

Mathematical Methods
EXAM 2 UNITS 3&4
2020 Written Trial Examination

Reading time: 15 minutes

Writing time: 2 hours

SOLUTIONS

SECTION A – Multiple-choice questions

Question 1

The set of values of k for which $2x^2 + (k+1)x + k = 0$ has two real solutions is

A. $k \in (-\infty, 3 + 2\sqrt{2})$

B. $k \in \mathbb{R}$

C. $k \in (3 - 2\sqrt{2}, \infty)$

D. $k \in (-\infty, 3 - 2\sqrt{2}) \cup (3 + 2\sqrt{2}, \infty)$

E. $k \in \{3 - 2\sqrt{2}, 3 + 2\sqrt{2}\}$

For the quadratic to have 2 real solutions, the discriminant must be greater than zero.

$$\Delta = b^2 - 4ac$$

$$(k+1)^2 - 4(2)k > 0$$

$$k^2 + 2k + 1 - 8k > 0$$

$$k^2 - 6k + 1 > 0$$

$$(k^2 - 6k + 9) - 9 + 1 > 0$$

$$(k-3)^2 - 8 > 0$$

$$(k-3-2\sqrt{2})(k-3+2\sqrt{2}) > 0$$

$$\therefore k \in (-\infty, 3 - 2\sqrt{2}) \cup (3 + 2\sqrt{2}, \infty)$$

Therefore, the answer is D.

Question 2

Let $g(x) = \frac{\log_e(ax)}{x^2}$, where a is non-zero real constant.

The derivative, $g'(x)$, is given by

A. $\frac{1}{2x^2}$

B. $\frac{2x\log_e(ax) - x}{x^4}$

C. $\frac{1 - 2\log_e(ax)}{x^3}$

D. $2x^2$

E. $x + 2x\log_e(ax)$

Using the quotient rule,

$$\text{Let } u(x) = \log_e(ax) \text{ and } v(x) = x^2$$

$$\text{So } u'(x) = \frac{1}{x} \text{ and } v'(x) = 2x$$

$$g'(x) = \frac{x^2 \left(\frac{1}{x} \right) - 2x(\log_e(ax))}{(x^2)^2}$$

$$g'(x) = \frac{x - 2x(\log_e(ax))}{x^4}$$

$$g'(x) = \frac{x(1 - 2\log_e(ax))}{x^4}$$

$$\therefore g'(x) = \frac{1 - 2\log_e(ax)}{x^3}$$

Therefore, the answer is C.

Question 3

A box contains five white tiles and six black tiles. Two tiles are drawn at random from the box without replacement. The probability that the tiles are the **same** colour is

A. $\frac{2}{11}$

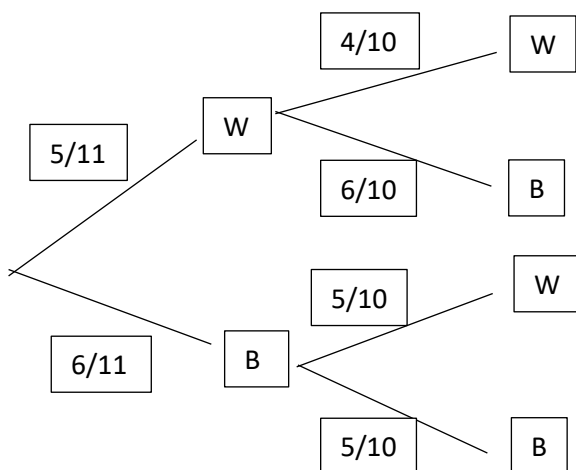
B. $\frac{5}{22}$

C. $\frac{3}{11}$

D. $\frac{5}{11}$

E. $\frac{6}{13}$

Using a tree diagram to model this sampling without replacement problem,



Two tiles the same colour means 2 whites or 2 blacks so,

$$\Pr(\text{same colour}) = \Pr(WW) + \Pr(BB)$$

$$\Pr(\text{same colour}) = \frac{5}{11} \times \frac{4}{10} + \frac{6}{11} \times \frac{5}{10} = \frac{5}{11}$$

Therefore, the answer is D.

Question 4

If $x - b$ is a factor of $3x^4 - 2x^3 - x^2$, then the value of b could be

A. $-\frac{1}{3}, 0, 1$

B. $0, 1$

C. $\frac{1}{3}, 0$

D. $0, \frac{1}{3}, 1$

E. $0, 1, 3$

Using the factor theorem and substituting in the value b , hence using the Null Factor Law,

$$\text{Let } f(x) = 3x^4 - 2x^3 - x^2$$

$$f(b) = 3b^4 - 2b^3 - b^2 = 0$$

$$b^2(3b^2 - 2b - 1) = 0$$

$$b^2(3b + 1)(b - 1) = 0$$

$$\therefore b = -\frac{1}{3}, 0, 1$$

Therefore, the answer is A.

Question 5

The simultaneous linear equations $y - (m + 2)x = 4$ and $my - 3x = k + 1$ have no solutions when

- A. $m = 3$ and $k = -13$ or $m = 1$ and $k \neq 3$
- B. $m = 3$ and $k \neq -13$ or $m \neq 1$ and $k \neq 3$
- C. $m = 1$ and $k = 3$ or $m = 1$ and $k = 3$
- D. $m = 1$ and $k = -13$ or $m \neq 1$ and $k \neq 3$
- E. $m = -3$ and $k \neq -13$ or $m = 1$ and $k \neq 3$

For simultaneous linear equations to have no solutions, they must have equal gradients but different y-intercepts. Restate equations as

$$y = (m + 2)x + 4 \quad \text{and} \quad y = \frac{3}{m}x + \frac{k + 1}{m}$$

Equating gradients

$$m + 2 = \frac{3}{m}$$

$$m^2 + 2m - 3 = 0$$

$$(m + 3)(m - 1) = 0$$

$$\therefore m = -3, 1$$

Checking y-intercepts

When $m = -3$,

$$4 = \frac{k + 1}{-3}$$

$$-12 = k + 1$$

$$\therefore k = -13 \text{ so we need } k \neq -13$$

When $m = 1$,

$$4 = \frac{k + 1}{1}$$

$$4 = k + 1$$

$$\therefore k = 3 \text{ so we need } k \neq 3$$

Therefore, the answer is E.

Question 6

Let $f(x) = \sqrt{4-x}$ and $g(x) = x^2$.

