m+2}{n+2}} - x^{\frac{m}{n}}\right) dx \\ &= 2 \left[\frac{x^{\frac{m+2}{n+2}+1}}{\frac{m+2}{n+2}+1} - \frac{x^{\frac{m}{n}+1}}{\frac{m}{n}+1} \right]_0^1 = 2 \left(\frac{1}{\frac{m+2}{n+2}+1} - \frac{1}{\frac{m}{n}+1} \right) = 2 \left(\frac{n+2}{m+n+4} - \frac{n}{m+n} \right) \\ &= \frac{4(m-n)}{(m+n+4)(m+n)} \end{aligned}$$

$$\text{Q9c } A = \frac{4(m-n)}{(m+n+4)(m+n)} < \frac{4(m+n)}{(m+n+4)(m+n)} = \frac{4}{m+n+4}$$

$$\text{i.e. } 0 < A < \frac{4}{m+n+4}$$

As $n \rightarrow \infty$, $m \rightarrow \infty$ (since $m > n$), $\therefore \frac{4}{m+n+4} \rightarrow 0$, $\therefore A \rightarrow 0$

Please inform mathline@itute.com re conceptual and/or mathematical errors