For $f(g(x))$ to exist, the domain and range of $g(x)$ respectively, must be changed to

A. $[-\infty, 0]$ and $[4, \infty]$

B. R and $[0, \infty]$

C. $[0, 4]$ and $[-2, 2]$

D. $[0, 2]$ and $[-4, 0]$

E. $[-2, 2]$ and $[0, 4]$

	Dom	Ran
$f(x)$	$(-\infty, 4]$	$[0, \infty)$
$g(x)$	R	$[0, \infty)$

For $f(g(x))$ to exist, the range of $g(x)$ must be a subset or equal to the domain of $f(x)$. The range of $g(x)$ must be restricted for $f(g(x))$ to exist so

	Dom	Ran
$f(x)$	$(-\infty, 4]$	$[0, \infty)$
$g(x)$	$[-2, 2]$	$[0, 4]$

Therefore, the answer is E.

Question 7

The point $A(4, -1)$ lies on the graph of the function $f(x)$. A transformation maps the graph of $f(x)$ to the graph of $g(x)$, where $g(x) = -3f(x+2)+1$.

If that same transformation maps the point A to the point P , the coordinates of the point P are

A. $(4, 2)$

B. $(2, 4)$

C. $(-2, 4)$

D. $(-4, 2)$

E. $(-1, 4)$

Looking at the series of transformations performed on $f(x)$ as

- dilation by a factor of 3 from the x -axis
- reflected in the x -axis
- translated -2 units horizontally
- translated +1 unit vertically

then the initial point at $A(4, -1)$ will now also transform as

$$(4, -1) \rightarrow (4, -3) \rightarrow (4, 3) \rightarrow (2, 3) \rightarrow (2, 4)$$

Therefore, the answer is B.

Question 8

A discrete random variable has a binomial distribution with a mean of 3.15 and a variance of 1.7325. The values of n (the number of independent trials) and p (the probability of success in each trial) are

A. $n = 9$ and $p = 0.45$

B. $n = 7$ and $p = 0.55$

C. $n = 7$ and $p = 0.45$

D. $n = 5$ and $p = 0.5$

E. $n = 7$ and $p = 0.65$

Set up two equations for mean and variance and solve simultaneously.

$$np = 3.15 \quad \text{and} \quad np(1-p) = 1.7325$$

$$1-p = \frac{1.7325}{3.15}$$

$$1-p = 0.55$$

$$\therefore p = 0.45$$

Substitute p

$$n(0.45) = 3.15$$

$$\therefore n = 7$$

Therefore, the answer is C.

Question 9

A six-sided die is loaded such that the chance of throwing a 1 is $\frac{x}{5}$, the chance of a 2 is $\frac{1}{5}$ and the chance of a 3 is $\frac{1}{5}(1+x)$. The chance of a 4, 5 or 6 is $\frac{1}{6}$ and the die is thrown twice. The probability of getting a sum of 6 on the dice is

A. $\frac{1}{75}(3x^2 + 11x + 8)$

B. $\frac{1}{30}(3x^2 + 11x + 8)$

C. $3x^2 + 11x + 8$

D. $\frac{1}{75}(6x^2 + 17x + 11)$

E. $\frac{1}{30}(6x^2 + 17x + 11)$

Using a lattice diagram to show all the possibilities for a sum of 6, we can get (1,5) or (5,1), (2,4) or (4,2) and (3,3).

$$\text{For } 6: (1,5) \text{ or } (5,1) \quad 2\left(\frac{x}{5} \times \frac{1}{6}\right) = \frac{2x}{30}$$

$$(2,4) \text{ or } (4,2) \quad 2\left(\frac{1}{5} \times \frac{1}{6}\right) = \frac{2}{30}$$

$$(3,3) \quad \frac{1}{5}(1+x) \times \frac{1}{5}(1+x) = \frac{1}{25}(1+x)^2$$

$$\text{Pr}(6) = \frac{2x}{30} + \frac{2}{30} + \frac{1}{25}(1+x)^2$$

$$\text{Pr}(6) = \frac{10x + 10 + 6(1+x)^2}{150}$$

$$\text{Pr}(6) = \frac{10x + 10 + 6 + 12x + 6x^2}{150}$$

$$\therefore \text{Pr}(6) = \frac{3x^2 + 11x + 8}{75} = \frac{1}{75}(3x^2 + 11x + 8)$$

Therefore, the answer is A.

Question 10

The average value of the function with the rule $f(x) = 2x^3 + 3x$ over the interval $[0, m]$, where $m > 0$ is

A. $2m^2 + 3$

B. $\frac{2m^3 + 3}{m}$

C. $\frac{2}{m}(m + 3)$

D. $\frac{m}{2}(m^2 + 3)$

E. $6m^2 + 3$

The average value of the function is the integral which represents the rectangle equivalent to the area.

$$\text{Average value} = \frac{1}{m} \int_0^m 2x^3 + 3x \, dx$$

$$\text{Avg value} = \frac{1}{m} \left[\frac{2x^4}{4} + \frac{3x^2}{2} \right]_0^m$$

$$\text{Avg value} = \frac{1}{m} \left[\frac{x^4}{2} + \frac{3x^2}{2} \right]_0^m$$

$$\text{Avg value} = \frac{1}{m} \left[\left(\frac{m^4}{2} + \frac{3m^2}{2} \right) - (0) \right]$$

$$\text{Avg value} = \frac{m^3}{2} + \frac{3m}{2}$$

$$\therefore \text{Avg value} = \frac{m}{2}(m^2 + 3)$$

Therefore, the answer is D.

Question 11

A tangent line to the graph of $y = xe^{2x}$ can be found at $x = 1$.
This tangent will cross the y -axis at

A. $-2e^2$

B. e^2

C. $\frac{-2}{e^2}$

D. $3e^2$

E. $\frac{e^2}{2}$

The coordinate at $x = 1$ will be $y = (1)e^{2(1)}$, $(1, e^2)$.

To find the gradient of the tangent

$$\frac{dy}{dx} = e^{2x}(1) + 2xe^{2x}$$

$$\frac{dy}{dx} = e^{2x} + 2xe^{2x}$$

When $x = 1$,

$$\frac{dy}{dx} = e^2 + 2(1)e^{2(1)} = 3e^2$$

For the equation of the tangent

$$y - e^2 = 3e^2(x - 1)$$

$$y - e^2 = 3e^2x - 3e^2$$

$$\therefore y = 3e^2x - 2e^2$$

Therefore, the answer is A.

Question 12

The domain and range of $f(x) = 3\log_e(x+1) - x$, respectively, is given by

- A. $(-1, \infty)$ and $\left(\log_e\left(\frac{27}{e^2}\right), \infty\right)$
- B. $(-1, \infty)$ and $\left(-\infty, \log_e\left(\frac{27}{e^2}\right)\right)$**
- C. $(-1, \infty)$ and $\left(-\infty, \log_e(27e^2)\right)$
- D. $(-1, \infty)$ and $\left(\log_e(27e^2), \infty\right)$
- E. $\left(-\infty, \log_e\left(\frac{27}{e^2}\right)\right)$ and $(-1, \infty)$

The graph can be observed to have an asymptote at $x = -1$ and a turning point as a maximum, hence shown in the corresponding domain and range.

$$f(x) = 3\log_e(x+1) - x$$

$$f'(x) = 3\left(\frac{1}{x+1}\right) - 1 = 0$$

$$\frac{3}{x+1} = 1$$

$$x+1 = 3$$

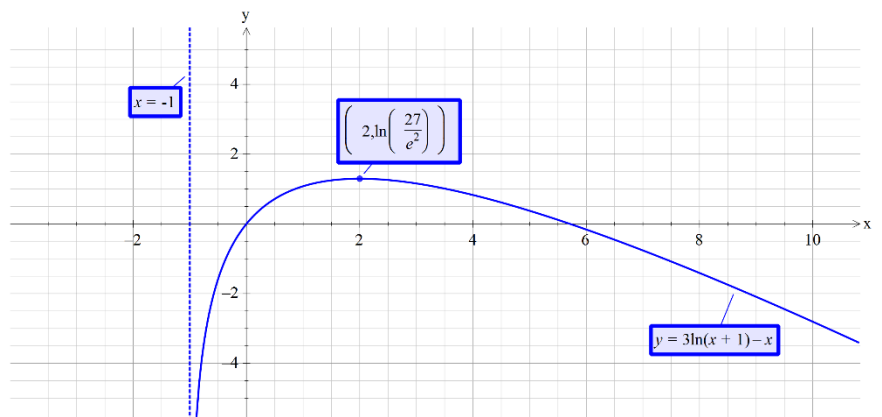
$$\therefore x = 2$$

$$f(2) = 3\log_e(3) - 2$$

$$f(2) = \log_e(27) - 2\log_e(e)$$

$$f(2) = \log_e(27) - \log_e(e^2)$$

$$\therefore f(2) = \log_e\left(\frac{27}{e^2}\right)$$



Therefore, the answer is B.

Question 13

If the point $(3a, -b)$ is transformed by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

then the image coordinate is

A. $(4 + 9a, 2b - 1)$

B. $(9a - 4, 2b + 1)$

C. $(4 - 9a, -2b - 1)$

D. $(3a, -b)$

E. $(-9a, -2b)$

Substitute the initial coordinate into the initial matrix

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3a \\ -b \end{bmatrix} + \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -9a \\ -2b \end{bmatrix} + \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

$$x' = -9a + 4$$

$$y' = -2b - 1$$

$$\therefore \text{image coordinate } (4 - 9a, -2b - 1)$$

Therefore, the answer is C.

Question 14

Which one of the following is the inverse function of $f : (-\infty, -4] \rightarrow \mathbb{R}$, $f(x) = x^2 + 8x - 1$?

A. $f^{-1} : [-17, \infty) \rightarrow \mathbb{R}$, $f^{-1}(x) = -4 + \sqrt{x+17}$

B. $f^{-1} : [17, \infty) \rightarrow \mathbb{R}$, $f^{-1}(x) = -4 - \sqrt{x-17}$

C. $f^{-1} : [-17, 17) \rightarrow \mathbb{R}$, $f^{-1}(x) = 4 - \sqrt{x+17}$

D. $f^{-1} : [-17, \infty) \rightarrow \mathbb{R}$, $f^{-1}(x) = -4 - \sqrt{x+17}$

E. $f^{-1} : [0, \infty) \rightarrow \mathbb{R}$, $f^{-1}(x) = -4 - \sqrt{x+17}$

By completing the square, the $\text{ran } f(x) \in [-17, \infty)$.

To find the inverse function,

Let $y = f(x)$, swap x & y

$$x = y^2 + 8y - 1$$

$$x + 1 = (y^2 + 8y + 16) - 16$$

$$x + 17 = (y + 4)^2$$

$$y + 4 = \pm\sqrt{x+17} \quad \text{but} \quad \text{dom } f(x) \in (-\infty, -4]$$

$$y = -\sqrt{x+17} - 4$$

$$\therefore f^{-1} : [-17, \infty) \rightarrow \mathbb{R}, f^{-1}(x) = -\sqrt{x+17} - 4$$

Therefore, the answer is D.

Question 15

Given that $\frac{d(2x \cos(3x))}{dx} = 2 \cos(3x) - 6x \sin(3x)$, then $\int x \sin(3x) dx$ is equal to

A. $\frac{1}{9} \left(\int 2 \cos(3x) dx - 2x \cos(3x) \right)$

B. $\frac{1}{3} \sin(3x) - \frac{1}{9} \cos(3x) + c$

C. $\frac{1}{3} \left(\int 2 \cos(3x) dx - 2x \cos(3x) \right)$

D. $\frac{1}{3} \cos(3x) - \frac{1}{9} \sin(3x) + c$

E. $\frac{1}{6} \left(\int 2 \cos(3x) dx - 2x \cos(3x) \right)$

Using integration by recognition,

$$\int 2 \cos(3x) - 6x \sin(3x) dx = 2x \cos(3x)$$

$$\int 2 \cos(3x) dx - \int 6x \sin(3x) dx = 2x \cos(3x)$$

$$\int 2 \cos(3x) dx - 2x \cos(3x) = \int 6x \sin(3x) dx$$

$$\frac{1}{6} \int 6x \sin(3x) dx = \frac{1}{6} \left(\int 2 \cos(3x) dx - 2x \cos(3x) \right)$$

$$\therefore \int x \sin(3x) dx = \frac{1}{6} \left(\int 2 \cos(3x) dx - 2x \cos(3x) \right)$$

Therefore, the answer is E.

Question 16

The waiting time at a kiosk, in minutes, is normally distributed with a mean of 8 and a standard deviation of 1.2.

When a customer arrives at the kiosk, the probability that they wait longer than 10 minutes is closest to

A. 0.7408

B. 0.4780

C. 0.8748

D. 0.0478

E. 0.7804

Let X represent the waiting time at the kiosk with the parameters

$$\Pr(X > 10) = \text{normCdf}(10, \infty, 8, 1.2)$$

$$\therefore \Pr(X > 10) = 0.0478$$

Therefore, the answer is D.

Question 17

Let $h:[0,4] \rightarrow R, h(x) = \frac{-2}{1-x} + 3$

Which one of the following statements about h is true?

A. An endpoint is $\left(4, \frac{5}{3}\right)$

B. The y -intercept is $(0,1)$

C. The asymptotes are $x = -1$ and $y = 3$

D. The range of $h(x)$ is R^+

E. $h(x)$ has a stationary point at $x = 2$

Going through the options individually,

A.

$$h(4) = \frac{-2}{1-4} + 3$$

$$h(4) = \frac{11}{3} \quad \text{coordinate} \left(4, \frac{11}{3}\right)$$

B.

$$h(0) = \frac{-2}{1-0} + 3$$

$$h(0) = 1 \quad \text{coordinate} (0,1)$$

C.

$$\text{The asymptotes are } x = 1 \text{ and } y = 3$$

D.

$$\text{The range of } h(x) \in R \setminus \{3\}$$

E.

Hyperbolas do not have stationary points.

Therefore, the answer is B.

Question 18

The average rate of change of $g(x) = x^2 \sin(2x)$ over the interval $\left[-\frac{\pi}{2}, \frac{\pi}{4}\right]$ is

A. π

B. $\frac{\pi}{12}$

C. $\frac{\pi}{2}$

D. $\frac{12}{\pi}$

E. $\frac{3\pi}{4}$

Using the gradient equation to find the average rate of change,

$$g\left(\frac{-\pi}{2}\right) = \left(\frac{-\pi}{2}\right)^2 \sin\left(\frac{-2\pi}{2}\right)$$

$$g\left(\frac{-\pi}{2}\right) = \frac{\pi^2}{4} \sin(-\pi) = 0 \quad \left(\frac{-\pi}{2}, 0\right)$$

$$g\left(\frac{\pi}{4}\right) = \left(\frac{\pi}{4}\right)^2 \sin\left(\frac{2\pi}{4}\right)$$

$$g\left(\frac{\pi}{4}\right) = \frac{\pi^2}{16} \sin\left(\frac{\pi}{2}\right) = \frac{\pi^2}{16} \quad \left(\frac{\pi}{4}, \frac{\pi^2}{16}\right)$$

$$\text{Average ROC} = \frac{\pi^2/16 - 0}{\pi/4 - \pi/2}$$

$$\therefore \text{Average ROC} = \frac{\pi^2/16}{3\pi/4} = \frac{\pi}{12}$$

Therefore, the answer is B.

Question 19

The area bounded by the graph of $f(x)$, the line $x = 0$, the line $x = 1$ and the x -axis, is $\frac{2}{e} + 1$.

A possible equation for $f(x)$ is

A. $f(x) = -2e^{x-1} + 3$

B. $f(x) = 2e^{x+1} + 3$

C. $f(x) = -2e^{x-1} - 3$

D. $f(x) = -e^{x-1} + 3$

E. $f(x) = -2e^x + 3$

Going through the options individually,

A.

$$\begin{aligned} & \int_0^1 -2e^{x-1} + 3 \, dx \\ &= \left[-2e^{x-1} + 3x \right]_0^1 \\ &= \left[(-2e^0 + 3(1)) - (-2e^{-1} + 3(0)) \right] \\ &= (-2 + 3) + 2e^{-1} = 1 + \frac{2}{e} \end{aligned}$$

B.

$$\begin{aligned} & \int_0^1 2e^{x+1} + 3 \, dx \\ &= \left[2e^{x+1} + 3x \right]_0^1 \\ &= \left[(2e^2 + 3(1)) - (2e^1 + 3(0)) \right] \\ &= (2e^2 + 3) - 2e = 2e^2 - 2e + 3 \end{aligned}$$

C.

$$\begin{aligned} & \int_0^1 -2e^{x-1} - 3 \, dx \\ &= \left[-2e^{x-1} - 3x \right]_0^1 \\ &= \left[(-2e^0 - 3(1)) - (-2e^{-1} - 3(0)) \right] \\ &= (-2 - 3) + 2e^{-1} = -5 + \frac{2}{e} \end{aligned}$$

D.

$$\begin{aligned}
& \int_0^1 -e^{x-1} + 3 \, dx \\
&= \left[-e^{x-1} + 3x \right]_0^1 \\
&= \left[\left(-e^0 + 3(1) \right) - \left(-e^{-1} + 3(0) \right) \right] \\
&= (-1 + 3) + e^{-1} = 2 + \frac{1}{e}
\end{aligned}$$

E.

$$\begin{aligned}
& \int_0^1 -2e^x + 3 \, dx \\
&= \left[-2e^x + 3x \right]_0^1 \\
&= \left[\left(-2e^1 + 3(1) \right) - \left(-2e^0 + 3(0) \right) \right] \\
&= (-2e + 3) + 2 = 5 - 2e
\end{aligned}$$

Therefore, the answer is A.

Question 20

Which one of the following statements is false for $f : (0,10] \rightarrow R, f(x) = \log_e(x) - \cos(x)$?

A. It has no y-intercept

B. It has an endpoint at $\frac{1}{\log_{10}(e)} - \cos(10)$

C. It has 3 turning points

D. $f'(x) = \frac{1 + x \sin(x)}{x}$

E. It has a stationary point of inflection

Going through the options individually,

A.

The domain is exclusive at the lower bound, so no y-intercept is possible.

B.

$$f(10) = \log_e(10) - \cos(10)$$

$$f(10) = \frac{\log_{10}(10)}{\log_{10}(e)} - \cos(10)$$

$$\therefore f(10) = \frac{1}{\log_e(10)} - \cos(10)$$

C.

Sketching $f : (0,10] \rightarrow R, f(x) = \log_e(x) - \cos(x)$, you can observe 3 clear turning points, 2 maximums and 1 minimum.

D.

$$f'(x) = \frac{1}{x} + \sin(x)$$

$$f'(x) = \frac{1 + x \sin(x)}{x}$$

E.

Sketching $f : (0,10] \rightarrow R, f(x) = \log_e(x) - \cos(x)$, you cannot observe any stationary points of inflection.

Therefore, the answer is E.

SECTION B – Extended Response questions

Question 1 (15 marks)

Let $f: R^+ \rightarrow R, f(x) = x^2 \log_e(x)$.

- a. Find $\{x: f(x) = 0\}$.

2 marks

$$x^2 \log_e(x) = 0$$

M1

Let $x^2 = 0, x = 0$ but not possible, $\therefore x \neq 0$

$$\text{Let } \log_e(x) = 0, \therefore x = 1$$

A1

- b. i. Find the stationary point of $f(x)$.

3 marks

$$f'(x) = \log_e(x)(2x) + x^2 \left(\frac{1}{x} \right)$$

$$f'(x) = 2x \log_e(x) + x$$

$$f'(x) = x(2 \log_e(x) + 1)$$

$$\text{Let } f'(x) = 0$$

$$x(2 \log_e(x) + 1) = 0$$

M1

Let $x = 0$ but not possible, $\therefore x \neq 0$

$$\text{Let } 2 \log_e(x) + 1 = 0$$

$$2 \log_e(x) = -1$$

$$\log_e(x) = \frac{-1}{2}$$

$$x = e^{-1/2} = \frac{1}{\sqrt{e}}$$

M1

For the stationary point

$$f\left(\frac{1}{\sqrt{e}}\right) = \left(\frac{1}{\sqrt{e}}\right)^2 \log_e\left(\frac{1}{\sqrt{e}}\right)$$

$$f\left(\frac{1}{\sqrt{e}}\right) = \frac{1}{e} \log_e\left(e^{-1/2}\right)$$

$$f\left(\frac{1}{\sqrt{e}}\right) = \frac{-1}{2e}$$

$$\therefore \text{coordinate } \left(\frac{1}{\sqrt{e}}, \frac{-1}{2e}\right)$$

A1

- ii. Show that the nature of the turning point is a minimum.

1 mark

x	$\frac{1}{2}$	$\frac{1}{\sqrt{e}}$	1
$f'(x)$	-0.19	0	1
slope	\	—	/

A1

Hence, the turning point is a local minimum.

- c. Find the equation of the tangent line at $x = e$.

2 marks

$$f(e) = e^2 \log_e(e)$$

$$f(e) = e^2 \quad \therefore (e, e^2)$$

$$f'(e) = 2e \log_e(e) + e$$

$$f'(e) = 3e$$

M1

$$y - e^2 = 3e(x - e)$$

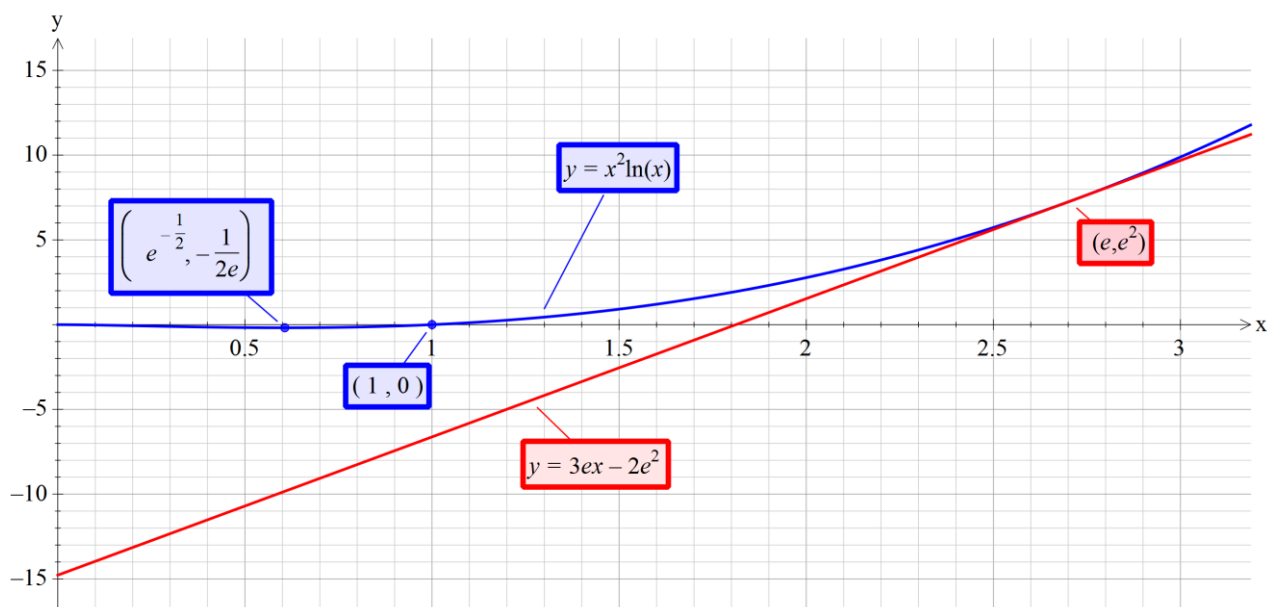
$$y - e^2 = 3ex - 3e^2$$

$$\therefore y = 3ex - 2e^2$$

A1

- d. Sketch $f(x)$ and the tangent line at $x = e$, for $x \in (0, 3]$.

3 marks



1 mark – $f(x)$ shape with endpoint $(3, 9\log_e(3))$

1 mark – $f(x)$ intercepts and turning point

1 mark – Tangent with endpoint $(3, 9e - 2e^2)$

- e. i. If $y = x^3 \log_e(x)$, find $\frac{dy}{dx}$.

1 mark

$$\frac{dy}{dx} = \log_e(x)(3x^2) + x^3\left(\frac{1}{x}\right)$$

$$\frac{dy}{dx} = 3x^2 \log_e(x) + x^2$$

A1

ii. Hence, algebraically find $\int_1^2 x^2 \log_e(x) dx$.

3 marks

$$\int_1^2 3x^2 \log_e(x) + x^2 dx = x^3 \log_e(x)$$

$$\int_1^2 3x^2 \log_e(x) dx + \int_1^2 x^2 dx = x^3 \log_e(x)$$

M1

$$\int_1^2 3x^2 \log_e(x) dx = x^3 \log_e(x) - \int_1^2 x^2 dx$$

$$\int_1^2 x^2 \log_e(x) dx = \frac{1}{3} \left[x^3 \log_e(x) - \frac{x^3}{3} \right]_1^2$$

M1

$$\int_1^2 x^2 \log_e(x) dx = \frac{1}{3} \left[\left(8 \log_e(2) - \frac{8}{3} \right) - \left(\log_e(1) - \frac{1}{3} \right) \right]$$

$$\int_1^2 x^2 \log_e(x) dx = \frac{1}{3} \left(8 \log_e(2) - \frac{8}{3} + \frac{1}{3} \right)$$

$$\therefore \int_1^2 x^2 \log_e(x) dx = \frac{1}{3} \left(8 \log_e(2) - \frac{7}{3} \right)$$

A1

Question 2 (16 marks)

An airport has been analysing their departure processes and have some information on three main stages: taking carry-on luggage, selection for random swabbing and proceeding through customs.

Carry-on luggage statistics has shown to follow a normal distribution with a mean of 7.2 kg and a variance of 4.70.

- a. If the top 10% of carry-on luggage is rejected as too heavy and needs to be placed in the cargo hold, what is the maximum acceptable weight for carry-on luggage, correct to 2 decimal places?

2 marks

The top 10% will occur at the upper end of the Normal Distribution as shown. This means that the lower 90% can be used as the area in our working.

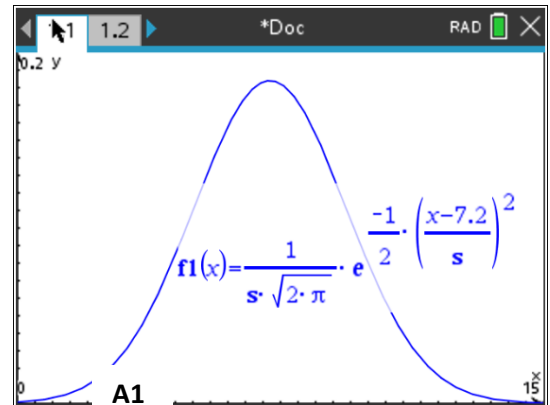
Let X represent the weight of carry-on luggage.

$$\Pr(X > a) = 0.1$$

Use $\text{invNorm}(0.9, 7.2, \sqrt{4.70})$ **M1**

$$\therefore a = 9.98$$

Hence, the maximum acceptable weight is 9.98 kgs.



- b. Find the value of c if $\Pr(-c < X < c) = 0.95$ for carry-on luggage weights, correct to 3 decimal places?

1 mark

For a Normal Distribution, for $\Pr(-c < X < c) = 0.95$ occurs 2 standard deviations from the mean, so

$$\mu - 2\sigma \leq X \leq \mu + 2\sigma$$

$$7.2 - 2(\sqrt{4.70}) \leq X \leq 7.2 + 2(\sqrt{4.70})$$

$$2.864 \leq X \leq 11.536$$

So carry-on luggage is between 2.864 and 11.536 kgs, with 95% confidence. **A1**

- c. A low-cost carrier has a different policy on carry-on luggage. Their limits of acceptability are imposed on the lowest 5% and highest 10%, being 3.1327 kg and 7.5223 kg respectively. Find the mean and standard deviation of this normally distributed policy, correct to 2 decimal places.

3 marks

Show the upper and lower bounds graphically, and find the equivalent standard Z-scores to the X values.

Lower bound, 5%

$$Z_1 = \frac{3.1327 - \mu}{\sigma}$$

$$-1.6449 = \frac{3.1327 - \mu}{\sigma} \quad \text{M1}$$

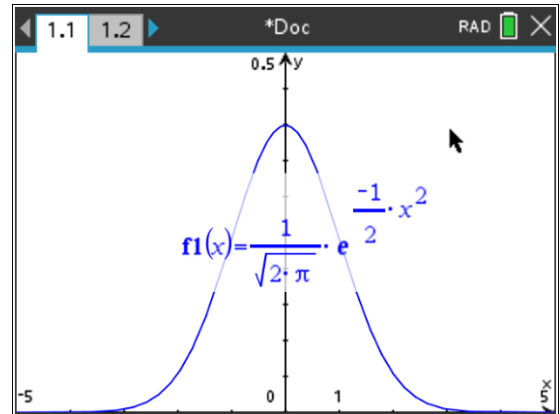
Upper bound, 10%

$$Z_2 = \frac{7.5223 - \mu}{\sigma}$$

$$1.2816 = \frac{7.5223 - \mu}{\sigma} \quad \text{M1}$$

$$\frac{-1.6449}{1.2816} = \frac{3.1327 - \mu}{7.5223 - \mu}$$

$$\therefore \mu = 5.60 \text{ and } \sigma = 1.50$$



Hence, the mean is 5.60 kgs and the standard deviation is 1.50 kgs. **A1**

Random swabbing is a way of checking for traces of explosives and other potential threats. The airport has set a target of checking 22% of the passengers as they pass through security.

- d. If a group of 10 travellers pass through security, find the probability that less than half are swabbed, correct to 4 decimal places.

2 marks

Let X represent the number of travellers swabbed. Using the Binomial Distribution where $n = 10$ and $p = 0.22$.

$$\Pr(X \leq 4) = \text{binomCdf}(10, 0.22, 0, 4) \quad \text{M1}$$

$$\Pr(X \leq 4) = 0.9521 \quad \text{A1}$$

- e. If the airport claims that there is more than a 98% chance of at least 2 passengers being swabbed when n travellers pass through security, find the smallest possible value of n .

2 marks

Setup the Binomial Distribution which represents this situation

$$\Pr(X \geq 2) > 0.98$$

$$1 - \Pr(X = 0) - \Pr(X = 1) > 0.98$$

$$0.02 < \Pr(X = 0) + \Pr(X = 1)$$

$$0.02 < \binom{n}{0}(0.22)^0(0.78)^n + \binom{n}{1}(0.22)(0.78)^{n-1} \quad \mathbf{M1}$$

$$0.02 < (0.78)^n + n(0.22)(0.78)^{n-1}$$

$$n > 23.9959\dots$$

$$\therefore n = 24$$

Hence, 24 passengers will need to pass through security for this claim to be true. **A1**

From the group of 10 travellers previously mentioned, 6 of them are locals and the rest are internationals. As they pass through the next stage and approach customs, they are asked to approach the desk 4 at a time.

- f. Find the probability that the 4 chosen are all internationals.

1 mark

Let X represent the number of local travellers. This is a sampling without replacement situation.

$$\Pr(X = 0) = \frac{\binom{6}{0}\binom{4}{4}}{\binom{10}{4}}$$

$$\therefore \Pr(X = 0) = \frac{1}{210} \quad \mathbf{A1}$$

- g. Find the probability that more locals are chosen than internationals in the selected 4 travellers to approach the desk.

2 marks

If there are more locals than internationals, it means that there can be 3 or 4 locals included in the selected 4 people.

$$\Pr(X \geq 3) = \frac{\binom{6}{3}\binom{4}{1} + \binom{6}{4}\binom{4}{0}}{\binom{10}{4}} \quad \mathbf{M1}$$

$$\Pr(X \geq 3) = \frac{(20)(4) + (15)(1)}{210}$$

$$\therefore \Pr(X \geq 3) = \frac{95}{210} = \frac{19}{42} \quad \mathbf{A1}$$

- h. Given that at least 3 locals were chosen to approach the desk, find the probability that no internationals were included in the selected 4 travellers.

3 marks

We know that 3 locals were chosen so this is the conditional element of the question which will be included. No internationals implies 4 locals.

$$\Pr(X = 4 | X \geq 3) = \frac{\Pr(X = 4 \cap X \geq 3)}{\Pr(X \geq 3)} \quad \mathbf{M1}$$

$$\Pr(X = 4 | X \geq 3) = \frac{\Pr(X = 4)}{\Pr(X \geq 3)}$$

$$\Pr(X = 4 | X \geq 3) = \frac{\binom{6}{4}\binom{4}{0}}{\Pr(X \geq 3)}$$

$$\Pr(X = 4 | X \geq 3) = \frac{15/210}{19/42} \quad \mathbf{M1}$$

$$\therefore \Pr(X = 4 | X \geq 3) = \frac{3}{19} \quad \mathbf{A1}$$

Question 3 (15 marks)

Temperature variation for a wintery town can be given by

$$T(t) = -3\cos\left(\frac{\pi}{12}(t-b)\right) + 1, \quad t \in [0, 24]$$

where t is given in hours, starting at 9pm on a Sunday, and T is measured in $^{\circ}\text{C}$.

- a. Find the amplitude and period of $T(t)$.

1 mark

Amplitude is 3° and period is $\frac{2\pi}{\pi/12} = 24$ hours. **A1**

A scientist takes a temperature reading at 10pm on the same day and records it as $\frac{-3\sqrt{2}}{2} + 1$

- b. Show that the smallest positive value of $b = 4$, if $b > 0$.

2 marks

Let $t = 1$

$$T(1) = -3\cos\left(\frac{\pi}{12}(1-b)\right) + 1 = \frac{-3\sqrt{2}}{2} + 1$$

$$-3\cos\left(\frac{\pi}{12}(1-b)\right) = \frac{-3\sqrt{2}}{2} \quad \textbf{M1}$$

$$\cos\left(\frac{\pi}{12}(1-b)\right) = \frac{\sqrt{2}}{2}$$

$$\frac{\pi}{12}(1-b) = \frac{-\pi}{4}$$

$$1-b = -3 \quad \textbf{A1}$$

$$\therefore b = 4$$

- c. State the initial temperature for the town.

1 mark

Initial temperature occurs when $t = 0$.

$$T(0) = -3\cos\left(\frac{\pi}{12}(0-4)\right) + 1$$

$$T(0) = -3\cos\left(\frac{-\pi}{3}\right) + 1$$

$$T(0) = -3\left(\frac{1}{2}\right) + 1 = \frac{-1}{2}$$

Hence, initial temperature is -0.5°C

A1

- d. If the temperature is taken from the initial time for one period, find the t intercepts, correct to 3 decimal places, hence state the time to the nearest minute.

3 marks

Find the t – intercepts when the temperature is zero.

$$-3\cos\left(\frac{\pi}{12}(t-4)\right)+1=0 \quad 0 \leq t \leq 24$$

$$\cos\left(\frac{\pi}{12}(t-4)\right)=\frac{1}{3} \quad -4 \leq t-4 \leq 20$$

$$(Basic\ Angle)\theta = \cos^{-1}\left(\frac{1}{3}\right), 1st \& 4th\ quads \quad \frac{-\pi}{3} \leq \frac{\pi}{12}(t-4) \leq \frac{5\pi}{3} \quad \mathbf{M1}$$

$$\frac{\pi}{12}(t-4) = \theta, 2\pi - \theta$$

$$t = \frac{12}{\pi}\theta + 4, \frac{12}{\pi}(2\pi - \theta) + 4$$

$$\therefore t = 8.702, 23.298 \quad \mathbf{M1}$$

This corresponds to 8 hours 42 minutes and 23 hours 18 minutes, hence the time will be Monday 5.42am and 8.18pm respectively.

A1

- e. Find the average value of the temperature over the first 12 hours.

3 marks

Use the integral to show the average value over the first 12 hours

$$Average\ value = \frac{1}{12-0} \int_0^{12} -3\cos\left(\frac{\pi}{12}(t-4)\right)+1\ dx$$

$$Avg\ Val = \frac{1}{12} \left[\frac{-36}{\pi} \sin\left(\frac{\pi}{12}(t-4)\right) + t \right]_0^{12} \quad \mathbf{M1}$$

$$Avg\ Val = \frac{1}{12} \left[\left(\frac{-36}{\pi} \sin\left(\frac{\pi}{12}(8)\right) + 12 \right) - \left(\frac{-36}{\pi} \sin\left(\frac{\pi}{12}(-4)\right) + 0 \right) \right]$$

$$Avg\ Val = \frac{1}{12} \left[\left(\frac{-36}{\pi} \sin\left(\frac{2\pi}{3}\right) + 12 \right) - \left(\frac{-36}{\pi} \sin\left(\frac{-\pi}{3}\right) \right) \right]$$

$$Avg\ Val = \frac{1}{12} \left[\left(\frac{-36}{\pi} \times \frac{\sqrt{3}}{2} + 12 \right) - \left(\frac{-36}{\pi} \times \frac{-\sqrt{3}}{2} \right) \right] \quad \mathbf{M1}$$

$$Avg\ Val = \frac{1}{12} \left(\frac{-18\sqrt{3}}{\pi} + 12 - \frac{-18\sqrt{3}}{\pi} \right)$$

$$\therefore Avg\ Val = 1 - \frac{3\sqrt{3}}{\pi} \quad \mathbf{A1}$$

- f. Find the stationary points of $T(t)$ over the whole day.

2 marks

Find when the derivative is equal to zero

$$T'(t) = \frac{\pi}{4} \sin\left(\frac{\pi}{12}(t-4)\right) = 0 \quad 0 \leq t \leq 24$$

$$\sin\left(\frac{\pi}{12}(t-4)\right) = 0 \quad -4 \leq t-4 \leq 20$$

$$(Basic \ Angle) \theta = 0, \pi \text{ 1st \& 2nd quads} \quad \frac{-\pi}{3} \leq \frac{\pi}{12}(t-4) \leq \frac{5\pi}{3} \quad \mathbf{M1}$$

$$\frac{\pi}{12}(t-4) = 0, \pi$$

$$t-4 = 0, 12$$

$$\therefore t = 4, 16$$

Hence, the stationary points occur at $(0, 4)$ and $(16, 0)$.

A1

- g. Find the equation of the normal at $t = 6$.

3 marks

Coordinate at $t = 6$

$$T(6) = -3 \cos\left(\frac{\pi}{12}(2)\right) + 1$$

$$T(6) = -3 \cos\left(\frac{\pi}{6}\right) + 1$$

$$T(6) = -3 \times \frac{\sqrt{3}}{2} + 1$$

$$\therefore T(6) = 1 - \frac{3\sqrt{3}}{2} \quad \left(6, 1 - \frac{3\sqrt{3}}{2}\right) \quad \mathbf{M1}$$

Gradient of the tangent at $t = 6$

$$T'(6) = \frac{\pi}{4} \sin\left(\frac{\pi}{12}(2)\right)$$

$$T'(6) = \frac{\pi}{4} \sin\left(\frac{\pi}{6}\right)$$

$$\therefore T'(6) = \frac{\pi}{4} \times \frac{1}{2} = \frac{\pi}{8} \quad \mathbf{M1}$$

Use perpendicular gradient for equation of the normal

$$T - \left(1 - \frac{3\sqrt{3}}{2}\right) = \frac{-8}{\pi}(t-6)$$

$$\therefore T(t) = \frac{-8}{\pi}t + \frac{48}{\pi} + 1 - \frac{3\sqrt{3}}{2} \quad \mathbf{A1}$$

Question 4 (14 marks)

Let $f : (-\infty, a) \rightarrow \mathbb{R}, f(x) = \frac{3}{(x-5)^2} - 1$.

- a. Find the largest possible value of a for $f^{-1}(x)$ to exist.

1 mark

For an inverse to exist, the original function must be one-to-one.

The maximum x value is the vertical asymptote, at $x = 5$, hence $a = 5$.

A1

- b. Find the inverse function, $f^{-1}(x)$, using full notation.

2 marks

For the original function, domain $f(x) \in (-\infty, 5)$ and range $f(x) \in (-1, \infty)$.

Let $y = f(x)$, swap x & y

$$x = \frac{3}{(y-5)^2} - 1$$

$$x + 1 = \frac{3}{(y-5)^2}$$

$$(y-5)^2 = \frac{3}{x+1}$$

M1

$$y - 5 = \pm \sqrt{\frac{3}{x+1}} \quad \text{but } \text{dom } f(x) \in (-\infty, 5)$$

$$y = 5 - \sqrt{\frac{3}{x+1}}$$

$$\therefore f^{-1} : (-1, \infty) \rightarrow \mathbb{R}, f^{-1}(x) = 5 - \sqrt{\frac{3}{x+1}}$$

A1

- c. Find the coordinates of the point(s) of intersection between $f(x) = f^{-1}(x)$, correct to 2 decimal places.

2 marks

Equate either functions given in $f(x) = f^{-1}(x) = x$

$$\text{Let } \frac{3}{(x-5)^2} - 1 = x \quad \text{OR} \quad 5 - \sqrt{\frac{3}{x+1}} = x \quad \textbf{M1}$$

On CAS

$$\therefore x = -0.91, 4.24 \quad \text{hence } (-0.91, 0) \text{ and } (4.24, 0) \quad \textbf{A1}$$

- d.** The function $h(x)$ is obtained by applying the transformation T to the function $f(x)$, where

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 6 \\ -2 \end{bmatrix}$$

- i.** Find the equation $h(x)$.

3 marks

Applying this transformation to this original equation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 6 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -x \\ 3y \end{bmatrix} + \begin{bmatrix} 6 \\ -2 \end{bmatrix}$$

$$x' = -x + 6 \quad \therefore x = 6 - x'$$

$$y' = 3y - 2 \quad \therefore y = \frac{y' + 2}{3} \quad \textbf{M1}$$

Substitute x and y into the original equation

$$\frac{y' + 2}{3} = \frac{3}{(6 - x' - 5)^2} - 1$$

$$\frac{y + 2}{3} = \frac{3}{(1 - x)^2} - 1 \quad \textbf{M1}$$

$$y + 2 = \frac{9}{(1 - x)^2} - 3$$

$$y = \frac{9}{(1 - x)^2} - 5$$

$$\therefore h(x) = \frac{9}{(1 - x)^2} - 5 \quad \textbf{A1}$$

- ii.** State the equations of the asymptotes for $h(x)$.

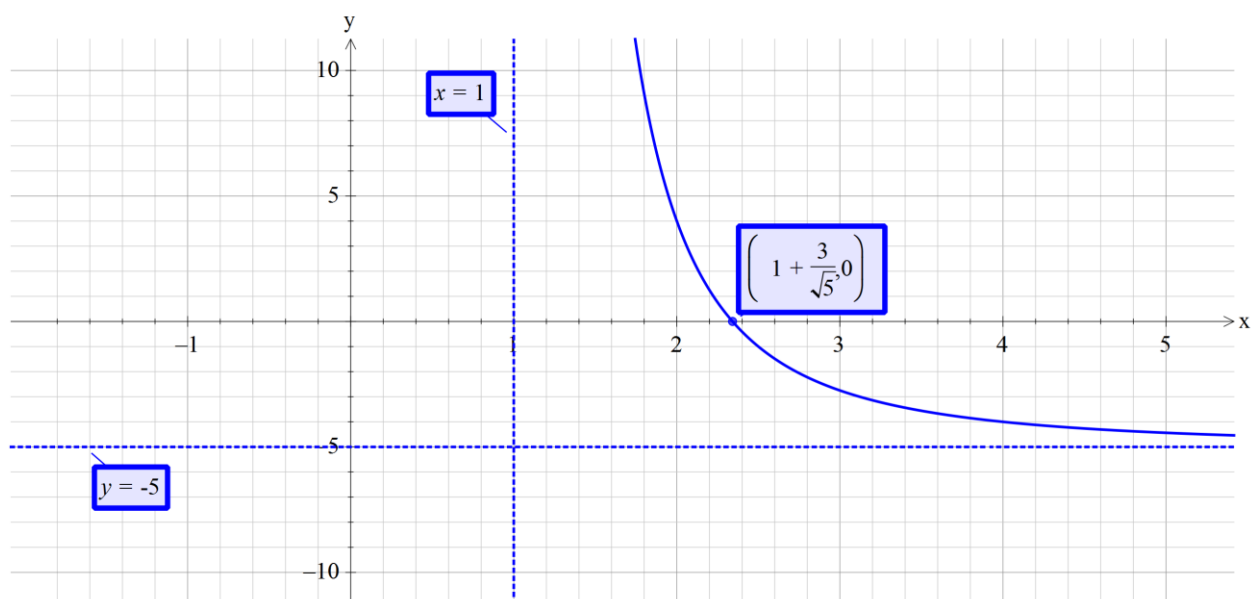
1 mark

From the previous question, asymptotes occur at $x = 1$ and $y = -5$.

A1

e. Sketch the graph of $h(x)$ within the relevant domain.

3 marks



1 mark – Shape

1 mark – x -intercept

1 mark – asymptotes

- f. Find the area for $h(x)$ enclosed between $x = 2.5, 5$ and the x -axis.

2 marks

The area in question is negative and so must be made positive

$$Area = \int_5^{5/2} 9(1-x)^{-2} - 5 \, dx$$

$$A = \left[\frac{9(1-x)^{-1}}{-1 \times -1} - 5x \right]_5^{5/2}$$

$$A = \left[\frac{9}{1-x} - 5x \right]_5^{5/2}$$

$$A = \left[\left(\frac{9}{1-5/2} - 5\left(\frac{5}{2}\right) \right) - \left(\frac{9}{1-5} - 5(5) \right) \right] \quad \mathbf{M1}$$

$$A = \left[\left(\frac{9}{-3/2} - \frac{25}{2} \right) - \left(\frac{-9}{4} - 25 \right) \right]$$

$$A = \left(-6 - \frac{25}{2} \right) - \left(\frac{-9}{4} - 25 \right)$$

$$\therefore A = 19 - \frac{41}{4} = \frac{35}{4} \, units^2 \quad \mathbf{A1}$$